Math464/564 O. Knill, **February 1997**

Homework 6

Due: Tuesday, March 4, 1997 in class

Topic: Expectation and Variance of discrete random variables.

Remark. This HW contains 5 questions with 50 points.

1) (10 points) Topic **Expectation and Variance** Consider the probability space of throwing a dice once, that is $\Omega = \{1,2,3,4,5,6\}$ and $\mathrm{P}[A] = |A|/|\Omega|$. a) What is the expectation and variance of the random variable $X(\omega) = \omega$?
b) What is the expectation and variance of the random variable $X(\omega) = \omega^2$?

2) (20 points) Topic: **Expectation of discrete random variables.**

> A physical system can be in different energy states and which is in contact with a heat reservoir of inverse temperature β. If the energies take discrete values $e_1, e_2, \dots$, the system will be in the **Bolzman distribution** $\mathrm{P}_\beta[\{j\}] = e^{-\beta e_j}/Z(\beta)$, where $Z(\beta) = \sum_{j=1}^\infty e^{-\beta e_j}$ is called the **partition function**. The probabilities $\mathrm{P}_\beta[\{j\}]$ are called the **Bolzmann-factors**. The **energy** of the state j is $e_j = X(j)$. The case $e_j = j$ is the situation of the **quantum mechanical oscillator**. The formula defined in d) is **Planck's formula** for the **total energy** of a quantum mechanical oscillator. In suitable physical units, this is **Planck's blackbody radiation formula**, which gives the average energy of the oscillator in dependence of the temperature.

Given for every real number $\beta > 0$ the discrete probability space $(\Omega, \mathcal{A}, P_\beta)$ with

$$\Omega = \mathbb{N} = \{1,2,\dots\}, \mathcal{A} = \{A \subset \Omega\}, \mathrm{P}_\beta[\{j\}] = \frac{e^{-\beta e_j}}{Z(\beta)} ,$$

where $Z(\beta) = \sum_{j=1}^\infty e^{-\beta e_j}$. and the real numbers e_j are such that the sum $Z(\beta)$ is finite. Consider the random variable X on $(\Omega, \mathcal{A}, P_\beta)$ defined by $X(j) = e_j$.

a) (5 points) Verify that P is a probability measure. (In the midterm, the Ω contained 0).
b) (5 points) Show that

$$\mathrm{E}_\beta[X] = -\frac{\frac{d}{d\beta} Z(\beta)}{Z(\beta)} ,$$

where E_β is the expectation with respect to $(\Omega, \mathcal{A}, P_\beta)$.
c) (5 points) Compute $Z(\beta)$ in the case, when $e_j = j$ for $j = 1, \dots$. Simplify as much as possible.
d) (5 points) Compute, using $b)$ and $c)$, the value of $\mathrm{E}_\beta[X]$, in the case $e_j = j$.

3) Topic: **Expectation of discrete random variables.** (10 points)

> In many tables of physical constants and statistical data, the **leading digit** of the data is not uniformly distributed among the digits (as might naivly be expected). Rather, the lower digits appear much more frequently than the higher ones. **Benford's law** says that the probability that the first significant digit is k is given by $\log_{10}(1 + k^{-1})$. Benford derived this law in 1938 from some statistics he did from twenty different tables. It is today quite evident that Benford manipulated the round off errors to obtain a better fit. Dispite this fraud, the law is called after him. The IRS is considering doing Bedford tests on the data obtained from the taxpayer.

Consider the finite set $\Omega = \{1,2,\dots,9\}$ and the Boolean algebra $\mathcal{A} = \{A \subset \Omega\}$.
a) (5) Show that $\mathrm{P} : \mathcal{A} \to [0,1]$

$$\mathrm{P}[A] = \sum_{k \in A} \log_{10}(\frac{k+1}{k})$$

is a probability measure, where $\log_{10}$ is the logarithm with respect to the base 10.
b) (5) Let X be the random variable, which gives the first digit $X(k) = k$. Compute $\mathrm{E}[X]$. (Simplify the result as much as possible).

4) (10 points). Assume X is geometrically distributed with parameter λ and Y is Poisson distributed with parameter μ.
a) What is the expectation of $X + Y$?
b) What is the variance of $X + Y$ if $\mathrm{E}[XY] = 0$?