CALCULUS AND DIFFERENTIAL EQUATIONS

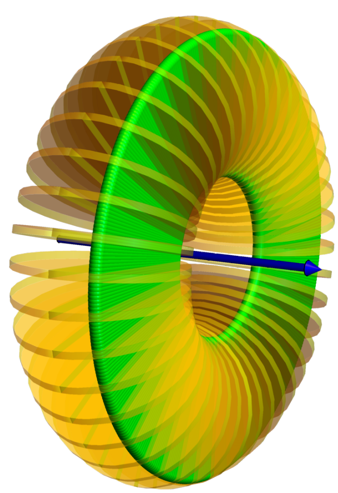
MATH 1B

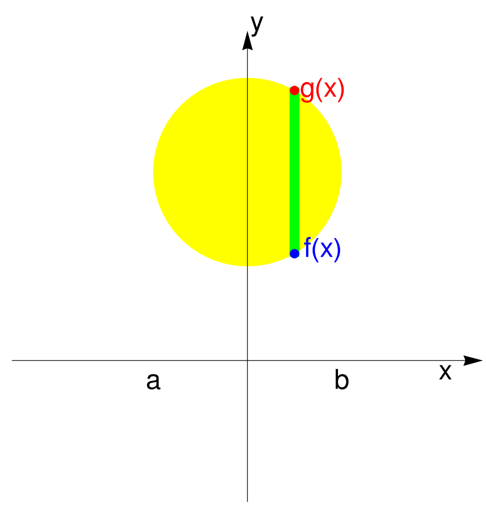
Lecture 5: Volumes of Revolution

5.1. Given a region G in two dimensions and a line, we can in three dimensional space rotate the region around that line. It produces a **solid of revolution**. What is the volume of this body? There are different ways to slice this.

Washer Method

5.2. Take a disk G of radius 1 centered at $(0, 2)$ and rotate it around the y-axes. We get a doughnut. To get the volume, slice this up perpendicular to the x-axes. We get **washer cross sections** of area $\pi(2+\sqrt{1-x^2})^2 - \pi(2-\sqrt{1-x^2})^2$. This simplifies to $4\pi\sqrt{1-x^2}$. If we integrate this from $x = -1$ to $x = 1$, we get $4\pi\pi = \boxed{4\pi^2}$.



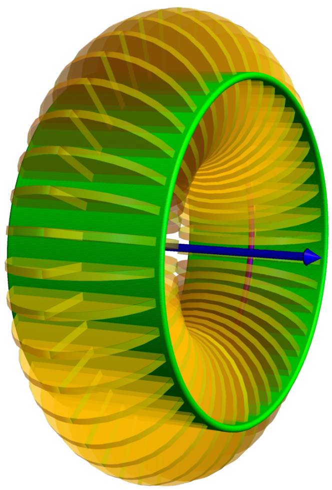


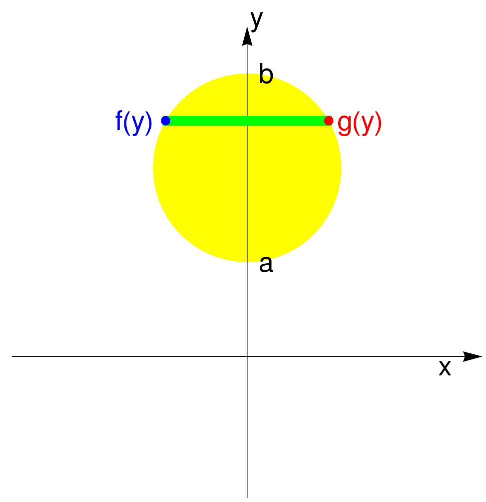
If $g(x)$ is the outer radius and $f(x)$ the inner radius, we add up the washer areas:

$$V = \int_a^b g(x)^2\pi - f(x)^2\pi \, dx \ .$$

Shell method

5.3. We can also slice horizontally and get a segment of length $L = 2\sqrt{1-(y-2)^2}$. If we rotate this around the y axes we get **cylindrical shells** of area $2\pi y L$ and so a infinitesimal shell volume $2\sqrt{1-(y-2)^2} \cdot 2\pi y$. If we integrate this from $y = 1$ to $y = 3$, the result is again $\boxed{4\pi^2}$.



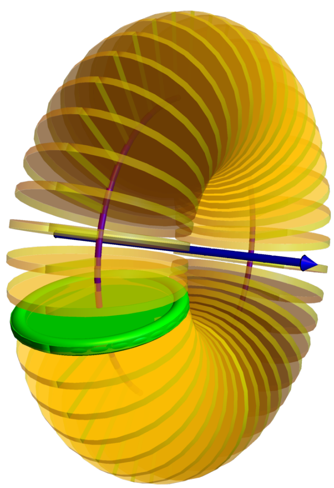


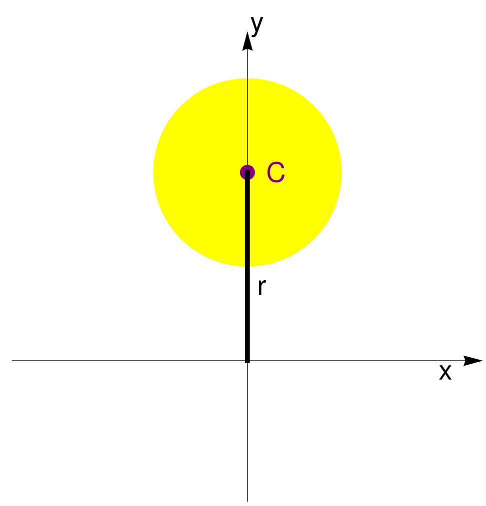
If $f(y)$ is the left bound and $g(y)$ the right bound, we add up the shell volumes

$$V = \int_a^b (g(y) - f(y)) 2\pi y \, dy \, .$$

Disk method

5.4. We can also look directly at the disk and rotate it around. The disc has area π. The center of the disk moves along a path of length $2\pi r = 4\pi$. The volume is $4\pi\pi = 4\pi^2$. It is a **formula of Pappus** assures that the volume of a solid of revolution is the length of the circle traced by the center of mass of the region times the area of the region. In the case of the doughnut just considered, the center of mass is the center of the disk. It traces a curve of length 4π. The area of the circle is π. We again have $\boxed{4\pi^2}$.





In general the volume is

$$V = A \cdot 2\pi r \, .$$

Oliver Knill, knill@math.harvard.edu, Math 1b, Harvard College, 9/15/2023