Characteristic functions

DEFINITION. The **characteristic function** of a random variable X is defined as $\phi_X(t) = \mathrm{E}[e^{itX}] = \int_\Omega e^{itx}\, dP(x)$. Mathematicians call it the **Fourier transform** of the measure P.

CASE: If X takes integer values, then $\phi_X(t) = \sum_{n \in \mathbb{Z}} \mathrm{P}[X = n] e^{inx}$ is a 2π-periodic function. The numbers $a_n = \mathrm{P}[X = n]$ are the **Fourier coefficients** of the function $\phi_X(t)$.
CASE: If X is absolutely continuous with density f_X, then $\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx$ is the **Fourier transform** of the density function f_X. Outside probability theory, one uses the notation $\hat{f}_X$ for the Fourier transform.

FOURIER INVERSION: The characteristic function determines P: if a, b are continuity points of F_X, then

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ita} - e^{-itb} .$$

CASE: If X takes integer values, then

$$\mathrm{P}[X = k] = \frac{1}{2\pi} \int_0^{2\pi} e^{-itk} \phi_X(t)\, dt .$$

CASE: If X is absolutely continuous with density f_X, then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t)\, dt .$$

PROPERTIES:
SUM OF INDEPENDENT RANDOM VARIABLES: If X and Y are independent, then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.
(The Fourier transform renders convolutions into multiplications: $\hat{f} \cdot \hat{g} = f \hat{\star} g$.)
CALCULATION OF MOMENTS:

$$\mathrm{E}[X^k] = (-i)^k \phi_X^{(k)}(0) .$$

LINEAR CHANGE OF VARIABLES:

$$\phi_{aX+b}(t) = e^{ib} \phi_X(at) .$$

RIEMANN-LEBESGUE: If X is absolutely continuous, then $\phi_X(t) \to 0$ for $|t| \to \infty$.
FOURIER-SERIES: If X takes integer values, then $\phi_X(t)$ is 2π-periodic.

EXAMPLES.

Distribution	Parameter	Characteristic function
Normal	$m \in \mathbb{R}$, $\sigma^2 > 0$	$e^{mit - \sigma^2 t^2/2}$
Standard normal	$m = 0, \sigma = 1$	$e^{-t^2/2}$
Uniform	$[-a, a]$	$\sin(at)/(at)$
Exponential	$\lambda > 0$	$\lambda/(\lambda - it)$
Gamma	$\beta > 0, \lambda$	$\lambda^\beta/(\lambda - it)^\beta$
Binomial	$n \in \mathbb{N}$, $p \in [0, 1]$	$(1 - p + pe^{it})^n$
Poisson	$\lambda > 0, \lambda$	$e^{\lambda(e^{it} - 1)}$
Geometric	$p \in (0, 1)$	$\frac{pe^{it}}{(1-(1-p)e^{it}}$
Uniform	$\{1, 2, \dots, n\}$	$\frac{1}{n} \frac{e^{i(n+1)t} - e^{it}}{e^{it} - 1}$