**Math 464/564, O. Knill,** **April 1997**

# The Borel-Cantelli lemma

The Borel-Cantelli lemma appeared in two places: in a proof of an advanced central limit theorem and for establishing the recurrence or transience of the random walk.

DEFINITION: Given a sequence of events $A_n$ in a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Define $A_\infty := \limsup_{n\to\infty} A_n := \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n$. We have $A_\infty = \{\omega \mid \omega \text{ is in infinitely many } A_i\}$.
INTERPRETATION. The set $A_\infty$ is is the event that infinitely many of the events $A_n$ happen.

BOREL CANTELLI LEMMA:

(1) If $\sum_n \mathrm{P}[A_n] < \infty$, then $\mathrm{P}[A_\infty] = 0$.

(2) If $A_n$ are independent and $\sum_n \mathrm{P}[A_n] = \infty$, then $\mathrm{P}[A_\infty] = 1$.

PROOF.

(1) $\mathrm{P}[A_\infty] = \lim_{n\to\infty} \mathrm{P}[\bigcup_{k\geq n} A_k] \leq \lim_{n\to\infty} \sum_{k \geq n} \mathrm{P}[A_k] = 0$.

(2) For every $n \in \mathbb{N}$, we have

$$\mathrm{P}[\bigcap_{k\geq n} A_k^c] = \prod_{k\geq n} \mathrm{P}[A_k^c] = \prod_{k \geq n} (1 - \mathrm{P}[A_k]) \leq \prod_{k\geq n} \exp(-\mathrm{P}[A_k]) = \exp(-\sum_{k\geq n} \mathrm{P}[A_k]) = 0 \ .$$

(We have used in the first equality the independence of the sets $A_i$ and in the inequality step the elementary estimate $1 - x \leq e^{-x}$ for $x \geq 0$.)

From

$$\mathrm{P}[A_\infty^c] = \mathrm{P}[\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k^c] \leq \sum_{n\in\mathbb{N}} \mathrm{P}[\bigcap_{k\geq n} A_k^c] = 0$$

follows $\mathrm{P}[A_\infty^c] = 0$ and so $\mathrm{P}[A_\infty] = 1$.

MONKEY TYPING SHAKESPEARE:
Sit a monkey at a typewriter. The probability that he will write Shakespeares collected work is one. Actually, according to the Borel-Cantelli lemma, he will write it infinitely often. The following Mathematica procedures allow to simulate the typing of the Monkey typing Shakespeare: We let him type 700 letters on the computer and save it in a file Sheakspeare. Enjoy reading.

```
Alphabet={a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z};
Monkey[n_]:=Table[Alphabet[[Random[Integer,{1,26}]]],{n}]

Shakespeare=Monkey[700];
Save["Shakespeare",Shakespeare];
```