Math 464/564, O. Knill, **April 1997**

Absolutely continuous distributions

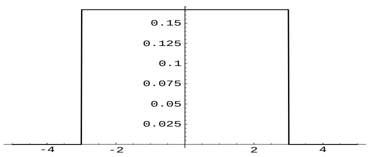
REMINDER: The **Distribution function** of a random variable X is $F_X(t) = \mathrm{P}[X \leq t]$. A random variable is **absolutely continuous** if the density $F'_X = f_X$ exists. The **expectation**, **variance** and $\mathrm{E}[g(X)]$ for $g(X)$ is in the continuous case

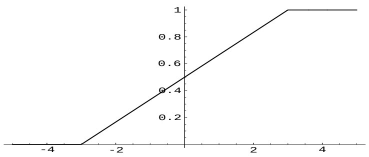
$$m = \mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx,\ \mathrm{Var}[X] = \int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx\ , \mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

UNIFORM DISTRIBUTION. $\Omega = [a,b]$.

$$\boxed{f(x) = (b-a)^{-1}}$$

The natural distribution on a finite interval $[a,b]$. Example: angle of a weel of a parking bike on campus is uniformly distributed on $[0, 2\pi)$.

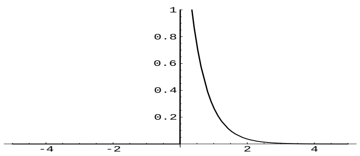


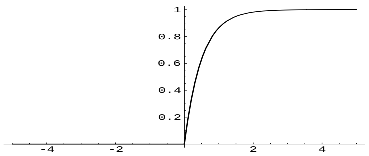


EXPONENTIAL DISTRIBUTION. $\Omega = [0, \infty)$.

$$\boxed{f(x) = \lambda e^{-\lambda x}}$$

The natural distribution on the positive real axes. Example: the decay times of radioactive isotopes is exponentially distributed

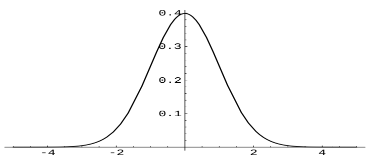


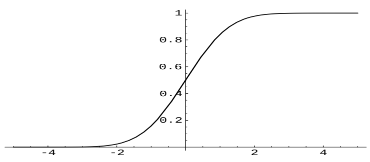


NORMAL DISTRIBUTION. $\Omega = \mathbb{R}$

$$f(x) = (2\pi\sigma^2)^{-1/2} e^{-(x-\mu)^2/2\sigma^2}.$$

The natural distribution on the real line. Example: measurement errors in scientific experiments.

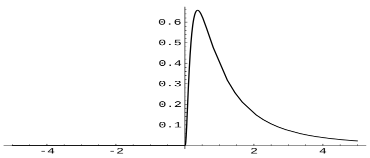


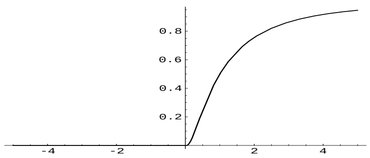


LOG-NORMAL DISTRIBUTION. $\Omega = [0, \infty)$.

$f(x) =$ (see HW 10)

The exponential of a normal distribued random variable is log-Normal distributed. (The terminology is because the LOGarithm of a log-normal distribued random variable has a NORMAL distribution).

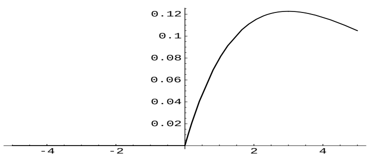


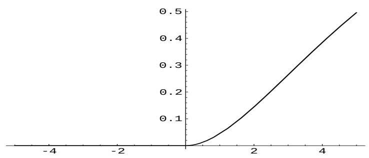


GAMMA DISTRIBUTION. $\Omega = [0, \infty)$. The function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$ is called the **Gamma function**. For integers $\Gamma(n) = (n-1)!$.

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

Useful in population statistics. The square of a standard normal distributed random variable is an example of a Gamma distributed random variable.

