Math 464/564, O. Knill, **March 1997**

Probability generating functions

Probability generating functions are useful for computing distributions of sums of independent random variables as well as to compute expectation and variance or higher moments $\mathrm{E}[X^n]$. Useful are also the moment generating function $\mathrm{E}[e^{-tX}]$ or the characteristic function $\mathrm{E}[e^{itX}]$ which we will see later.

DEFINITION: Let X be a discrete random variable taking values in $\mathbb{N} = \{0, 1, \dots\}$. The **probability generating function** of X is $\phi_X(t) = \sum_{n=0}^{\infty} \mathrm{P}[X = n] t^n = \sum_{n=0}^{\infty} f_X(n) t^n$.

THE SUM OF TWO INDEPENDENT RANDOM VARIABLES.
If X, Y are independent, then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.

PROOF OF THIS FACT.

$$f_{X+Y}(n) = \mathrm{P}[X+Y = n] = \sum_k \mathrm{P}[X = k, Y = n-k] = \sum_k \mathrm{P}[X = k]\mathrm{P}[Y = n-k] = \sum_k f_X(k) f_Y(n-k)$$

and we obtain

$$\phi_X(t)\phi_Y(t) = (\sum_k f_X(k)t^k)(\sum_m f_Y(m)t^m) = \sum_n (\sum_k f_X(k) f_Y(n-k))\, t^n = \sum_n f_{X+Y}(n)\, t^n$$

COMPUTATION OF EXPECTATION AND VARIANCE.

$$\mathrm{E}[X] = \phi_X'(1) .$$

$$\mathrm{Var}[X] = \phi_X''(1) + \phi_X'(1)(1 - \phi_X'(1)) .$$

PROOF OF THESE FACTS.

$$\phi_X(t) = \sum_n \mathrm{P}[X = n]\, t^n .$$

Differentiation gives

$$\phi_X'(t) = \sum_n n\mathrm{P}[X = n]\, t^{n-1} .$$

Evaluate at $t = 1$ gives the formula for the expectation.
Again differentiation gives

$$\phi_X''(t) = \sum_n n(n-1)\mathrm{P}[X = n]\, t^{n-2} .$$

For $t = 1$, we obtain

$$\phi_X''(1) = \mathrm{E}[X^2] - \mathrm{E}[X] .$$

Adding $\mathrm{E}[X] - \mathrm{E}[X]^2$ gives the variance.

PROBABILITY GENERATING FUNCTIONS OF SOME DISCRETE DISTRIBUTIONS:

Distribution	$f_X(k) = \mathrm{P}[X = k] =$	Parameters	Domain	Probability generating function
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	$n \in \mathbb{N}$, $p \in [0, 1]$	$\{0, \dots, n\}$	$(pt + 1 - p)^n$
Uniform	$1/n$	$n \in \mathbb{N}$	$\{1, \dots, n\}$	$\frac{(t - t^{n+1})}{n(1-t)}$
Poisson	$\frac{\lambda^k}{k!} e^{-\lambda}$	$\lambda > 0$	$\{0, 1, \dots\}$	$e^{\lambda(t-1)}$
Geometric	$(1-p)^k p$	$p \in (0, 1)$	$\{0, 1, \dots\}$	$p/(1 - (1-p)t)$