Math 464/564, O. Knill, **March 1997**

CARATHEODORY EXTENSION THEOREM

The construction of general probability spaces needs some measure theory. If you are interested into the foundations of probability theory, here is the tool which is necessary for constructing general probability spaces like the unit interval with the Lebesgue measure. The idea is to define the measure first on a Boolean algebra one extend this to a σ-algebra. For example, one first defines the Lebesgue measure on sets which are finite unions of disjoint intervals $[a_i, b_i)$ by $\mu([a_i, b_i)) = b_i - a_i$. The Carathéodory extension theorem (proven here) assures then that this can be extended consistently to a σ-algebra of sets containing these intervals.

REPETITION DEFINITION.
A set of subsets of Ω $\mathcal{A}$ is an **Boolean algebra** if $\Omega \in \mathcal{A}$,
$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.
If $A_n \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mathcal{A}$ is called a **σ-algebra** and the pair $(\Omega, \mathcal{A})$ is called a **measurable space**.

DEFINITION. A σ-additive map from a Boolean algebra $\mathcal{A}$ to $[0, \infty)$ is called a **measure**.

CARATHEODORY CONTINUATION THEOREM. Any measure on a Boolean algebra $\mathcal{R}$ can be continued uniquely to a measure on $\sigma(\mathcal{R})$, the smallest σ-algebra containing $\mathcal{R}$.

DEFINITION. Let $\mathcal{A}$ be a Boolean algebra and let $\lambda : \mathcal{A} \mapsto [0, \infty]$ be a measure with $\lambda(\emptyset) = 0$. A set $A \in \mathcal{A}$ is called a **λ-set**, if $\lambda(A \cap G) + \lambda(A^c \cap G) = \lambda(G)$ for all $G \in \mathcal{A}$.

LEMMA. The set $\mathcal{L}$ of λ-sets of a Boolean algebra $\mathcal{A}$ is again a Boolean algebra and satisfies $\sum_{k=1}^n \lambda(A_k \cap G) = \lambda((\bigcup_{k=1}^n A_k) \cap G)$ for all finite disjoint families $\{A_k\}_{k=1}^n$ and all $G \in \mathcal{A}$.

PROOF OF THE LEMMA. From the definition, it is clear that $\Omega \in \mathcal{L}$ and that if $B \in \mathcal{L}$, then $B^c \in \mathcal{L}$. Given $B, C \in \mathcal{L}$. Then $A = B \cap C \in \mathcal{L}$. Proof. Since $C \in \mathcal{L}$, we get

$$\lambda(C \cap A^c \cap G) + \lambda(C^c \cap A^c \cap G)u = \lambda(A^c \cap G)$$

This can be rewritten with $C \cap A^c = C \cap B^c$ and $C^c \cap A^c = C^c$ as

$$\lambda(A^c \cap G) = \lambda(C \cap B^c \cap G) + \lambda(C^c \cap G) . \tag{1}$$

Since B is a λ-set, we get using $B \cap C = A$.

$$\lambda(A \cap G) + \lambda(B^c \cap C \cap G) = \lambda(C \cap G) . \tag{2}$$

Since C is a λ set, we have

$$\lambda(C \cap G) + \lambda(C^c \cap G) = \lambda(G) . \tag{3}$$

Adding up these three equations gives that $B \cap C$ is a λ set. We have shown that Λ is a Boolean algebra. If B and C are disjoint in $\mathcal{L}$ we get since B is a λ set

$$\lambda(B \cap (B \cup C) \cap G) + \lambda(B^c \cap (B \cup C) \cap G) = \lambda((B \cup C) \cap G) .$$

This can be rewritten as $\lambda(B \cap G) + \lambda(C \cap G) = \lambda((B \cup C) \cap G)$. The analogue statement for finitely many sets is obtained by induction.

DEFINITION. Let $\mathcal{A}$ be a σ-algebra. A map $\lambda : \mathcal{A} \Rightarrow [0, \infty]$ is called an **outer measure**, if

$\lambda(\emptyset) = 1$,
$A, B \in \mathcal{A}$ with $A \subset B \Rightarrow \lambda(A) \leq \lambda(B)$.
$A_n \in \mathcal{A} \ \Rightarrow \lambda(\bigcup_n A_n) \leq \sum_n P(A_n)$ (σ subadditivity)

CARATHEODORY LEMMA. If λ is an outer measure on a measurable space $(\Omega, \mathcal{A})$, then the λ sets $\mathcal{L} \subset \mathcal{A}$ form a σ-algebra on which λ is countably additive.

