Math 464/564, O. Knill, **May 1997**

Definition and existence of Brownian motion

Brownian motion is an important object in mathematics. Not part of the current course we give here as a preview the definition and existence of Brownian motion.

DEFINITION. A collection of random vectors X_t, $t \in T \subset \mathbb{R}$ is called a **stochastic process**. If T is a discrete set then it is a discrete time stochastic process. An example is the random walk, where $\mathbb{T} = \mathbb{N}$. If T is an interval, then it is a continuous time stochastic process. The process is called **continuous** if $(t, \omega) \mapsto X_t(\omega)$ is continuous for almost all ω.

RECALL. A $\mathbb{R}^n$-valued random variable X is called **Gaussian** (= normal) if it has the characteristic function $E[e^{it \cdot X}] = e^{-(t,Vt)/2 + imt}$, where V is the covariance matrix and $m = E[X]$ is the mean vector. A **Gaussian processs** is a stochastic process such that $(X_{t_0}, X_{t_1}, \dots X_{t_n})$ is a Gaussian random vector for any $t_0 \leq t_1 < \dots < t_n$. It is called centered, if $m = 0$. The importance of the Gaussian distribution is that two Gaussian random vectors X, Y are independent if and only if they are uncorrelated.

DEFINITION. A continuous Gaussian process X_t with values in $\mathbb{R}^d$ having the mean vector $m_t = E[X_t]$ and the covariance matrix $V(s,t) = Cov[X_s, X_t] = E[(X_s - m_s)(X_t - m_t)^*]$ is called **Brownian motion** if for any $0 \leq t_0 < t_1 < \dots < t_n$, the random variables $X_{t_0}, X_{t_{i+1}} - X_{t_i}$ are independent and the covariance matrix V satisfies $V(s,t) = V(r,r)$, where $r = \min(s,t)$ and $s \mapsto V(s,s)$ is increasing. It is called the **standard Brownian motion** if $m_t = 0$ for all t and $V(s,t) = \min\{s,t\}$.

LEMMA. A Gaussian process with covariance $V(s,t) = V(r,r)$ with $r = \min(s,t)$ is a Brownian motion.
PROOF. For independence it is enough to check that $Cov[X_{t_0}, X_{t_{j+1}} - X_{t_j}] = 0$, $Cov[X_{t_{i+1}} - X_{t_i}, X_{t_{j+1}} - X_{t_j}] = 0$ which follows from $Cov[X_{t_0}, X_{t_{j+1}} - X_{t_j}] = V(t_0, t_{j+1}) - V(t_0, t_j) = V(t_0, t_0) - V(t_0, t_0) = 0$ and $Cov[X_{t_{i+1}} - X_{t_i}, X_{t_{j+1}} - X_{t_j}] = V(t_{i+1}, t_{j+1}) - V(t_{i+1}, t_j) - V(t_i, t_{j+1}) + V(t_i, t_j) = V(t_{i+1}, t_{i+1}) - V(t_{i+1}, t_{i+1}) - V(t_i, t_i) + V(t_i, t_i) = 0$.

THEOREM. Brownian motion exists!

SOME HISTORY. Water under a microscope contains little things moving around. One first thought that these particles were alive. **Brown** was studying the fertilization process in a species of flowers. Looking at the pollen in water through a microscope, he observed small particles in "rapid oscillatory motion". Brown's explanation to this was that matter is composed of small particles, which he calls active molecules, which exhibit a rapid, irregular motion having its origin in the particles themselves and not in the surrounding fluid. Brown's contribution was to establish Brownian motion as an important phenomenon, to demonstrate its presence in inorganic as well as organic matter and to refute by experiment wrong explanations of the phenomenon.
The topic was neglected in the first part of the 19'th century but awareness of the pheomenon remained widespread. From 1860 on, many scientists worked on the phenomenon. The first one, to express a notion close to the modern theory of Brownian motion was **N. Wiener** in 1863. Careful experiments and arguments lead to the kinetic theory that Brownian motion is caused by bombardment by the molecules of fluid. But the results failed the theory by a factor of about 100'000. The difficulty was the fact that the motion is very irregular composed of translations and rotations and that the trajectory appears to have no tangent. So, any attempt to determine the velocity of the particles failed. The success of **Einstein**'s theory of Brownian motion was largely due to his go around this question.
Einstein himself was unaware of the phenomenon Brownian motion. He predicted it on theoretical grounds and formulated a correct quantitative theory of it. Einsteins's arguments do not give a dynamical theory of Brownian motion. It only determines the nature of the motion and the value of the diffusion coefficients on the basis of some assumptions.
A modern probabilistic treatment of Brownian motion became possible in this century with an axiomatically developed theory of probability and stochastic processes.

PROOF. CONSTRUCTION OF BROWNIAN MOTION.

I) LEMMA. Given a Hilbert space $(H, ||\cdot||)$ that is a vector space on which there is a scalar product. There exists a probability space $(\Omega, \mathcal{A}, P)$ and a family $X(h), h \in H$ of random variables such $h \mapsto X(h)$ is linear, and $X(h)$ is Gaussian with mean zero and $E[X(h)^2] = ||h||^2$.
PROOF. Pick an orthonormal basis e_n in H and attach to each e_n a centered Gaussian IID random variable g_n satisfying $||g_n||_2 = 1$. Given a general $h = \sum h_n e_n \in H$, define $X(h) = \sum_n h_n g_n$ which converges in $\mathcal{L}^2$. Since g_n are independent, they are orthonormal in $\mathcal{L}^2$ so that $||X(h)||_2^2 = \sum_n h_n^2 ||g_n||_2 = \sum_n h_n^2 = ||h||_2^2$.

