Math 464/564, O. Knill, **April 1997**

Recurrence of random walks

QUESTION. A walker starts at the origin 0 of the d dimensional lattice $\mathbb{Z}^d$ and makes each second a random step into one of the $2d$ directions. Does the walker return back to the origin again and again?

THEOREM. The walker returns to the origin infinitely often with probability one in one or two dimensions. In dimensions $d \geq 3$, the walker returns only finitely many times to zero and escapes to infinity with probability one.

SETUP. Consider IID random vectors X_n taking values in $\{e \in \mathbb{Z}^d \mid |e| = \sum_{i=1}^d |e_i| = 1\}$ with uniform distribution $\mathrm{P}[X_n = e] = (2d)^{-1}$. The random vector $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$ is the position of the walker at time n. The random variable $Y_n = 1_{A_n}$ with $A_n = \{X_n = 0\}$ tells, whether the walker is at the origin at time n or not. The sum $B_n = \sum_{i=0}^n Y_n$ counts the number of visits of the walker up to time n and $B = \sum_{i=0}^\infty Y_n$ is the random variable which gives the total number of visits during the evolution. The expectation $\mathrm{E}[B] = \sum_{n=0}^\infty \mathrm{P}[S_n = 0]$ is the expected number of returns to the origin.

LEMMA OF POLYA. $\mathrm{E}[B] = \infty$ for $d = 1, 2$ and $\mathrm{E}[B] < \infty$ for $d > 2$.

PROOF.

(i) For $n \in \mathbb{N}$, we know that $\mathrm{P}[S_n = k]$ for $k \in \mathbb{Z}^d$. vanishes for large $|k|$ because the walker can only reach a finite region in finite time. The characteristic function $\phi_{S_n} = \sum_k \mathrm{P}[S_n = k] e^{ixk}$ is therefore a smooth function on the torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ (each coordinate of $\phi_{S_n}(x)$ is a 2π-periodic function).

(ii) $\phi_{X_k}(x) = \frac{1}{2d}\sum_{|j|=1} e^{ix_j} = \frac{1}{d}\sum_{i=1}^d \cos(x_i)$. Because the S_n is a sum of n IID random variables we get $\phi_{S_n} = \phi_{X_k}^n = \frac{1}{d^n}(\sum_{i=1}^d \cos(x_i))^n$.

(iii) By the Fourier inversion formula, we have $\mathrm{P}[S_n = 0] = (2\pi)^{-d}\int_0^{2\pi} \dots \int_0^{2\pi} \phi_{S_n}(x)\, dx_1 \dots dx_d$. Therefore

$$\mathrm{E}[B] = \int_{\mathbb{T}^d} \sum_n \phi_X^n(x)\, dx = \int_{\mathbb{T}^d} \frac{1}{1 - \phi_X(x)}\, dx \ .$$

With a Taylor expansion $\phi_{X_k}(x) = \frac{1}{d}\sum_{i=1}^d \cos(x_i) = 1 - \sum_j x_j^2/(4d) + \dots$ we can estimate $\frac{1}{8d}|x|^2 \leq 1 - \phi_X(x) \leq \frac{1}{2d}|x|^2$. The lemma follows because $\int_{\{|x|<\epsilon\}} \frac{1}{|x|^2}\, dx$ is finite if and only if $d \geq 3$. (For the later fact see the Homework).

PROOF OF THE THEOREM.

(i) $d > 2$: define $A_n = \{S_n = 0\}$. The event $A_\infty = \limsup_n A_n$ is the subset of Ω, for which the walker returns to 0 infinitely many times. Since by the lemma $\mathrm{E}[B] = \sum_{n=0}^\infty \mathrm{P}[A_n] < \infty$, the Borel-Cantelli lemma gives $\mathrm{P}[A_\infty] = 0$ for $d > 2$. The walker returns therefore back to 0 only finitely many times and in the same way it visits each lattice point only finitely many times. The walker eventually leaves every bounded set and escapes to infinity.

(ii) $d = 1$ or $d = 2$: if $p = \mathrm{P}[\bigcup_n A_n]$ is the probability that the walker returns to 0 at least once, then p^{m-1} is the probability that there are at least m visits of the origin and $p^{m-1} - p^m = p^{m-1}(1-p)$ is the probability that there are exactly m visits. We can write

$$\mathrm{E}[B] = \sum_{m \geq 1} m p^{m-1}(1-p) = 1/(1-p) \ .$$

Since by Polya's lemma $\mathrm{E}[B] = \infty$ we see that $p = 1$.