Definition and existence of Poisson processes

DEFINITION. Let S be a region of $\mathbb{R}^n$, $\mathcal{B}$ the set of measurable sets and μ a measure (called **mean measure**) on S. A **Poisson process on** S (S,μ,N) is a family of random variables N_B, $B \in \mathcal{B}$ such that
(i) N_B is Poisson distributed with parameter $\mu(B)$.
(ii) If $B_j, j = 1, \dots k$ are disjoint, then N_{B_j} are independent.

INTERPRETATION. The random variables N_B can be defined over a probability space Ω where each point ω is a random collection of points in S. $N_B(\omega)$ is the number of points of ω which ly in B.
EXAMPLES. a) If B is a sky region in sky and N_B is the number of stars in that region. b) If B is a region in the Arizona desert and N_B is the number of Saguari in that region. c) If B is a time interval then N_B, the number of cars passing a certain spot on the $I10$ is a Poisson process.

SUPERPOSITION LEMMA. If $(S,\mu^{(i)},N^{(i)})$ are independent Poisson processes on S, then $(S,\sum_i \mu^{(i)}, \sum_i N^{(i)})$ is a Poisson process on S.
PROOF. We check the conditions.
(i) The sum of independent Poisson distributed random variables is Poisson distributed.
(ii) If $B_k, k = 1, \dots, n$ are disjoint, then $\{N^{(i)}_{B_k}\}_{i \in \mathbb{N}, k=1,\dots n}$ are all independent and so are therefore the random variables $N_{B_k} = \sum_{i=1}^\infty N^{(i)}_{B_k}, k = 1, \dots n$.

DEFINITION. A measure μ is called **nonatomic** if $\mu(\{x\}) = 0$ for all $x \in S$. A measure μ is called σ**-finite** if $\mu = \sum_n \mu^{(n)}$ with $\mu^{(n)}(S) < \infty$.

EXISTENCE THEOREM. If μ is a nonatomic σ-finite measure on S, then there is a Poisson process (S,μ,N) with mean measure μ.

PROOF. Write $\mu = \sum_n \mu^{(n)}$, where $\mu^{(n)}$ are nonatomic measures with $0 < \mu^{(n)}(S) < \infty$.
Take IID N_n, $n = 1, 2, \dots$ which are Poisson-$(\mu^{(n)}(S))$ distributed. Independent from them take IID $\mathbb{R}^d$-valued random vectors X_{nr}, $r = 1, 2, \dots$ which have joint distribution of a vector $X(x) = x$ on the probability space $(S, \mathcal{B}, \mu^{(n)}(\cdot)/\mu^{(n)}(S))$. All these independent random variables N_n, X_{nr} are defined on some probability space Δ with elements δ. Define $N^{(n)}_B(\delta)$ as the number of points in the finite set $\{X_{n1}(\delta), \dots, X_{nN(\delta)}(\delta)\} \cap B$. Claim: $(S,\mu^{(n)},N^{(n)})$ is a Poisson process with mean measure $\mu^{(n)}$.
(i) $N^{(n)}_B$ is $\mu^{(n)}(B) := \mu^{(n)}(B)$-Poisson distributed because $\mathrm{P}[N^{(n)}_B = k] = \mathrm{P}[N_n = k]$.
(ii) Let $B_k, k = 1, \dots, m$ be disjoint. For $l = l_0 + \dots + l_m$ and $B_0 := S \setminus \bigcup_{k=1}^m B_k$, one has $\mathrm{P}[N^{(n)}_{B_1} = l_1, \dots, N^{(n)}_{B_m} = l_m \mid N_n = l = l_0 + \sum_{j=1}^m l_j] = \frac{l!}{l_0! \cdots l_m!} \prod_{j=1}^m (\mu^{(n)}(B_j)/\mu^{(n)}(S))^{l_j}$. Therefore

$$\begin{aligned}
\mathrm{P}[N^{(n)}_{B_1} = l_1, \dots, N^{(n)}_{B_m} = l_m] &= \sum_{l=\sum_j l_j}^\infty \frac{e^{-\mu^{(n)}(S)} \mu^{(n)}(S)^l}{l!} \frac{l!}{l_0! \cdots l_m!} \prod_{j=1}^m (\mu^{(n)}(B_j)/\mu^{(n)}(S))^{l_j} \\
&= [\sum_{l_0=0}^\infty \frac{e^{-\mu^{(n)}(B_0)} \mu^{(n)}(B_0)^{l_0}}{l_0!}] \prod_{j=1}^m \frac{e^{-\mu^{(n)}(B_j)} \mu^{(n)}(B_j)^{l_j}}{l_j!} \\
&= \prod_{j=1}^m \frac{e^{-\mu^{(n)}(B_j)} \mu^{(n)}(B_j)^{l_j}}{l_j!} = \prod_{j=1}^m \mathrm{P}[N^{(n)}_{B_j} = l_j] \, .
\end{aligned}$$

The superposition lemma shows now that $(S, \mu = \sum_i \mu^{(i)}, N = \sum_i N^{(i)})$ is a Poisson process with mean measure μ.