II) Take $H = L^2(\mathbb{R}^+, dx)$. For a measurable set $A \subset \mathbb{R}^+$, define $X(A) = X(1_A)$. The vector space $X(H) \subset \mathcal{L}^2$ is a Hilbert space isomorphic to H and in particular $E[X(h)X(h')] = (h, h')$. From I), we know that h and h' are orthogonal if and only if $X(h)$ and $X(h')$ are independent and that $E[X(A)X(B)] = Cov[X(A), X(B)] = (1_A, 1_B) = |A \cap B|$. Especially $X(A)$ and $X(B)$ are independent if and only if A and B are disjoint.

III) DEFINITION OF B_t. Define $B_t = X([0,t])$. This process has independent increments $B_{t_i} - B_{t_{i-1}}$ and is a Gaussian process. For each t, we have $E[B_t^2] = t$ and for $s < t$, the increment $B_t - B_s$ has variance $t - s$ so that $E[B_s B_t] = E[B_s^2] + E[B_s(B_t - B_s)] = E[B_s^2] = s$. This model of Brownian motion has therefore everything except continuity!

IV) KOLMOGOROV LEMMA. Given a process $X_t, t \in [a,b]$ for which there exists $p > r, K$ such that $E[|X_{t+h} - X_t|^p] \leq K \cdot h^{1+r}$ for every $t, t+h \in [a,b]$. Then X_t has a modification Y_t which is continuous: $|Y_t(\omega) - Y_s(\omega)| \leq C(\omega)\ |t-s|^\alpha$, where $0 < \alpha < r/p$
PROOF. We can assume $a = 0, b = 1$. Define $\epsilon = r - \alpha p$. By Chebychev-Markov inequality $P[|X_{t+h} - X_t|] \geq |h|^\alpha] \leq |h|^{-\alpha p} E[|X_{t+h} - X_t|^p] \leq K|h|^{1+\epsilon}$, so that $P[|X_{(k+1)/2^n} - X_{k/2^n}| \geq 2^{-n\alpha}] \leq K 2^{-n(1+\epsilon)}$. Therefore

$$\sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} P[|X_{(k+1)/2^n} - X_{k/2^n}| \geq 2^{-n\alpha}] < \infty .$$

By Borel-Cantelli's lemma, there exists $n(\omega) < \infty$ almost everywhere such that for all $n \geq n(\omega)$ and $k = 0, \dots, 2^n - 1$

$$|X_{(k+1)/2^n}(\omega) - X_{k/2^n}(\omega)| < 2^{-n\alpha} .$$

Let $n \geq n(\omega)$ and $t \in [k/2^n, (k+1)/2^n]$ of the form $t = k/2^n + \sum_{i=1}^m \gamma_i/2^{n+i}$ with $\gamma_i \in \{0,1\}$. Then $|X_t(\omega) - X_{k2^{-n}}(\omega)| \leq \sum_{i=1}^m \gamma_i 2^{-\alpha(n+i)} \leq d 2^{-n\alpha}$ with $d = (1 - 2^{-\alpha})^{-1}$. Similarly

$$|X_t - X_{(k+1)2^{-n}}| \leq d 2^{-n\alpha} .$$

Given $t, t+h \in D = \{k2^{-n} \mid n \in \mathbb{N}, k = 0, \dots n-1\}$. Take n so that $2^{-n-1} \leq h < 2^{-n}$ and k so that $k/2^{n+1} \leq t < (k+1)/2^{n+1}$. Then $(k+1)/2^{n+1} \leq t+h \leq (k+3)/2^{n+1}$ and $|X_{t+h} - X_t| \leq 2d2^{-(n+1)\alpha} \leq 2dh^\alpha$. For almost all ω, this holds for sufficiently small h. We know now that for almost all ω, the path $X_t(\omega)$ is uniformly continuous on the dense set of dyadic numbers D. Such a function can be extended to a continuous function on $[0,1]$ by defining

$$Y_t(\omega) = \lim_{s \in D \to t} X_s(\omega) .$$

Since the inequality in the assumption of the theorem implies $E[X_t(\omega) - \lim_{s \in D \to t} X_s(\omega)] = 0$ and by Fatou's lemma in measure theory $E[Y_t(\omega) - \lim_{s \in D \to t} X_s(\omega)] = 0$ we know that $X_t = Y_t$ almost everywhere. Y is therefore a modification of X. Moreover, Y satisfies for all s, t,

$$|Y_t(\omega) - Y_s(\omega)| \leq C(\omega)\ |t-s|^\alpha .$$

V) PROOF OF THE CLAIM. In one dimensions, take B_t from above. Since $X_h = B_{t+h} - B_t$ is centered with variance h, we have $E[X_h^4] = \frac{d^4}{dx^4} \exp(-x^2 h/2)_{|x=0} = 3h^2$, so that $E[(B_{t+h} - B_t)^4] = 3h^2$. By Kolmogorov's lemma there is a continuous modification of B. To define standard Brownian motion in n-dimension, we take the joint motion $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ of n independent one- dimensional Brownian motions.