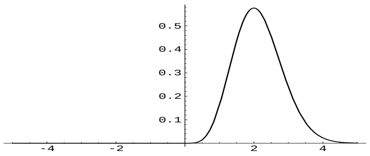


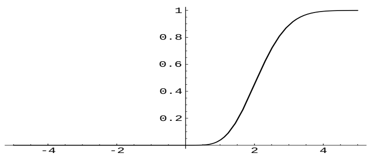


χ^2-DISTRIBUTION. $\Omega = [0, \infty)$.

$$\boxed{f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.}$$

Special case of Gamma distribution. The sum of n squares of independent standard normal distributed random variable is χ^2-distributed. A Γ distributed random variables with parameter $\alpha = n/2, \lambda = 1/2$ is χ^2 distributed.

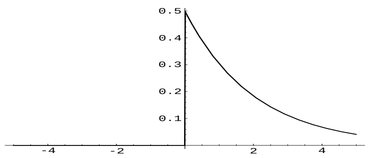


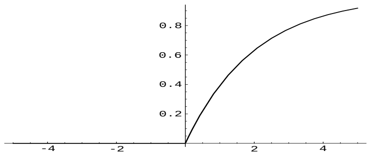


WEIBULL-DISTRIBUTION. $\Omega = [0, \infty)$.

$$\boxed{f(x) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha}.}$$

Generalisation of the exponential distribution which is the special case $\alpha = 1$. Is useful to model life-times of items.

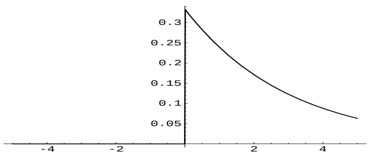


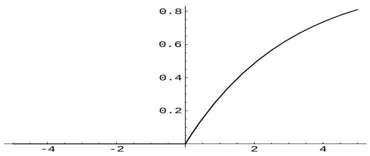


ERLANG-DISTRIBUTION. $\Omega = [0, \infty)$.

$$\boxed{f(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x} \,,}$$

Special case of Γ distribution for integer values of α. The sum of n independent exponential distribued random variables is Erlang distributed.

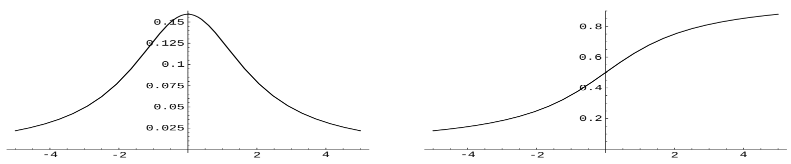




CAUCHY DISTRIBUTION. $\Omega = \mathbb{R}$.

$$\boxed{f(x) = \frac{1}{\pi(1+x^2)}.}$$

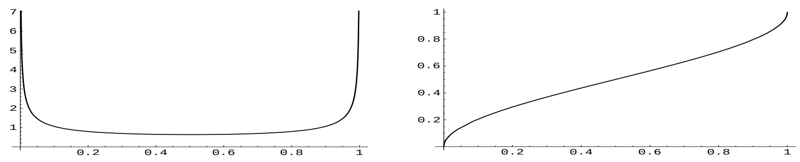
The tangent of a uniformly distributed angle is Cauchy distributed.



ARC-SIN DISTRIBUTION. $\Omega = \mathbb{R}$.

$$\boxed{f(x) = \frac{1}{\pi\sqrt{x(1-x)}}.}$$

The spectral density of a free particle in a pure crystal has the arc-sin distribution.



Mathematica program which generated all the pictures on these pages:

```
<<Statistics`ContinuousDistributions`;
PlotPDFandCDF[distr_]:=Show[GraphicsArray[
      {Plot[PDF[distr, x],{x,-5,5},
          DisplayFunction->Identity],
       Plot[CDF[distr, x],{x,-5,5},
          DisplayFunction->Identity]},
      DisplayFunction->$DisplayFunction]];
PlotPDFandCDFArcSin:=
Show[GraphicsArray[
      {Plot[1/(N[Pi] Sqrt[x(1-x)]),{x,0.0000001,1-0.0000001},
          DisplayFunction->Identity],
       Plot[2/N[Pi] ArcSin[Sqrt[x]],{x,0,1},
          DisplayFunction->Identity]},
      DisplayFunction->$DisplayFunction]];
ErlangDistribution=GammaDistribution;

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Display["!psfix -land>cdist02.ps",PlotPDFandCDF[ExponentialDistribution[2]]];
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Display["!psfix -land>cdist07.ps",PlotPDFandCDF[WeibullDistribution[1,2]]];
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Display["!psfix -land>cdist09.ps",PlotPDFandCDF[CauchyDistribution[0,2]]];
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```