CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 12: Taylor Polynomials

Finite degree approximations

12.1. The **Taylor approximation** of a function f at a point c is the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!}$$

In sum notation, this reads

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!} \ .$$

We will practice to switch back and forth from the "dot dot dot" notation to the "sum" notation and back.

12.2.

Example: Approximate $f(x) = e^x$ as a Taylor polynomial of degree 7 at $c = 0$:
Answer: As $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$, we have

$$P_7(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \ .$$

In sum notation, this is

$$P_7(x) = \sum_{k=0}^{7} \frac{x^k}{k!} \ .$$

12.3.

Example: Write down the degree 3 Taylor polynomial of $f(x) = x^5$ at $c = 1$. **Answer:** As $f(1) = 1, f'(2) = 5, f''(1) = 5 * 4, f'''(1) = 5 * 4 * 3$,

$$P_3(x) = 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 \ .$$

Technology

12.4. Technology helps to compute the Taylor polynomials. One tool is Mathematica a computer algebra system. The Taylor series is already built in. You can say $Series[Exp[x], x, 0, 10]$ for example or $Series[Sin[x], x, Pi/2, 4]$ to get the degree 4 polynomial of the Sin function at $c = \pi/2$. Here is an implementation done by hand which shows what the software is actually doing:

```
Taylor[f_,c_,n_]:=Function[x,Sum[D[f[y],{y,k}](x-c)^k/k! /. y->c, {k,0,n}]];
f = Function[x, Sin[x] + Cos[3 x]];
g = Taylor[f, 0, 5]; g[x]
```

```
Here is the output

                  2      3        4      5
               9 x      x     27 x      x
      1 + x - ---- - -- + ----- + ---
                2       6       8      120
```

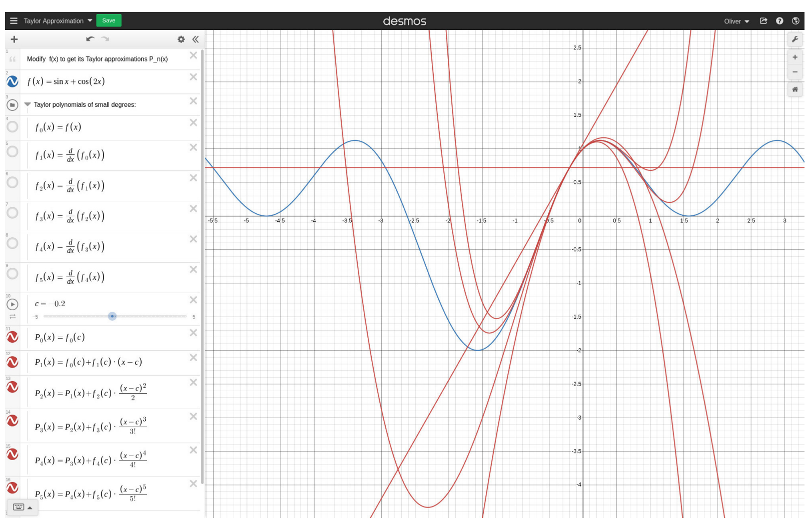
12.5. An here is an example, where the expansion is done at an other point:

```
g = Taylor[f, Pi, 4]; g[x]

                            2              3                   4
                 9 (-Pi + x)     (-Pi + x)      27 (-Pi + x)
     -1 + Pi - x + ------------- + ----------- - ---------------
                        2              6                8
```

Desmos

12.6. Desmos is a popular online graphing calculator. There are many applets for Taylor series already built. Here is an adaptation to our course:


Taylor Approximation
Save
desmos
Modify f(x) to get its Taylor approximations P_n(x)
$f(x) = \sin x + \cos(2x)$
Taylor polynomials of small degrees:
$f_0(x) = f(x)$
$f_1(x) = \frac{d}{dx}(f_0(x))$
$f_2(x) = \frac{d}{dx}(f_1(x))$
$f_3(x) = \frac{d}{dx}(f_2(x))$
$f_4(x) = \frac{d}{dx}(f_3(x))$
$f_5(x) = \frac{d}{dx}(f_4(x))$
$c = -0.2$
$P_0(x) = f_0(c)$
$P_1(x) = f_0(c)+f_1(c)\cdot(x-c)$
$P_2(x) = P_1(x)+f_2(c)\cdot\frac{(x-c)^2}{2}$
$P_3(x) = P_2(x)+f_3(c)\cdot\frac{(x-c)^3}{3!}$
$P_4(x) = P_3(x)+f_4(c)\cdot\frac{(x-c)^4}{4!}$
$P_5(x) = P_4(x)+f_5(c)\cdot\frac{(x-c)^5}{5!}$

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