Math 464/564, O. Knill, **February 1997**

Combinatorics in Music

Are there combinatorical restrictions in creating new music? How does it come that one plays today also music from say 200 years ago while in mathematics for example, it is unthinkable to read Gauss or Eulers work? Something is different and maybe there are not so many melodies to invent and stealing of tunes is a necessity. This brings us to combinatorial music theory evenso we can not answer the above question.
A first interesting question is what are the choices to using frequencies, rythms etc? Let's stick to frequencies.
The western music divides the octave into 12 equal steps in the sense that going up one steps corresponds to a multiplication of the frequency by a factor $2^{1/12}$. The frequencies form a geometric sequence.
Why does one do that? The mulitplicative structure is because our ear listens logarithmically (which is both physically and physiologically reasonable). Frequency relations, which are rational are considered "harmonic".
Euler defined in his music theory for a frequency ration p/q the "gradus suavitatis" (degree of pleasure) $\Gamma(p/q) = (1 + \sum_i (p_i - 1))$, where p_i are the prime factors of $p * q$ (with multiplicity). For example, the little decime 12/5 has the degree of pleasure $(1+(2-1)(2-1)(3-1)(5-1)) = 9$. Now, choosing a geometric scale of 12 tunes interpolating a frequency doubling allows to approximate quite well the most harmonic relations.

$$
\begin{aligned}
2 &= \Gamma(1/2) \\
3 &= \Gamma(1/3) = \Gamma(1/4) \\
4 &= \Gamma(2/3) = \Gamma(1/6) = \Gamma(1/8) \\
5 &= \Gamma(1/5) = \Gamma(1/9) = \Gamma(1/12) = \Gamma(3/4) = \Gamma(1/16)
\end{aligned}
$$

As a matter of fact, the square of $2^{-1/12}$ which is $2^{-1/6} = 1.122462048...$ is close to $9/8 = 1.125$. The "antique doric tune" is obtained by taking from $c - d$ and $d - e$ the frequency relation 9/8 and from $e - f$ the relation 128/117, ($c - f$ has then a relation 4/3) then again 9/8 from $f - g$, $g - a$ and $a - h$ and again 128/117 from h to c. Other tunes are the "diatonic tune", $4/3 = (9/8) * (8/7) * (28/27)$ the "chromatic tune" $4/3 = (32/27) * (36/35) * (28/27)$, or the "enharmonic tune" using $4/3 = (5/4) * (36/35) * 28/27)$. The frequency relation $2^{1/12}$ for the smaller steps has the advantage that it is "translational invariant", that is one can compose music pieces translated. This is changing the possibilities for a composer of piano muscic because the scale on a piano is not translational invariant (black and white keys).
Assume, we have a building block of 12 frequencies, one can choose sub scales, which have better "degrees of pleasure". Examples are the baby scale $c - d - e - f, g, a, h$ consisting of 7 elements and used for most songs. Taking the same scale but starting with an other initial point gives so called "church-scales" or the scale "minor".
Our ear does not like frequency relations which large "degree of pleasure". That's why one has a smaller set of frequencies 7 instead of 12, with which one can play simpler melodies.
Now, we look at some problems to play accords (tunes played together), and melodies (tunes played in a sequence):
Question 1: A piano has 88 tunes. How many accords consisting of 3 tunes can one play? Answer: $c(88, 3) = 109736$.

Some of the accords are translational invariant and the tunes have to be taken modulo 12.
Question 2: How many accords of 3 tunes can one play modulo 12. Answer: put the first tune at c. Now, can choose 2 elements of 11. So, we have only 55 such accords.
Question 3 (to combinations with symmetry): How many accords can one play on the stair of 7 tunes $\{c, d, e, f, g, a, h\} = \{1, 2, 3, 4, 5, 6, 7\}$? Answer 2^7. To make the question more difficult, we ask for translational invariant different accords.
This question is now more difficult bug can be answered with the

> LEMMA OF BURNSIDE: Let a group G act on a set H. Then the number of cycles of the group action is
> $$|G|^{-1} \sum_{g \in G} \mathrm{Fix}(g) \; .$$

To apply this lemma to the problem of accords: let G be the group of translations consisting of 7 elements. The number of fixed points is 2^7 for the identity, 2 for every other element. The answer is $(2^7 + 2 * 6)/20$. Most of these accords are used in music: $\{"pause", "one-tune", "second", "tertz", "quart", "second-quart", "second-tertz", "second-quint", "three-accord", "tertz-quart", "quarte-ajoutee", "second-tertz-quarte", "second-quart-quint", "sixth-ajoutee", "second-ajoutee", "second-quart-ajoutee", "quart-sixth-ajoutee", "second-sith-ajoutee", "second-quart-sixth-ajoutee", "septime-second-quarte-sixth-ajoutee"\}$.

Question 4 (to permutations): How many pieces can one play in the "twelf-tone music"? Answer $12! = 479'001'600$.
Question 5 (to sampling) Anton Webern's Sinphonie op 21 the second "satz" consisists of a "theme" with 8 "variations" and a "coda". Each, "theme", "variation" and "coda" is a "twelf-tone music" and they have to be different. How many "sinphonies" can one write like this? Answer: we have to choose 11 permutations out of the 12! permutations. Therefore

$$c(12!, 11) = 7630462054792415927705549450172273571842040394984563168904316741381986181085708708377600 \sim$$

I remind you that our universe is a about 10^{17} seconds old! So even if we could play 10^{10} themes in a second (with future 1000x CD playes using future 10 GHz Pentium processors!) and every person of the soon 10^{10} people in the world would play with 10^{10} computers. One would still have to play 10^{40} ages of universes long to enjoy all possibile sinphonies. A reason to switch the music taste?

Question 6 (to sampling without replacement): How many melodies of length n can one play on a piano with 88 keys? 88^n.

Question 7 (to combinations with repetition): Assume three saxophones play together. How many accords can one form in a scale of 12 frequencies? $c^*(12, 3) = c(14, 3) = 364$.
Question 8 (to permutations with identification): an orchestra consists of 20 strings, 10 cellos and 5 bass. We permute them randomly and let one after the other play the tune c. How many "songs" can one hear like that? Answer: $p(35; 20, 10, 5) = 9753573309480$.