Math 464/564, O. Knill, **March 1997**

Integration

In order to define the expectation for general random variables, one needs integration $\mathrm{E}[X] = \int_\Omega X(\omega) dP(\omega)$. For $\Omega = \mathbb{R}^n$ and $d\mathrm{P} = \prod_i dx_i$, this is the usual (multiple) integral in calculus.

DEFINITION. Let $(\Omega, \mathcal{A}, P)$ be a proability space. Let $\mathcal{B}$ be $\sigma-$ algebra of all measurable sets on $\mathbb{R}$. A function $X : \Omega \to \mathbb{R}$ is called **a random variable** if for all $B \in \mathcal{B}$, $X^{-1}(B) \in \mathcal{A}$.

EXAMPLE.
1) If $\mathcal{A}$ is the set of subsets of $\mathcal{A}$, then every function X is measurable.
2) Let $\Omega = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}$ the Borel σ-algebra. A continuous function $X : \Omega \to \mathbb{R}$ is a random variable.

DEFINITION. A **step function** is a random variable which is of the form $X = \sum_{i=1}^n \alpha_i \cdot 1_{A_i}$ with $\alpha_i \in \mathbb{R}$ and where $A_i \in \mathcal{A}$ are disjoint. Call $\mathcal{S}$ the set of such random variables. These functions can alternatively be defined as random variables which take only finitely many values. For $X \in \mathcal{S}$ we can define the *integral*

$$\mathrm{E}[X] := \int_\Omega X \, dP = \sum_{i=1}^n \alpha_i P(A_i) = \sum_{a \in X(\Omega)} a \cdot \mathrm{P}[X = a] \; .$$

DEFINITION. Define $\mathcal{L}^1$ as the set of random variables X, for which

$$\sup_{Y \in \mathcal{S}, Y \leq |X|} \int Y \, dP$$

is finite. For $X \in \mathcal{L}^1$, we can define the **integral** or **expectation**

$$\mathrm{E}[X] := \int X \, dP = \sup_{Y \in \mathcal{S}, Y \leq X^+} \int Y \, dP - \sup_{Y \in \mathcal{S}, Y \leq X^-} \int Y \, dP$$

where $X^+ = X \vee 0 = \max(X, 0)$ and $X^- = -X \vee 0 = \max(-X, 0)$. The vector space $\mathcal{L}^1$ is called the space of *integrable random variables.*

EXAMPLES. 1) If Ω is a finite, then the expectation is the expectation we had considered in the first Chapter.
2) If Ω is a countable set, then

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \mathrm{P}[X = x] \; .$$

3) Let $\Omega = [a, b]$ with $a < b$, $\mathcal{A}$ the Borel σ algebra and $P[(c, d)] = (d - c)/(b - a)$.