Math 464/564, O. Knill, **April 1997**

Multidimensional distributions (Summary II)

The distribution of a sum

PROPOSITION. If X and Y are independent and absolutely continuous, then
$f_{X+Y}(z) = f_X \star f_Y(y) := \int_{-\infty}^{\infty} f_X(x) f_Y(y-x)\, dx$.

EXAMPLES.
1) The sum of independent $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$ distributed random variables is $\Gamma(\alpha_1+\alpha_2, \lambda)$ distributed.
2) If X and Y are independent normal $N(m_1, \sigma_1^2)$ and $N(m_2, \sigma_2^2)$ distributed random variables, then $X+Y$ is $N(m_1+m_2, \sigma_1^2+\sigma_2^2)$ distributed.

Transformations of densities

The usual transformation rule for volume integrals gives:

PROPOSITION: Given a continuous random vector X with density f and a differentiable invertible function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with inverse ψ. The random variable $Y = \phi(X)$ has the density

$$g(y) = f(\psi(y))|\mathrm{Det}(D\psi(y))| ,$$

where $[D\psi(y)]_{ij} = \frac{\partial}{\partial x_j} \psi_i(y)$ is the Jacobian matrix.

EXAMPLES.
1) The formula generalizes the formula for $n = 1$, where $D\psi(y) = \psi'(y)$.
2) Given a random vector (X, Y) with joint density f. Problem: determine the density of $(U, V) = \phi(X, Y) = (X+Y, X-Y)$.
The inverse is $(x, y) = \psi(u, v) = ((u+v)/2, (u-v)/2)$ which has the determinant $-1/2$ since $\mathrm{Det}(D\psi) = \mathrm{Det}(D\phi^{-1}) = 1/\mathrm{Det}(D\phi)$. Therefore

$$f_{(U,V)}(u, v) = \frac{1}{2} f_{(X,Y)}(\frac{u+v}{2}, \frac{u-v}{2}) .$$

Conditional densities

DEFINITION. Given an absolutely continuous random vector (X, Y), The **conditional probability density function** for X given Y is defined as

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)} .$$

It is a density function of a random variable.

EXAMPLE. If X and Y are independent, then $f_{X|Y}(x|y) = f_X(x)$ because then $f_{(X,Y)}(x, y) = f_X(x) f_Y(y)$.

CONTINUOUS BAYES RULE.

$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(x) f_{Y|X}(y|x)\, dx} .$$

Proof. Plug in $f(x, y) = f_X(x) f_{Y|X}(y|x)$ into $f_Y(x) = \int_{-\infty}^{\infty} f(x, y)\, dx$ and put this into the formula $f_X(x) f_{Y|X}(y|x) = f_{X|Y}(x|y) f_Y(x)$.