Math 464/564, O. Knill, **May 1997**

Feynman path integrals

THE PROBLEM. In quantum mechanics, the Schrödinger equation $i\hbar\dot{u} = Hu$ defines the evolution of the wave function $f(t) = U^t f = e^{-itH/\hbar} f(0)$ in a Hilbert space $\mathcal{H}$. The operator H is the **Hamiltonian**. If $\mathcal{H} = L^2(\mathbb{R}^d)$, $H = H_0 + V$, where $H_0 = -\Delta/2$ is the Hamiltonian of a free particle and $V : \mathbb{R}^d \to \mathbb{R}$ is the potential the operator is called a **Schrödinger operator**. How do we compute $(f, U^t g)$?

DEFINITION. A linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is **symmetric** if $(Au, v) = (u, Av)$ for all $u, v \in D(A)$ and **selfadjoint**, if it is symmetric and $D(A) = D(A^*)$.

BACKGROUND. One has to restrict the opeator H to a vector space $D(H) \subset \mathcal{H}$ called **domain** containing the dense set $C_0^\infty(\mathbb{R}^d)$ of all smooth functions vanishing at infinity. Define $D(A^*) = \{u \in \mathcal{H} \mid v \mapsto (Av, u) \textit{ is a bounded linear functional on } D(A)\}$. If $u \in D(A^*)$, then there exists a unique function $w = A^* u \in \mathcal{H}$ such that $(Av, u) = (v, w)$ for all $u \in D(A)$. This defines the **adjoint** A^* of A with domain $D(A^*)$.

FEYNMAN'S IDEA. Assume $H = H_0 + V$ is selfadjoint. Then

$$e^{-itH} u(x_0) = \lim_{n\to\infty} (\frac{2\pi i t}{n})^{-d/2} \int_{(\mathbb{R}^d)^n} e^{iS_n(x_0, x_1, x_2, \dots, x_n, t)} u(x_n) \, dx_1 \dots dx_n$$

where

$$S_n(x_0, x_1, \dots, x_n, t) = \frac{t}{n} \sum_{i=1}^n \frac{1}{2} (\frac{|x_i - x_{i-1}|}{t/n})^2 - V(x_i) \; .$$

This is essentially a consequence of the following formula which generalizes the Lie product formula

$$\lim_{n\to\infty} (\exp(A/n)\exp(B/n))^n = \exp(A+B)$$

for finite dimensional matrices A, B. One says X_n **converges strongly** to X if $X_n f \to X f$ for all $f \in \mathcal{H}$ and writes $X = s - \lim_{n\to\infty} X_n$.

TROTTER PRODUCT FORMULA. Given selfadjoint operators A, B defined on $D(A), D(B) \subset \mathcal{H}$. Assume $A + B$ is selfadjoint on $D = D(A) \cap D(B)$, then $e^{it(A+B)} = s - \lim_{n\to\infty} (e^{itA/n} e^{itB/n})^n$. If A, B are bounded from below, then $e^{-t(A+B)} = s - \lim_{n\to\infty} (e^{-tA/n} e^{-tB/n})^n$.

Unfortunately, there are problems to perform this summation in general over all paths in the limit when the time intervals $[x_n, x_{n-1}]$ goes to zero. However, when replacing t by t/i, one gets $(f, e^{-tH} g)$ which can be computed using Brownian motion B_t defined over a probability space $(\Omega, \mathcal{A}, P)$. If $V \in C_0^\infty(\mathbb{R}^n)$, the integral $\int_0^t V(B_s(\omega)) \, ds$ can be taken for each ω as a limit of Rieman sums. Then, $\int_0^t V(B_s) \, ds$ is a random variable. The existence of Brownian motion defines a probability measure on the space of all continuous paths on $\mathbb{R}^n$. Integration with respect to this measures is denoted by $\int \cdot dB$.

FEYNMAN-KAC FORMULA.

$$(f, e^{-tH} g) = \int \overline{f}(B_0) g(B_t) e^{-\int_0^t V(B_s)) \, ds} \, dB \; .$$

This is an integral over all continuous paths B_s starting at zero.

The Feynman-Kac formula is useful in mathematical physics: it allows to treat operators with magnetic fields or to compute groundstates and groundstate energies perturbatively. The concept of functional integration is a way of quantisation which generalizes to more situations, where canonical quantisation (replacing classical variables by operators) is not available.