Math464/564 O. Knill, **April 1997**

Homework 12

Due: Tuesday, April 22, 1997 in class

Topic: Convergence of random variables. The central limit theorem.

Remark. This HW contains a question with 20 points and three questions giving 10 points so that you can get a maximal credit of 50 points for this HW.

1) (20 points) Topic: Convergence of random variables. Recall that X_n converges in distribution to X if and only if $\mathrm{P}[X_n \leq t] \to \mathrm{P}[X \leq t]$ for all t. Find in each of the following cases a random variable X to which X_n converges in distribution.

   a) (4 points) X_n is Normal distributed with parameters $N(1/(1+n^2), 1+1/n)$.
   b) (4 points) X_n is Binomial distributed with parameter $B(n, 1/n)$.
   c) (4 points) $X_n = (S_n - np)/\sqrt{np(1-p)}$, where S_n is a sum of n independent $B(n,p)$-distributed random variables.
   d) (4 points) X_n has a exponential distribution with parameter $\lambda = n$.
   e) (4 points) X_n is uniformly distributed on the interval $[-n, n]$.

2) (10 points) Topic: Convergence of random variables. Assume X_n is uniformly distributed on $\{0, 1, \ldots, n\}$ satisfying $\mathrm{P}[X_n = k/n] = 1/(n+1)$. Prove that X_n converges in distribution to the uniform distribution on $[0, 1]$.

3) (10 points) Topic: Convergence of random variables. Assume X_n is a sequence of pairwise uncorrelated random variables with mean $\mathrm{E}[X_n] = 0$ and $\mathrm{Var}[X_n] \leq 5$.

   a) Prove that S_n/n^α converges to 0 in distribution for all $\alpha > 1/2$.

   b) Under the additional condition that $\mathrm{Var}[X_n] \geq 1$, prove that S_n/n^α does not converge in distribution to a random variable with finite variance if $\alpha < 1/2$.

4) (10 points) Topic: Central limit theorem. Let X_n be a sequence of independent random variables and let $\Phi(t)$ is the distribution function of a standard normal distributed random variable. The central limit theorem implies that $\mathrm{P}[S_n \leq t]$ is for large n close to $\Phi(\frac{x-E[S_n]}{\sigma(S_n)})$.

   Denote by X_n the number of strokes, a golf player has to do in order to drive the ball into one hole and assume X_n are independent. Assume the course has 18 teeing areas and the performance of a golf player called Tiger Woods is $\mathrm{E}[X_n] = 3$ and $\mathrm{Var}[X_n] = 2$. What is the probability that Woods needs more than 50 strokes in order to finish a course having 18 holes.

   Hint. If the function Φ is not on your computer, there is a table on page 252 in the book.