Math 464/564, O. Knill, **April 1997**

Some Inequalities

DEFINITION. A function $h : \mathbb{R} \to \mathbb{R}$ is called *convex*, if there exists for all $x_0 \in \mathbb{R}$ a linear map $l(x) = ax + b$ such that $l(x_0) = h(x_0)$ and for all $x \in \mathbb{R}$ the inequality $l(x) \leq h(x)$ holds.

DEFINITION. Let $\mathcal{L}^p$ be the set of random variables which satisfy $\mathrm{E}[|X|^p] < \infty$ for $p \in [1, \infty)$ and $|X| \leq K$ almost everywhere for $p = \infty$. On the vector space $\mathcal{L}^p$ define $||X||_p = \mathrm{E}[|X|^p]^{1/p}$ rsp. $||X||_\infty = \inf\{K \in \mathbb{R} \mid |X| \leq K$ almost everywhere $\}$.

JENSEN INEQUALITY. Given $X \in \mathcal{L}^1$. For any convex function $h : \mathbb{R} \to \mathbb{R}$, one has

$$\mathrm{E}[h(X)] \geq h(\mathrm{E}[X]) .$$

PROOF. Let l be the linear map at $x_0 = E[X]$. By the linearity and monoticity of the expectation, we get

$$h(\mathrm{E}[X]) = l(\mathrm{E}[X]) = \mathrm{E}[l(X)] \leq \mathrm{E}[h(X)] .$$

HOELDER INEQUALITY. Given $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$ and $X \in \mathcal{L}^p$ and $Y \in \mathcal{L}^q$. Then $XY \in \mathcal{L}^1$ and

$$||XY||_1 \leq ||X||_p ||Y||_q .$$

PROOF. We can restrict to $X, Y \geq 0$ and $||X||_p > 0$. Define the probabilty measure $Q = \frac{X^p P}{\mathrm{E}[X^p]}$ and define $u = 1_{\{X>0\}} Y / X^{p-1}$. Jensen gives $Q(u)^q \leq Q(u^q)$ so that

$$\mathrm{E}[|XY|] \leq ||X||_p ||1_{\{Z>0\}} Y||_q \leq ||X||_p ||Y||_q .$$

CAUCHY BUNYAKOWSKY-SCHWARZ INEQUALITY. $||XY||_1 \leq ||X||_2 ||Y||_2$.

MINKOWSKI INEQUALITY. Given $p \in [1, \infty]$ $X, Y \in \mathcal{L}^p$. Then $||X + Y||_p \leq ||X||_p + ||Y||_p$.

PROOF. We use Hölder's inequality to get

$$\mathrm{E}[|X + Y|^p] = \mathrm{E}[|X||X + Y|^{p-1}] + \mathrm{E}[|Y||X + Y|^{p-1}] \leq ||X||_p C + ||Y||_p C ,$$

where $C = |||X + Y|^{p-1}||_q = \mathrm{E}[|X + Y|^p]^{1/q}$ which leads to the claim.

PROOF. Given $p \leq q$. Define $h(x) = |x|^{q/p}$. Jensen's inequality gives $\mathrm{E}[|X|^q] = \mathrm{E}[h(|X|^p)] \leq h(\mathrm{E}[|X|^p] = \mathrm{E}[|X|^p]^{q/p}$. This implies that $||X||_q := \mathrm{E}[|X|^q]^{1/q} \leq \mathrm{E}[|X|^p]^{1/p} = ||X||_p$ for $p \leq q$. In other words, if $\mathcal{L}^p$ is the set of random variables with $\mathrm{E}[|X|^p] < \infty$, then $\mathcal{L}^p \subset \mathcal{L}^q$ for $p \geq q$.

CHEBYCHEV-MARKOV INEQUALITY Let h be a monotone function on $\mathbb{R}$ with $h \geq 0$. For every $c > 0$, and $h(X) \in \mathcal{L}^1$ we have

$$h(c) \cdot \mathrm{P}[X \geq c] \leq \mathrm{E}[h(X)] .$$

PROOF. Integrate the inequality $h(c) 1_{X \geq c} \leq h(X)$ using the monotonicity and linearity of the expectation.