Math 464/564, O. Knill, **May 1997**

Summary and further directions

SUMMARY. Probability theory is the study of random variables (or random vectors) on a probability space. A distribution of a variable is determined in the discrete case by $P[X = x]$ and in the continuous case by the (joint) density $f_X(x)$. The distribution is determined by the characteristic function $\Phi_X(x) = E[\exp(it \cdot X)]$. It determines properties like the (vector) mean $E[X]$ the covariance matrix $Cov[X_i, X_j]$ etc. For independent X, Y, one has $f_{X+Y}(z) = f_X \star f_Y)(z)$ and $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.
Calculations for discrete distributions need some combinatorial tools, for continuous distributions, we rely on analytic (calculus) tools. Combinatorics (e.g. the Monty-Hall problem) or problems using continuous distributions (e.g. the Bertrand paradox) can be puzzling without a solid foundation of probability theory.
A bunch of distributions with their properties, characteristic functions (make a comprehensive list!) help to model and solve practical problems.
Some important theorems in probability theory are the theorem of Caratheodory, the central limit theorem, the (weak) law of large numbers. The Chebychev-Markov inequalities as well as other inequalities (Jensen, Hölder, Cauchy-Schwartz) are useful for theory or practical estimate.
We have seen that probabilistic tools can give easy access to analytic results like the theorem of Weierstrass or the Stirling formula.
Probability theory hits at several places the foundations of mathematics. Examples are the foundation of integration (the Lebesgue integral) or the impossibility to define a probability space on the interval $[0, 1]$ using all subsets as σ-algebra (Banach-Tarsky paradox).
Studiing the sum of independent random variables (like the random walk) lead to interesting questions and beautyful results (like e.g. limit theorems, Polya's theorem) and applications (e.g. ruin problems, Wald identities). Beside the random walk, other processes like the Poisson process or the Brownian motion (not treated here) are important.

FURTHER DIRECTIONS. A continuation of the theory involves the study of stochastic processes. The later is a set of random variables which are labeled by time. For continuous time, the setup needs some care. An important study is the theory of Brownian motion or more general martingales. With such processes, one can define stochastic integrals and therefore stochastic PdE's. This is useful in mathematical physics and some problems in quantum mechanics can be solved quite nicely using path integrals. Stochastic differential equations appear also in Economics (e.g. the Black-Scholl option pricing formula).
Some directions in mathematical physics like statistical physics, quantum mechanics or transport theory involve and motivate research in probability theory. Examples are percolation problems, the statistical theory of lattice gases, gauge theory in particle physics, the theory of random Schrödinger operators, the statistical mechanics of hyperbolic dynamical systems or problems in nonequilibrium statistical mechanics and fluid dynamics. A branch of dynamical system theory called ergodic theory can be viewed as an extension of probability theory: there, a measure preserving map T on the probability space and a random variable X define a sequence of random variables $\omega \mapsto X_n(\omega) = X(T^n\omega)$ which are in general not independent.
An other direction important in applications is statistics which has a relation to probability theory similar than numerical mathematics to analysis. The problem is to model data by distributions and get information about the reliablity of these models. This is in applications mostly used in descriptive statistics, where one wants to describe, illustrate and interpret data obtained for example in a laboratory, determine correlations between different quantities, determine the parameters of the distributions, the confidence intervals, the error probability etc.

FOR THE FINAL: we went through all relevant sections all except 6.5, 6.6 in the book. The comprehensive final will be on topics, which we have treated in class and can include topics about distributed material (not the proofs of theorems like Caratheodory or the advanced central limit theorem or Polya's theorem). A good preparation includes a personal summary of the material, especially to make a reliable list of all distributions with properties and a list of definitions and results as well as an organization of the notes, a review of the homework and the midterms. As in the midterms, you can use all your notes but no book.