Math 464/564, O. Knill, **April 1997**

Stirling's formula

The Stirling's formula can be derived using the central limit theorem.

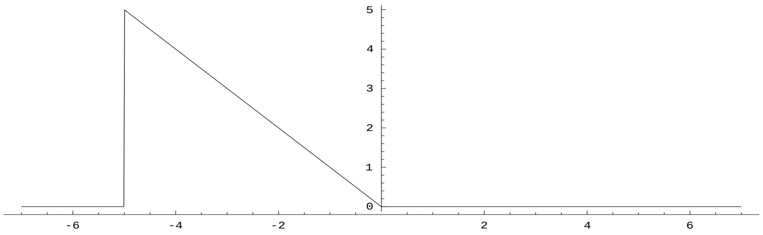
STIRLING'S FORMULA:

$$\boxed{n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}}$$

PROOF OF STIRLING'S FORMULA.

Let X_i be IID random variables, which are Poisson distributed with parameter $\lambda = 1$. That is $\mathrm{P}[X_i = k] = \frac{1}{k!} e^{-1}$. We know that the sum $S_n = \sum_{j=1}^n X_j$ is Poisson distributed with parameter $\lambda = k$. That is $\mathrm{P}[S_n = k] = \frac{n^k}{k!} e^{-n}$.

Let $g_M(x)$ be the function on $\mathbb{R}$ which is $g_M(x) = -x$ for $x \in [-M, 0]$ and 0 else.



We compute

$$\begin{aligned}
\mathrm{E}[g_M(\frac{S_n - n}{\sqrt{n}})] &= \sum_{n - M\sqrt{n} \le j \le n} g(\frac{j-n}{\sqrt{n}}) \frac{n^j e^{-n}}{j!} \\
&= \sum_{n - M\sqrt{n} \le j \le n} \frac{n-j}{\sqrt{n}} \frac{n^j e^{-n}}{j!} \\
&= \sum_{n - M\sqrt{n} \le j \le n} \frac{e^{-n}}{\sqrt{n}} \left(\frac{n^{j+1}}{j!} - \frac{n^j}{(j-1)!} \right) \\
&= \frac{e^{-n}}{\sqrt{n}} \left(\frac{n^{n+1}}{n!} - \frac{n^{[n - M\sqrt{n} + 1]}}{[n - M\sqrt{n} + 1]} \right) .
\end{aligned}$$

(The square bracket $[r]$ for a real number detnotes the largest integer smaller or equal to r.).
We see that for $M \to \infty$,

$$\mathrm{E}[g_M(S_n^*)] = \mathrm{E}[g_M(\frac{S_n - n}{\sqrt{n}})] \to \frac{e^{-n}}{\sqrt{n}} \frac{n^{n+1}}{n!} = n^{n+\frac{1}{2}} e^{-n} \frac{1}{n!} . \qquad (1)$$

The central limit theorem implies that $\mathrm{E}[g_M(S_n^*)] \to \mathrm{E}[g_M(X)]$ for $n \to \infty$, where X is a standard normal distributed random variable. Because

$$\mathrm{E}[g_M(X)] = \int_{-M}^{0} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} (1 - e^{-\frac{1}{2}M^2}) ,$$

we see that for $M \to \infty$,

$$\mathrm{E}[g_M(X)] \to \frac{1}{\sqrt{2\pi}} \qquad (2)$$

Putting the facts (1) and (2) together with $\mathrm{E}[g_M(S_n^*)] \to \mathrm{E}[g_M(X)]$ gives the Stirling formula.