Math 464/564, O. Knill, **February 1997**

A drunken Sailor, Euler's formula and a Devilish staircase.

DEFINITION. A discrete random variable is a function $X : \Omega \to \mathbb{R}$ with a discrete image, such that $X^{-1}(x) = \mathrm{P}[X = x]$ is an event. The function $f_X(x) = \mathrm{P}[\{X = x\}]$ is the discrete density function of X, while $F_X(x) = \mathrm{P}[X \leq x]$ is the discrete distribution function of X.
INTERPRETATION. Random variables describe measurements. Every experiment $\omega \in \Omega$ gives the outcome $X(\omega)$, like the value of a thrown dice.
MANIPULATION OF RANDOM VARIABLES. One can add, multiply and manipulate discrete random variables: if X, Y are random variables and λ is a real number, then $X + Y, XY, \lambda X$ are discrete random variables. Also, if $f : \mathbb{R} \to \mathbb{R}$ is a function, then $f \circ X$ is a random variable. For example $\sin(X)$ is a random variable.

EXAMPLE 1): RANDOM WALK (DRUNKEN SAILOR) The one-dimensional **random walk** models the random motion of a particle on the lattice $\mathbb{Z}$. The particle starts at 0 and makes in each period a time step $+1$ or -1 with probability 1/2. It is also used for discribing the balance development of a **gambler in a fair game** or the **value of a stock** in the market or the **steps of a drunken sailor**.
THE PROBABILITY SPACE OF THE RANDOM WALK is the set of all possible steps

$$\Omega = \{-1, 1\}^N = \{(\omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{-1, 1\}\}$$

the set of events $\mathcal{A} = \{A \subset \Omega\}$ and P is the probability measure $\mathrm{P}[A] = |A|/|\Omega|$.
THE LOCATION OF THE WALKER. Consider the random variables $X_k(\omega) = \omega_k$ and $S_n = \sum_{k=1}^n X_k(\omega)$. The value $S_n(\omega)$ is the location of the walker at time n having done the sequence of steps ω. In a gambling interpretation X_k is the win rsp. loss at the k'th step and S_n is the total win until the n'th step. For each $\omega \in \Omega$, one gets a trajectory $\{S_n(\omega)\}_{n=1}^N$.
THE DISTRIBUTION OF THE RANDOM VARIABLE S_n is a **Bernoulli distribution**

$$\mathrm{P}[S_n = k] = \left(\begin{array}{c} n \\ \frac{n+k}{2} \end{array} \right) p^k (1-p)^{n-k}$$

for $p = 1/2$. The walker can only reach places k at time n, if $n + k$ is even.

EXAMPLE 2): A FORMULA OF EULER ABOUT PRIMES. Probability theory is useful in number theory.
THE ZETA DISTRIBUTION. Consider a random variable with distribution $\mathrm{P}[X = n] = \zeta(s)^{-1} n^{-s}$, where ζ is the **Rieman zeta function**

$$s \mapsto \zeta(s) = \sum_{n=1}^{\infty} n^{-s} .$$

The famous open **Riemann Hypothesis** predicts that all zeros of $\zeta(s)$ in the complex line are on the line $\mathrm{Re}(s) = 1/2$. We know from theory that the random variable X defines a probability space $\Omega = \{1, 2, \dots, \}, \mathcal{A} = \{A \subset \Omega\}, \mathrm{P}[\{n\}] = X(\omega))$. We have only to check that $\mathrm{P}[\Omega] = \sum_n \mathrm{P}[\{n\}] = \zeta(s)^{-1} \sum_n n^{-1} = 1$.
SOME INDEPENDENT EVENTS. The sets $A_p = \{n \in \Omega \mid p \text{ divides } n\ \}$ with prime p are independent: we have $\mathrm{P}[A_p] = p^{-s}$ and if $p_i, i = 1, \dots, k$ are primes, then

$$\mathrm{P}[A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_k}] = \zeta(s)^{-1} \sum_n (n p_1 \cdots p_k)^{-s} == (p_1 \cdots p_k)^{-s} \zeta(s)^{-1} \sum_n n^{-s} = \mathrm{P}[A_{p_1}] \cdots \mathrm{P}[A_{p_k}] .$$

EULER'S FORMULA. Consider the event $A = \{n \mid n \text{ is not divisible by any prime number }\}$. Clearly, $A = \{1\}$. And $\mathrm{P}[A] = \zeta(s)^{-1}$. On the other hand, A is the interesection of all events $A_p^c = \Omega \setminus A_p$, where p runs over the prime numbers: from $A = \prod_{p \text{ prime}} A_p^c$, the independence of A_p^c, we get the **formula of Euler**

$$\frac{1}{\zeta(s)} = \prod_{p \in \mathbb{N}\ p \text{ is a prime}} (1 - p^{-s}) .$$

EXAMPLE 3): A DEVILISH DISTRIBUTION. The random variable

$$
\begin{aligned}
X(\frac{1}{2}) &= \frac{1}{2} \\
X(\frac{1}{4}) &= X(\frac{3}{4}) = \frac{1}{8} \\
X(\frac{1}{8}) &= X(\frac{3}{8}) = X(\frac{5}{8}) = X(\frac{7}{8}) = \frac{1}{32} \\
&\cdot \\
&\cdot \\
X(\frac{1}{2^n}) &= X(\frac{3}{2^n}) = \ldots = X(\frac{2^n-1}{2^n}) = \frac{2}{4^n} \\
&\cdot \\
&\cdot
\end{aligned}
$$

takes the values $(2k-1)/2^n, k = 1, \ldots, 2^{n-1}$ which is a dense set in the interval $[0,1]$.

The distribution function of X is a **devilish stair case**: you have to take in every time interval through infinitely many steps.

```
DevilSkin := {Line[{{0,0},{1,0}}]};
Hair[n_]:=Table[Line[{{(2k-1)/2^n,0},{(2k-1)/2^n,2/4^n}}],{k,2^(n-1)}];
DevilHair:=Show[Graphics[{DevilSkin,Table[Hair[n],{n,DeepnesLevel}]}],
  PlotRange->{{0,1},{0,2}},DisplayFunction->Identity];
LevelValues[n_]:=Table[{(2k-1)/2^n,2/4^n},{k,2^(n-1)}];
UpToLevel[nn_]:=Sort[Module[{s={}}, Do[s=Union[s,LevelValues[n]],{n,nn}];s]];
DevilStair[x_]:=Module[{v=.0},Do[If[u[[k,1]]<=x,v=v+u[[k,2]]],{k,Length[u]}];v];
DevilStairCase :=Plot[DevilStair[x],{x,0,1},DisplayFunction->Identity];
HellsBells[DL_]:=Module[{}, DeepnesLevel=DL; u=UpToLevel[DeepnesLevel];
  Show[GraphicsArray[{DevilHair, DevilStairCase}],
  DisplayFunction->$DisplayFunction]];
Display["!psfix -land>devilstair.ps",HellsBells[6]];
```