Math464/564 O. Knill, **February 1997**

Homework 4

Due: Tuesday, February 18, 1997 in class

Topics: Discrete random variables, discrete densities, distribution functions, independence, geometric densities.

Remark. This HW contains 5 questions. Each question can give 10 points, so that you can get a maximal credit of 50 points for this HW.

> REMINDER OF SOME DEFINITIONS:
> A **discrete random variable** is a real-valued function X on a probability space $(\Omega, \mathcal{A}, P)$, which has the property that $X(\Omega)$ is a discrete set $\{x_1, x_2, \dots, \}$ in the real line $\mathbb{R}$ and such that the sets $A_j = X^{-1}(x_j) = \{\omega \in \Omega \mid X(\omega) = x_j\}$ are in $\mathcal{A}$. If we replace $\mathbb{R}$ by $\mathbb{R}^d$, we speak of a **random vector** $X = (X_1, \dots, X_d)$.
> The **discrete density function** f of a random variable X is the function on $\mathbb{R}$ given by $f(x) = \mathrm{P}[X = x]$. For a random vector $X = (X_1, \dots, X_d)$ the **discrete density function** is the function $f(x_1, \dots, x_d) = \mathrm{P}[X_i = x_i]$ on $\mathbb{R}^d$.
> The **distribution function** of a random variable X is the function $F_X : \mathbb{R} \to \mathbb{R}$ given by $F_X(y) = \mathrm{P}[X \leq y]$. For a random vector, the distribution function is the function $F_X(x_1, \dots, x_d) = \mathrm{P}[X_i \leq x_i]$ on $\mathbb{R}^d$.
> Two random variables or random vectors X, Y are **independent** if the sets $A_i = X^{-1}(x_i)$ and $B_j = Y^{-1}(y_j)$ are independent events for all pairs $i \neq j$.

1) (10 points) Topic: **Definition of discrete random variables** .
a) How many random variables do there exist on the probability space

$$(\Omega = \{1, \dots, 37\}, \mathcal{A} = \{A \subset \Omega\}, \mathrm{P}[A] = |A|/|\Omega|]) \ .$$

b) Find all random variables on the probability space $(\Omega = \{1, 2, \dots, 1000\}, \mathcal{A} = \{\emptyset, \Omega\}, \mathrm{P}[\emptyset] = 0, \mathrm{P}[\Omega] = 1)$.
c) Is it true that on every finite probability space, where $\mathcal{A}$ is the set of subsets of Ω, and P is an arbitrary probability measure, every real function is a random variable?
d) Let $(\Omega = [0, 1], \mathcal{A} = \{\text{measurable sets}\}, \mathrm{P}[[a, b]] = b - a)$.
Is $X(x) = x$ a discrete random variable?
e) Find a discrete random variable X on $(\Omega = [0, 1], \mathcal{A} = \{\text{measurable sets}\}, P[[a, b]] = b - a)$ such that $X(\Omega) = \mathbb{N}$.

2) (10 points) Topic: **Density and distribution functions**. Let $\Omega = \{\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \mid \omega_i \in \{0, 1\}\ \}$ be the probability space of throwing a dime 4 times. Let $X(\omega) = \sum_{i=1}^4 \omega_i$. Draw the discrete density function and the distribution function of X.

3) (10 points) Topic: **Independent random variables.** Let $\Omega = \{\omega = (\omega_1, \omega_2) \mid \omega_i, \omega_2 \in \{1, \dots, 6\}\}, \mathcal{A} = \{A \subset \Omega\}, P[A] = |A|/|\Omega|\}$ be the probability space of throwing two dices. Consider the two random variables $X(\omega) = \omega_1, Y(\omega) = \omega_2$. Determine explicitely the events $X^{-1}(3) = \{$ the first dice is 3 $\}$ and $Y^{-1}(5) = \{$ the second dice is 5 $\}$ and verify that they are indeed independent events.

4) (10 points) Topic: **Indendent random variables** Show that if two random varables X, Y are independent, then also X^2 and Y^2 are independent.

5) (10 points) Topic: **Geometric density**. Let X, Y be independent random variables having geometric densities with parameters p and q. Find
a) $\mathrm{P}[X \geq Y]$.
c) The density of $X + Y$.