BASIC FORMULAS IN COMBINATORICS

PERMUTATIONS. Let $X = \{1, \dots, n\}$ be a finite set. A permutation π of X is a bijective function $\pi : X \to X$. There are

$$p(n) = n! = n(n-1)\cdots 1$$

such permutations. Proof: by induction, if one of the n possible values $\pi(1)$ is fixed, there are $p(n-1)$ possibilities to fix the other values $\pi(k)$.
EXAMPLE: There are $n!$ possible ways to redistribute n coats of n people.

PERMUTATIONS WITH IDENTIFICATION. Take $X = \{1_1, \dots, 1_{n_1}, 2_1, \dots, 2_{n_2}, \dots, k_1, \dots, k_{n_k}\}$. Elements like 2_3 and 2_5 are identified. There are

$$p(n; n_1, \dots, n_k) = \frac{n!}{n_1! \cdots n_k!}$$

permutations without distinguishing identified elements. Among all $n!$ permutations, there are $n_1!$ which are distinguished only by a permutation of identified elements 1_k so that $n!/n_1!$ permutations have 1-elments not distinguished. Proceed inductively.
EXAMPLE: From the letters A, A, B, B, B, B, E, U one can make $p(8; 2, 4, 1, 1)$ different words.

SAMPLING. We choose k elements from $X = \{1, \dots, n\}$. we look for k elements and do distinguish the picking order.

$$v(n,k) = \frac{n!}{(n-k)!}$$

if the picking order is distinguished.
EXAMPLE. 6 persons can in $v(10, 6)$ possible ways take place on 10 chairs. See also the birthday paradox from class.

SAMPLING WITH REPLACEMENT. We choose k elements out of a finite set $X = \{1, \dots, n\}$ but put the element back each time. The number of such elements is

$$v^*(n,k) = n^k$$

EXAMPLE: There are 6^{10} possible ways to throw 10 distinguishible dices.

COMBINATIONS. We pick k elements from n elements not distinguishing the picking order. We have to divide $v(n,k)$ by $k!$ to get

$$c(n,k) = \frac{n!}{(n-k)!k!}$$

possibilities to choose k elements out of n elements.
EXAMPLE: We can choose in $c(52, 10)$ possible ways a set of 10 cards from 52 cards.

COMBINATIONS WITH REPETITIONS. If in the combinations, we allow to pick several times the same elements, the number of possibilities is

$$c^*(n,k) = \frac{(n+k-1)!}{k!(n-1)!}$$

EXAMPLE: We can throw 5 indistinguishible dices in $c^*(6, 5)$ different ways.