Math 464/564, O. Knill, **January 1997**

Why is a solid foundation of probability theory necessary?

Colloquial language is not precice enough to treat many probabilistic problems. Paradoxons or difficulties appear, when the definition of objects allows different interpretations. This leads to headache, to funny or wrong conclusions even by smart people.

BERTRAND'S PARADOX (Bertrand 1889)
We throw at random straight lines onto the unit disc. What is the probability that the straight line intersects the disc with a length $\geq \sqrt{3}$, the length of the equilateral triangle inscribed in the circle?

Answer Nr 1: take an arbitrary point P in the disc. The set of all lines passing through that point is parametrized by an angle ϕ. In order that the chord is longer than $\sqrt{3}$, the line has to lie within an angle of $60°$ out of $180°$. The probability is thus $1/3$.
Answer Nr 2: consider all lines perpendicular to a fixed diameter. The chord is longer than $\sqrt{3}$, when the point of intersection lies on the middle half of the diameter. The probability is thus $1/2$.
Answer Nr. 3: if the intersections of the line with the disc lies in the disc of radius $1/2$ which has area $1/4$ times the area of the whole disc, the chord is longer than $\sqrt{3}$. The probability is thus $1/4$.

PETERSBURG PARADOX (D. Bernoulli 1738)
In this casino, you pay an entrence fee c dollars and you get the prize 2^T dollars, where T is the number of times, one has to throw a coin until "head" appears.

Fact Nr. 1: Fair would be an entrence fee of

$$c = \sum_{k=1}^{\infty} 2^k P[T = k] = \sum_{k=1}^{\infty} 1 = \infty ,$$

which is the expected win.
Fact Nr. 2: Nobody would agree to pay $c = 10$ dollars, would you? Try it out!

THE THREE DOOR PROBLEM (1991)
Suppose you're on a game show and you are given a choice of three doors. Behind one door is a car and behind the others are goats. You pick a door-say No. 1 - and the host, who knows what's behind the doors, opens another door-say, No. 3-which has a goat. (In all games, the host opens a door to reveal a goat). He then says to you, "Do you want to pick door No. 2?" (In all games he always offers an option to switch). Is it to your advantage to switch your choice?

The paradox is that most people analyze this simple problem in a wrong way. The problem was discussed by Marilyn vos Savant in a "Parade" column in 1991 and provoked a big controversy in the next months. Thousends of letters were written. The problem is that intuitive argumentation can easily lead to the conclusion
What do you think? Check out the web links on the course websites.