Math 464/564, O. Knill, **May 1997**

Option pricing: Black-Scholes formula

The theory of stochastic processes both with discrete or continuous time can be applied to problems in financial economics. A celebrated example is the Black-Scholes option pricing formula (1973) which is usually formulated in terms of continuous time stochastic processes.

Given $a < r < b < \infty$, define $p = (r-a)/(b-a)$. Let E_n be a random walk, that is a sequence of IID random variables satisfying $P[E_n = 1] = p, P[E_n = -1] = 1-p$.

DEFINITION OF AN ECONOMY OF STOCKS AND BONDS. We define two processes B_n, S_n, where B stands for **bonds** with fixed **interest rate** r, and S for **stocks** with **fluctuating interest rates** $R_n = (a+b)/2 + E_n(a-b)/2$. The definition is

$$\begin{aligned} B_n &= (1+r)^n B_{n-1}, B_0 = 1 \\ S_n &= (1+R_n)S_{n-1}, S_0 = 1 \, . \end{aligned}$$

We see that B_n satisfies the **difference equations** $B_n - B_{n-1} = rB_n$ and S_n satisfies the **stochastic difference equation** $S_n - S_{n-1} = R_n S_{n-1}$.

DEFINITION OF A PORTFOLIO AND FORTUNE. A with respect to R_n previsible sequence of pairs of random variables A_n, V_n is called a **portfolio**. Just after time n, you have A_n units of stock and V_n units of bonds. (A negative value of A_n means "short selling" of stocks, a negative value of V means borrowing with a fixed interest rate r.) We define our **fortune** X_n by $X_0 = x$ and $X_n - X_{n-1} = A_n(S_n - S_{n-1}) + V_n(B_n - B_{n-1})$. Using $B_n - B_{n-1} = rB_n$ and $S_n - S_{n-1} = R_n S_{n-1}$, we have the recursion

$$X_n = (1+r)X_{n-1} + A_n S_{n-1}(R_n - r) \, .$$

COMPARISON WITH THE GAMBLING SYSTEM MET EARLIER. To relate this with a gambling system, we write $R_n - r = \frac{1}{2}(b-a)(Z_n - Z_{n-1})$ with $Z_n = \sum_{k=1}^n (E_k - 2p + 1)$ having expectation $E[Z_n] = 0$. The process $Y_n := (1+r)^{-n} X_n$ satisfies then

$$\begin{aligned} Y_n - Y_{n-1} &= (1+r)^{-n} A_n S_{n-1}(R_n - r) \\ &= \frac{1}{2}(b-a)(1+r)^{-n} A_n S_{n-1}(Z_n - Z_{n-1}) \\ &= C_n(Z_n - Z_{n-1}) \end{aligned}$$

showing that $Y = \int C \, dZ$ is your winning in a fair random walk Z_n with **gambling system** C_n.

DEFINITION OF THE EUROPEAN OPTION. The **European option** is a contract made at time 0 which will allow you to buy one unit of stock later at time N with prize K. If you buy such an option, then you will make the buy of the stock if the value S_N of the stock at time N satisfies $S_N > K$ and not if $S_N < K$. Your win is $(S_N - K)^+$.

DEFINITION OF A HEDGING STRATEGY. A **hedging strategy** with initial fortune x, **strike time** N and **prize** K is a **portfolio management scheme** $\{(A_n, V_n)\}_{1 \leq n \leq N}$ defined above, where $X_0 = x$, $X_n \geq 0$ (you never go bankrupt) and $X_N = (S_N - K)^+$, where $z^+ = \max\{0, z\}$.

BLACK-SCHOLES FORMULA. For $x = E[(1+r)^{-N}(S_N - K)^+]$, a unique hedging strategy exists.

PROOF. Define Y_n be the expectation of $(1-r)^{-N}(S_N - K)^+$ under the condition that $R_1, \dots, R_n$ are known. The recursion

$$Y_n - Y_{n-1} = (1-r)^{-n} A_n S_{n-1}(R_n - r)$$

defines the previsible process A_n. With $X_n = (1+r)^n Y_n$, define $V_n = (X_n - A_n S_n)/B_n$. One can actually show $A_n \geq 0$ so that no short selling is necessary.