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The Banach-Tarsky paradox

QUESTION.
Let $\mathcal{A}$ be the σ-algebra consisting of all subsets of a cube $\Omega \subset \mathbb{R}^3$. Is there a function $P : \mathcal{A} \to [0,1]$ such that $(\Omega, \mathcal{A}, P)$ is a probability space such that if two sets $Y, Z \subset \Omega$ which are congruent satisfy $\mathrm{P}[Y] = \mathrm{P}[Z]$?

There is a dramatic theorem showing that the answer to this question is no:

BANACH-TARSKY THEOREM (1924).
One can decompose a ball $Y \subset \Omega$ into 5 disjoint pieces Y_1, Y_2, Y_3, Y_4, Y_5 satisfying $\cup_{i=1}^5 Y_i = Y$ and rotate and move the 5 pieces in Ω in such a way that two of the moved pieces Y_1', Y_2' fit together to a ball with the same radius than the ball Y and that the remaining three moved pieces Y_3', Y_4', Y_5' fit together to a second ball with the same radius than Y.

This gives the answer to the question because it follows from the axioms for P that $\mathrm{P}[Y_1 \cup Y_2]Y_3 + Y_4 + Y_5] = \mathrm{P}[Y_1 + Y_2 + Y_3 + Y_4 + Y_5] = \mathrm{P}[Y]$ and $\mathrm{P}[Y_1 \cup Y_2] = \mathrm{P}[Y_1] + \mathrm{P}[Y_2] = \mathrm{P}[Y_1'] + \mathrm{P}[Y_2'] = \mathrm{P}[Y]$ as well as $\mathrm{P}[Y_3 \cup Y_4 \cup Y_5] = \mathrm{P}[Y_3] + \mathrm{P}[Y_4] + \mathrm{P}[Y_5] = \mathrm{P}[Y_3'] + \mathrm{P}[Y_4'] = \mathrm{P}[Y_5'] = \mathrm{P}[Y]$. However $\mathrm{P}[Y] + \mathrm{P}[Y] = \mathrm{P}[Y]$ implying $\mathrm{P}[Y] = 0$. Since Ω can be covered with finitely many translated pieces Y or part of such pieces also $1 = \mathrm{P}[\Omega] = 0$. This contradiction shows and the assumption on the existence of P can not hold.
Remark. Banach has proven in 1923 that in dimensions $d \leq 2$, there exists a function $\mathrm{P} : \mathcal{A} \to [0, \infty]$ such that $\mathrm{P}[\emptyset] = 0$ and $\mathrm{P}[\bigcup_{i=1}^n A_i] = \sum_{i=1}^n \mathrm{P}[A_i]$ if A_i are disjoint sets in $\mathbb{R}^d$ and such that $\mathrm{P}[R(A)] = \mathrm{P}[A]$ for any rotation or translation R and $\mathrm{P}[\prod_{i=1}^n [a_i, b_i]] = \prod_i (b_i - a_i)$. So, there exists no Banach-Tarsky paradox in one or two dimensions.

TO THE PROOF.
The proof uses the axiom of choice which is part of the axiom system ZFC (Zermelo-Fraenkel with Axiom of Choice) all conventional mathematics relies on. (There is a branch of mathematical logic where the game is to figure out, what one can do without the axiom of choice. However, since most models for nature we have use heavily the axiom of choice in its foundations (for example in quantum mechanics through probability theory or functional analysis) nobody would like to live without this axiom.)

A proof of the paradox can be found in the article K. Stromberg, Americ. Math. Mongthly 86 (1979) p. 151-161. There is also a book about the paradox: S. Wagon, The Banach-Tarski Paradox, Cambridge 1985

VITALI'S EXAMPLE (1905).
Let $\Omega = [0,1]$. Call $x, y \in \Omega$ equivalent if $x - y$ is a rational number. Let $\mathcal{K}$ be the set of equivalence classes with respect to this relation. The axiom of Choice assures that one can choose for each equivalence class K an element k. That is, there exists a "choice function" $f : \mathcal{K} \to \Omega$ with $f(K) \in K$. Define $Y = f(\mathcal{K})$. Let $R = \{r_n\}$ be an enumeration of the rational numbers in Ω and define $Y_n = (r_n + Y)$. Then $[0,1] = (\bigcup_n Y_n) \cap [0,1]$.

Also this result shows that the σ-algebra has to be smaller. Vitali's result is less spectacular because of a decomposition into countabley many elements. Because the sets Y_n are all translations to each other, $\mathrm{P}[Y_n] = a$ is independent of a. $\sum_n a = \sum_n \mathrm{P}[Y_n] = \mathrm{P}[\bigcup_n Y_n] = \mathrm{P}[\Omega] = 1$ implies $a = 0$ in which case also $1 = \mathrm{P}[\Omega] = 0$ which is not possible.

CONCLUSION.
In probability theory, if Ω is uncountable, we have to take the σ-algebra smaller than the set of all subsets.