PROOF OF THE CARATHEODORY LEMMA. Given a disjoint sequence $A_n \in \mathcal{L}$. We have to show that $A = \bigcup_n A_n \in \mathcal{L}$ and $\lambda(A) = \sum_n \lambda(A_n)$. By the above lemma, $B_n = \bigcup_{k=1}^n A_k$ is in $\mathcal{L}$. We have therefore using the monotonicity, the additivity proved in the above lemma and the σ-subadditivity

$$\begin{aligned} \lambda(G) &= \lambda(B_n \cap G) + \lambda(B_n^c \cap G) \geq \lambda(B_n \cap G) + \lambda(A^c \cap G) \\ &= \sum_{k=1}^n \lambda(A_k \cap G) + \lambda(A^c \cap G) \geq \lambda(A \cap G) + \lambda(A^c \cap G) . \end{aligned}$$

Subadditivity for λ gives $\lambda(G) \leq \lambda(A \cap G) + \lambda(A^c \cap G)$. All the inequalities in this proof are therefore equalities. We conclude that $A \in \mathcal{L}$ and that λ is σ additive on $\mathcal{L}$.

PROOF OF THE CARATHEODORY CONTINUATION THEOREM.
Given an algebra $\mathcal{R}$ with a measure μ. Define $\mathcal{A} = \sigma(\mathcal{R})$ and the σ-algebra $\mathcal{P}$ consisting of all subsets of Ω. Define on $\mathcal{P}$ the function

$$\lambda(A) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \mid \{A_n\}_{n \in \mathbb{N}} \text{ sequence in} \mathcal{R} \text{ satisfying } A \subset \bigcup_n A_n\} .$$

(i) λ is an outer measure on $\mathcal{P}$.
$\lambda(\emptyset) = 0$ and $\lambda(A) \geq \lambda(B)$ for $B \geq A$ are obvious. To see the σ-subadditivity take a sequence $A_n \in \mathcal{P}$ with $\lambda(A_n) < \infty$ and fix $\epsilon > 0$. For all $n \in \mathbb{N}$, one can by the definition of λ find a sequence $\{B_{n,k}\}_{k \in \mathbb{N}}$ in $\mathcal{R}$ such that $A_n \subset \bigcup_{k \in \mathbb{N}} B_{n,k}$ and $\sum_{k \in \mathbb{N}} \mu(B_{n,k}) \leq \lambda(A_n) + \epsilon 2^{-n}$. Define $A = \bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{n,k \in \mathbb{N}} B_{n,k}$, so that $\lambda(A) \leq \sum_{n,k} \mu(B_{n,k}) \leq \sum_n \lambda(A_n) + \epsilon$. Since ϵ was arbitrary, the σ-subadditivity is proven.

(ii) $\lambda = \mu$ on $\mathcal{R}$.
Given $A \in \mathcal{R}$. Clearly $\lambda(A) \leq \mu(A)$. Suppose that $A \subset \bigcup_n A_n$, with $A_n \in \mathcal{R}$. Define a sequence $\{B_n\}_{n \in \mathbb{N}}$ of disjoint sets in $\mathcal{R}$ defined inductivly by $B_1 = A_1$, $B_n = A_n \cap (\bigcup_{k<n} A_k)^c$ such that $B_n \subset A_n$ and $\bigcup_n B_n = \bigcup_n A_n \supset A$. From the σ-additivity of μ on $\mathcal{R}$, we get

$$\mu(A) \leq \bigcup_n \mu(B_n) \leq \bigcup_n \mu(A_n)$$

so that $\mu(A) \geq \lambda(A)$.

(iii) Let $\mathcal{L}$ be the set of λ-sets in $\mathcal{P}$. Then $\mathcal{R} \subset \mathcal{L}$.
Given $A \in \mathcal{R}$ and $G \in \mathcal{P}$. There exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ in $\mathcal{R}$ such that $G \subset \bigcup_n B_n$ and $\sum_n \mu(B_n) \leq \lambda(G) + \epsilon$. By the definition of λ

$$\sum_n \mu(B_n) = \sum_n \mu(A \cap B_n) + \sum_n \mu(A^c \cap B_n) \geq \lambda(A \cap G) + \lambda(A^c \cap G)$$

because $A \cap G \subset \bigcup_n A \cap B_n$ and $A^c \cap G \subset \bigcup_n A^c \cap B_n$. Since ϵ is arbitary, we get $\lambda(A) \geq \lambda(A \cap G) + \lambda(A^c \cap G)$. On the other hand, since λ is subadditive, we have also $\lambda(A) \leq \lambda(A \cap G) + \lambda(A^c \cap G)$ and A is a λ-set.

(iv) By Carathéodory's lemma, λ is a measure on $(\Omega, \mathcal{L})$. Since by step (iii) $\mathcal{R} \subset \mathcal{L}$, we know by Caratheodories lemma that $\mathcal{A} \subset \mathcal{L}$ so that we can define μ on $\mathcal{A}$ as the restriction of λ to $\mathcal{A}$. By step (ii), this is an extension of the measure μ on $\mathcal{R}$.