FOUNDATION OF PROBABILITY THEORY

DEFINITION: Set theoretical operations.

Ω any set	"The set of all experiments"
$A, B, A_n \subset \Omega$	"Events"
$A \cap B = \{\omega \in \Omega \mid \omega \in A \text{ and } \omega \in B\}$	"Both events A and B happen"
$A \cup B = \{\omega \in \Omega \mid \omega \in A \text{ or } \omega \in B\}$	"Either A or B happens"
$A \Delta B = \{\omega \in \Omega \mid \omega \in A \text{ or } \omega \in B \text{ but not in both}\}$	"One of the events A or B happens"
$A \setminus B = \{\omega \in \Omega \mid \omega \in A \text{ but not } \omega \in B\}$	"A but not B happens"
$A^c = \{\omega \in \Omega \mid \omega \notin A\}$	"A does not happen"
$\bigcap_n A_n = \{\omega \in \Omega \mid \omega \in A_n, \text{ for all } n\}$	"All events A_n happen"
$\bigcup_n A_n = \{\omega \in \Omega \mid \omega \in A_n, \text{ for at least one } n\}$	"At least one event A_n happens"

DEFINITION: Boolean algebra

Ω be a set. A set $\mathcal{A}$ of subsets of Ω is a **Boolean algebra**, if

$$\begin{array}{l} \Omega \in \mathcal{A}, \\ A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}, \\ A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}. \end{array}$$

DEFINITION: σ-**field**

A σ-**field** or σ-**algebra** on Ω is a set $\mathcal{A}$ of subsets of Ω satisfying

$$\begin{array}{l} \Omega \in \mathcal{A}, \\ A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}, \\ \{A_1, A_2, \dots\} \subset \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}. \end{array}$$

PROPERTIES: A Boolean algebra $(\Omega, \mathcal{A})$ is closed under all set theoretical operations: $A, B \in \mathcal{A}$, then

$$\begin{array}{cc} \emptyset \in \mathcal{A} & A \cap B \in \mathcal{A}. \\ A \setminus B \in \mathcal{A} & A \Delta B \in \mathcal{A}. \end{array}$$

A σ-field $(\Omega, \mathcal{A})$ is closed under all set theoretical operations which can be enumerated.

DEFINITION: Probability measure

A function $P : \mathcal{A} \to \mathbb{R}$ on a σ-field $(\Omega, \mathcal{A})$ is a **probability measure** if

$\mathrm{P}[A] \geq 0,$	(nonnegativity)
$\mathrm{P}[\Omega] = 1,$	(normalisation)
$\mathrm{P}[\bigcup_{i=1}^n A_i] = \sum_{i=1}^n \mathrm{P}[A_i]$, if $A_i \cap A_j = \emptyset$, for all i, j,	(additivity)

"$\mathrm{P}[A]$ is the probability that the event A happens."

PROPERTIES:
$A \subset B \Rightarrow \mathrm{P}[A] \leq \mathrm{P}[B]$.
$\mathrm{P}[A^c] = 1 - \mathrm{P}[A]$, $\quad \mathrm{P}[\emptyset] = 0$
$\mathrm{P}[B] = \mathrm{P}[A \cap B] + \mathrm{P}[A^c \cap B]$
$\mathrm{P}[\bigcup_n A_n] = 1 - \mathrm{P}[\bigcap_n A_n^c]$.

SWITCH ON, SWITCH OFF formula:

$$\mathrm{P}[\bigcup_{i=1}^{n} A_i] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathrm{P}[A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}]$$