Math464/564 O. Knill, **April 1997**

Homework 10

Due: Tuesday, April 8, 1997 in class

Topic: Transformation of absolutely continuous random variables.

Remark. This HW contains 4 questions. Two questions give 10 points, the other two 15 points. You can get a maximal credit of 50 points for this HW.

1) (15 points) Topic: **The Arc-Sin distribution**. Background: A quantum mechanical particle moving freely in a one-dimensional discrete crystal has an energy density in $[-2,2]$ given by $f(x) = \frac{1}{\pi}(4-x^2)^{-1/2}$. This is called the density of states of the system. The probability that a particle has its energy in an interval $[a,b]$ is $\int_a^b f(x)\,dx$. The probability distribution function $F(t) = \int_{-\infty}^t f(s)\,ds$ is called the integrated density of states and is accessible to measurements in solid state physics.

   Consider a random variable X with probability density function f_X which vanishes outside $(-2,2)$ and which is $f_X(x) = 1/(\pi\sqrt{4-x^2})$ for $x \in (-2,2)$.

   a) (5 points) Verify that f is a probability density function and compute the distribution function $F_X(t) = \int_{-\infty}^t f(s)\,ds$. Hint: remember the name of the distribution.
   b) (5 points) Compute $\mathrm{E}[X]$, the average energy of a particle.
   c) (5 points) Compute $\mathrm{Var}[X]$. (Hint: your symbolic computing software tells you that $\int x^2/\sqrt{4-x^2}\,dx = 2\arcsin(x/2) - x\sqrt{4-x^2}/2$.)

2) (10 points) Topic: **The Log-Normal distribution**. It appears often in applied statistics that logarithms of observed datas are normal distributed. In this case, one deals with log-Normal distributed random variables.

   Assume X is normal distributed with mean m and variance σ^2. Compute the density of the random variable $Y = e^X$. This is called the **log-Normal** density.

3) (10 points) Topic: **Exponential distributed random variable**. The time X with which an isotop decays is an exponential distribued random variable with parameter $\lambda = 10$. You look up λ in a table where it is given in the case, when a time unit is 1 second. For some reason, you want to know λ, when the time unit is 1 year = 31'536'000 seconds. In other words, determine the density of the random variable cX for $c = 1/31'536'000$.

4) (15 points) Topic: **Bertrand paradox revisited**. Remember the Bertrand paradox? There Ω was the set of all possible ways with which one can put a line intersecting the unit disc $\mathbb{D} = \{x^2+y^2 \leq 1\}$. We considered the random variable $X(\omega)$="length of the segment cut by a random line in Ω". The problem was to determine $\mathrm{P}[X \leq \sqrt{3}]$. The probability measure P on Ω and so the distribution of X was not determined and lead to different answers.
   a) (5 points) In the first case, $\omega \in \Omega$ was represented by the distance of the line to the origin, assuming $\mathrm{P}[\{\omega \in [0,1] \mid \omega \in [a,b]\}\}] = b-a$. Determine the probability density function f_X of the random variable $X : [0,1] \to \mathbb{R}$ in this case and compute $\mathrm{E}[X]$.
   b) (5 points) In the second case, $\omega\Omega$ was represented by the angle it hits the circle assuming $\mathrm{P}[\{\omega \in [0,\pi]\ \omega \in [a,b]\}] = b-a$. Determine the proability density function f_X random variable $X : [0,\pi] \to \mathbb{R}$ in this case and compute $\mathrm{E}[X]$.
   c) (5 points) In the third case, $\omega \in \Omega$ was represented by that point in the disc, which is the center of the segment cut by the line and P was the normalized area measure on the disc $\mathbb{D}$. Determine the probability density function of the random variable $X : \mathbb{D} \to \mathbb{R}\}$ and compute $\mathrm{E}[X]$.