1D walk (1): gamblers ruin, beating the system, Wald identities

DEFINITION: Consider IID $\{-1,1\}$-valued random variables X_k. The random variables $S_n = \sum_{k=1}^n X_k(\omega)$ is the location of the random walker at time n. If X_k is the win or loss in a game at time k, then S_n is the total win or loss up to time n.

SIMPLE PROPERTIES: $\mathrm{E}[X_k] = 0$ and so $\mathrm{E}[S_k] = 0$. If $n+x$ even, $\mathrm{P}[X_n = x] = 2^{-n} \begin{pmatrix} n \\ \frac{n+x}{2} \end{pmatrix}$, if $n+x$ odd then $\mathrm{P}[X_n = x] = 0$.

DEFINITION: A **gambling system** or betting strategy to the random walk X_k is sequence of random variables V_k such that every event $\{V_n = c\}$ depends only on the values of X_j for $j < k$ (one says V is a previsible process with respect to the "martingale" S_n). Define

$$S_n^V = \sum_{i=1}^n V_k X_k ,$$

the **winning** with this system. Remark. S_n^V is also written as $V \cdot S$ or $\int V dS$. You can also use the fancy name like "martingale transform" or (totally impressive) "discrete stochastic integral".

YOU CAN'T BEAT THE SYSTEM.
If $\mathrm{E}[X_k] = 0$, then $\mathrm{E}[S_N^V] = 0$. PROOF. Because V_k is independent of X_k, we have $\mathrm{E}[V_k X_k] = \mathrm{E}[V_k X_k] = \mathrm{E}[V_k]\mathrm{E}[X_k]$.

DEFINITION. A **stopping time** is a random variable T taking values in $\overline{\mathbb{N}} := \{0, 1, \dots, \infty\}$ such that the event $\{T \leq k\}$ can be defined using the random variables $X_1, \dots, X_k$. Examples are

$$T_a = \min\{n \in \mathbb{N} \setminus \{0\} \mid S_n = a\} .$$

or

$$T_{a,b} = \min\{n \in \mathbb{N} \mid S_n = b \text{ or } S_n = -a \} .$$

Gambling interpretation: T_a is the time until a player gets broke or the time, when the player decides to stop the game. Whether or not to stop immediatly after the n'th game depends only on the history up to time n. For example, $T_{a,b}$ is the time until one of two competing players gets broke. $\mathrm{P}[X_T = -a]$ is the probability that the first player with capital a gets broke, $\mathrm{P}[X_T = b]$ is the probability that the second player gets broke.

The proofs of the following theorems are in the book.

FIRST WALD'S IDENTITY. If T be a stopping time, then

$$\mathrm{E}[S_T] = \mathrm{E}[T] \cdot \mathrm{E}[X] .$$

Especially, for a symmetric random walk, where $\mathrm{E}[X] = 0$, one has $\mathrm{E}[S_T] = 0$.

SECOND WALD IDENTITY. If T is a stopping time, then

$$\mathrm{Var}[S_T] = \mathrm{E}[T] \cdot \mathrm{Var}[X] .$$

APPLICATION OF FIRST WALD IDENTITY: FORMULA FOR THE RUIN PROBABILITY.

$$\mathrm{P}[X_{T_{a,b}} = -a] = 1 - \mathrm{P}[X_{T_{a,b}} = b] = b/(a+b) .$$

PROOF. $0 = \mathrm{E}[S_T] = -a\mathrm{P}[X_T = -a] + b\mathrm{P}[X_T = b] = -a\mathrm{P}[X_T = -a] + b(1 - \mathrm{P}[X_T = -a])$.