Math 464/564, O. Knill, **February 1997**

Expectation and variance for discrete random variables

DEFINITION: Let X be a discrete random variable.
The **expectation** of X is $\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \cdot \mathrm{P}[X = x]$.
The **variance** of X is $\mathrm{Var}[X] = \mathrm{E}[(X - E[X])^2]$.

SOME PROPERTIES OF EXPECTATION AND VARIANCE For random variables X, Y and $\lambda \in \mathbb{R}$

a) $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$
b) $\mathrm{E}[\lambda X] = \lambda \mathrm{E}[X]$
c) $X \leq Y \Rightarrow \mathrm{E}[X] \leq \mathrm{E}[Y]$
d) $\mathrm{E}[X^2] = 0 \Leftrightarrow X = 0$
e) $\mathrm{E}[X] = c$ if $X(\omega) = c$ is constant
f) $\mathrm{E}[X - \mathrm{E}[X]] = 0$
g) $\mathrm{Var}[X] \geq 0$
h) $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$
i) $\mathrm{Var}[\lambda X] = \lambda^2 \mathrm{Var}[X]$

PROOF OF THE ABOVE PROPERTIES:

a) $\mathrm{E}[X + Y] = \sum_{z \in (X+Y)(\Omega)} z \cdot \mathrm{P}[X + Y = z] = \sum_{x \in X(\Omega), y \in Y(\Omega)} (x + y) \mathrm{P}[X = x, Y = y]$
$= \sum_{x \in X(\Omega)} x \mathrm{P}[X = x] \sum_{y \in Y(\Omega)} y \mathrm{P}[Y = y] = \mathrm{E}[X] + \mathrm{E}[Y]$
b) $\mathrm{E}[\lambda X] = \sum_{x \in X(\Omega)} x \mathrm{P}[X = x] = \sum_{x \in X(\Omega)} \lambda x \mathrm{P}[X = x] = \lambda \mathrm{E}[X]$
c) $X \leq Y \Rightarrow X(\omega) \leq Y(\omega) \Rightarrow \mathrm{E}[X] \leq \mathrm{E}[Y]$
d) $\mathrm{E}[X^2] = 0 \Leftrightarrow P[X^2 = x] = 0 \;\; \forall x \quad \Leftrightarrow X = 0$
e) $X(\omega) = c$ is constant $\Rightarrow \mathrm{E}[X] = c \cdot \mathrm{P}[X = c] = c \cdot 1 = c$
f) $\mathrm{E}[X - \mathrm{E}[X]] = \mathrm{E}[X] - \mathrm{E}[\mathrm{E}[X]] = \mathrm{E}[X] - \mathrm{E}[X] = 0$
g) $(X - \mathrm{E}[X]) \geq 0 \Rightarrow \mathrm{Var}[X] \geq 0$
h) $\mathrm{E}[(X - E[X])^2] = \mathrm{E}[X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2] = \mathrm{E}[X^2] - 2\mathrm{E}[X]\mathrm{E}[X] + E[X]^2 = \mathrm{E}[X^2] - \mathrm{E}[X]^2$.
i) $\mathrm{E}[\lambda^2 X^2] = \lambda^2 \mathrm{E}[X^2]$ and $\mathrm{E}[\lambda X]^2 = \lambda^2 \mathrm{E}[X]^2$

EXPRESSIONS FROM THE DISCRETE DENSITY FUNCTION. Let $f_X(x)$ be the discrete density function of X, then

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \mathrm{P}[X = x] \sum_{x \in X(\Omega)} x f_X(x) \, .$$

$$\mathrm{Var}[X] = \sum_{x \in X(\Omega)} x^2 f_X(x) - (\sum_{x \in X(\Omega)} x f_X(x))^2 \, .$$

EXPECTATION OF $h(X)$ like for example $h(X) = \sin(X)$.

$$\mathrm{E}[h(X)] = \sum_{x \in X(\Omega)} h(x) \mathrm{P}[h(X) = h(x)] = \sum_{x \in X(\Omega)} x \mathrm{P}[X = x] = \sum_{x \in X(\Omega)} h(x) f_X(x) \, .$$

EXAMPLES OF DISCRETE DISTRIBUTIONS:

Distribution	$f_X(k) = \mathrm{P}[x = k] =$	Parameters	Domain	Expectation	Variance
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	$n \in \mathbb{N}$, $p \in [0, 1]$	$\{0, \dots, n\}$	np	$np(1-p)$
Uniform	$1/n$	$n \in \mathbb{N}$	$\{1, \dots, n\}$	$(1+n)/2$	$(n^2 - 1)/12$
Poisson	$\frac{\lambda^k}{k!} e^{-\lambda}$	$\lambda > 0$	$\{0, 1, \dots\}$	λ	λ
Geometric	$(1-p)^k p$	$p \in (0, 1)$	$\{0, 1, \dots\}$	$(1-p)/p$	$(1-p)/p^2$