CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 10: Improper Integrals

Improper integrals for unbounded functions

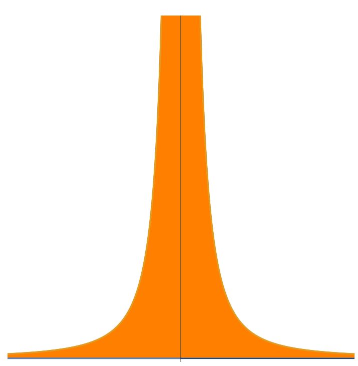
10.1. In this second lecture on improper integrals we look at integrals of the form $\int_a^b f(x)\ dx$, where f can become unbounded at some end point of the interval $a \leq x \leq b$. An other theme are integrals of the form $\int_{-\infty}^{\infty} f(x)\ dx$.

Definition: If f is continuous except at $x = a$ and the limit $\lim_{a \to 0} \int_a^b f(x)\ dx$ exists we say the integral $\int_0^b f(x)\ dx$ **converges**. Otherwise, we say that it **diverges**.

Definition: In the case of $\int_{-1}^1 f(x)\ dx$, where f is unbounded at 0, we ask that both improper integral limits $\lim_{a \to 0} \int_a^1 f(x)\ dx$ and $\lim_{a \to 0} \int_{-1}^{-a} f(x)\ dx$ do exist.

10.2. In the following example, we see what can go wrong if one integrates blindly:

Example: What is $\int_{-1}^1 \frac{1}{x^2}\ dx$? We see the function $1/x^2$ on the interval $[-1, 1]$. Integrating gives $-\frac{1}{x}|_{-1}^1 = -2$. This is obviously nonsense as the function is positive.



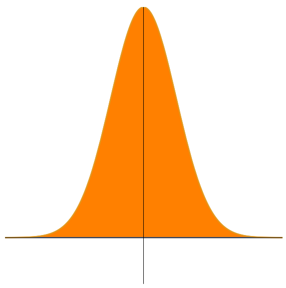
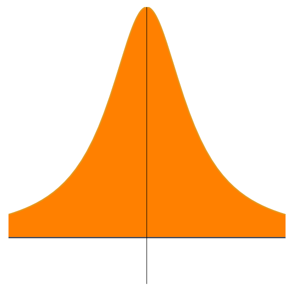
Definition: In the case of an integral like $\int_{-\infty}^{\infty} f(x)\ dx$, we need both sides $\int_0^{\infty} f(x)\ dx$ and $\int_{-\infty}^0 f(x)\ dx$ to exist.

10.3.

Example: What is $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$? In this case, we can verify that the integral converges and is π. Do it!

10.4.

Example: What is $\int_{-\infty}^{\infty} e^{-x^2}\, dx$. In this case we can not integrate but we can see that the integral converges using a comparison. We know from a movie that the integral is $\sqrt{\pi}$.

The functions $\exp(-x^2)$ and $1/(1+x^2)$. Both are important in statistics. The first one leads to the **normal distribution**, the second leads to the **Cauchy distribution**.

Example: For $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\, dx$, there are singularities at $x = 0$ and $x = 1$. The integrand has the anti derivative $2\arcsin(\sqrt{x})$. The integral converges and gives the value π. This example produces the **arcsin** distribution in statistics.

p-integrals

10.5. By explicit integration of the p-integral $\int_a^1 \frac{1}{x^p}\, dx$, (do it yourself!), we see:

$\int_0^1 \frac{1}{x^p}\, dx$ converges for $p < 1$ and diverges for $p \geq 1$.

10.6. The value $p = 1$ is a threshold. As before, for $p = 1$ we still have divergence.

Comparison test

10.7. The same comparison test works as in the case of the last lecture. If $0 \leq g(x) \leq f(x)$ and the integral $\int_0^1 f(x)\, dx$ converges then the integral $\int_0^1 g(x)\, dx$ converges.

10.8.

Example: What is

$$\int_0^1 \frac{(\sin(4x) + 3x + \cos(x))}{\sqrt{x}}\, dx\ .$$

In this case we can not integrate but we see that the integral converges by applying the comparison test. Can you see the p-integral which is an upper bound?