**Math464/564 O. Knill,** **March 1997**

# Second Midterm

**Time: Thursday, March 28, 1997, 14:00-15:15**

**Topics: Chapter 3-4: Discrete random variables, expectation and variance.**

<u>**Material**</u>: your notes (as much you have), homework, distributed text, no books (to prevent loosing time with reading and encourage good personal summaries of the material)

<u>**Points**</u>: There are 100 points for 5 questions of 20 points. There is a "super-choice type" question on the fourth page with 20 points. The points achieved in this six'th problem will replace the lowest score of the first five questions. You have the choice to either replace one of the first five questions with the six'th, or (if your time permits) to do all 6 questions and have the minimal number among the 6 subtotals cancelled.

<u>**Form**</u>: The answer to the first question has to be answered on this page, the answer to the six'th question, (if you make use of it) on the fourth page. You can use the space in the boxes on these pages as well as additional pages for the rest of the questions. Please sign those additional pages and indicate the problem number on each of them. For Problems 2-5, the derivations for the result must be provided.

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1) (20 points) Not necessarily sorted (True/False) questions. Each correct answer gives two points.

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01) The variance of a random variable which is not identically zero is positive.

02) Chebychev's inequality allows to give an upper bound on probabilities $P[|X - E[X]| \geq \epsilon]$.

03) The probability generating function of a random variable is a polynomial if and only if the random variable takes only finitely many values.

04) The covariance of two independent random variables vanishes.

05) The sum of two independent Poisson distributed random variables is Poisson distributed.

06) If $X$ takes the value 6 with probability 1/3 and the value 3 with probability 2/3, then $E[X] = 4$.

07) If $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$, then $X$ and $Y$ are uncorrelated.

08) There are random variables taking finitely many values for which the expectation is not defined even so the variance is finite.

09) Given a sequence of random variables, then $(X_1 + X_2 + \ldots + X_n)/n$ converges.

10) $E[(X - E[X])Y] = 0$ if $X$ and $Y$ are independent

2) (20 points) Topic: **The Cramer-Bernoulli solution to the Petersburg paradox by the idea of utility.**

Recall the Petersburg paradox (suggested by Daniel Bernoulli in 1713): you enter the game with an entrance fee of $c$ dollars. Then a sequence of dimes are thrown and you win in that game $X = 2^Y$ dollars if $Y$ subsequent heads occur. For example, for the outcome $\omega = (HHHT\ldots)$, you would win $X(\omega) = 2^{Y(\omega)} = 2^3 = 8$ dollars. The total profit per game is $2^Y - c$ and the game is fair if the expectation of $2^Y - c$ is zero. The paradox was that no matter which entrance fee $c$ you pay, the game will be favorable to you because the expectation of $X = 2^Y$ is infinite. This contradicts the the fact that after having done some games, nobody would agree to pay an entrance fee of say $c = 20$. Gabriel Cramers suggested in 1728 (similarly than James Bernoulli (an oncle of a cousin of Daniel Bernoulli) the following "solution" to the problem. Because for "a person with a lot of money a dollar is less worth), the effective win in a game should be replaced by $\mathrm{E}[\sqrt{X}]^2$. (This suggestion makes sense because for example a win of say one gogool (which is more than the number of atoms in the universe) dollars is certainly useless).

Problem: Compute $\mathrm{E}[\sqrt{X}]^2$ which is the value of a new fair entrance fee $c$. (You should get a number).

3) (20 points) Topic: **The Lotka model for population statistics.**

Based on experimental data, Lotka proposed the following model for the number $X$ of children in a random family: $\mathrm{P}[X = k] = \beta p q^k$ if $k \neq 0$ and $\mathrm{P}[X = k] = 1 - \beta q$ if $k = 0$, where $q = 1 - p$. The case $k = 0$ is teated as a special case because by biological or personal reasons some families are not able or do not want to have children. Based on data from 1930 for American families, one obtains a good fit for $\beta = 9/10$ and $p = 1/4$ (and therefore $q = 3/4$).

Problem. Compute the expected number $\mathrm{E}[X]$ of children with these parameters. (The answer which is a number).

4) (20 points) Topic: **A casino buys an assurance**.

In roulette, while betting on one number, a customer wins $70'000$ dollars with a probability $1/37$ and loses $2'000$ dollars with probability $36/37$. To soften the loss in the case of a win of the customer, the manager of the casino plans to sign a contract with an assurance company which proposes the following option: if the casino loses to a customer, then assurance pays $40'000$ dollars to the casino, each time, the casino wins, it has to pay the assurance company $1'000$ dollars.

a) What is the expected win of the casino in one game with assurance. Compare it with the expected win in one game without assurance.
b) What is the expected win of the assurance company in one game?

5) (20 points) Topic: **Chevalier de Méré's problem with mathematics.**

Let $X$ be the number of 1's in 4 rolls of a fair dice and $Y$ the number of double 1's in 24 rolls of two fair dice.

a) Find the probability that $X$ is positive. b) Find the probability that $Y$ is positive.
(Hint to a) and b): compute the probability of the complementary event).

Chevalier de Méré was a high roller who thought that the answers to part a) and b) should be the same. Accordingly, he usually bet on $Y$ being positive and blamed mathematics when he lost money over a long sequence of bets.

What might have led Méré to think wrongly? Maybe the answers to the following question c) and d) explain:

c) What is the expectation of $X$? d) What is the expectation of $Y$?

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6) (20 points) The points in this question will replace the lowest score of the previous 5 questions. Each correct gives 5 points and you can choose 4 of the five following questions (or do all and get the lowest of the scores in a)-e) dropped. Write the answers into the boxes.

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a) (5 points)

If $X$ is a $(5, 1/3)$-Bernoulli distributed random variable, find using Chebychev's inequality a number $C$ such that $\mathrm{P}[|X - \mathrm{E}[X]| \geq 2] \leq C$.

b) (5 points)

What is the probability generating function of $X + Y$, where $X$ and $Y$ are independent and $X$ is $\lambda = 3$ has a geometric distribution and $Y$ is $\lambda = 5$ Poisson distributed.

c) (5 points)

Compute $\mathrm{E}[(X+Y)^4]$, where $X$ and $Y$ independent random variables with $\mathrm{E}[X^2] = \mathrm{E}[Y^2] = 1, \mathrm{E}[X^3] = \mathrm{E}[Y^3] = 0$ and $\mathrm{E}[X^4] = \mathrm{E}[Y^4] = 2$.

d) (5 points)

Write down $\mathrm{Var}[X]$ for a random variable, for which you know that the probability generating function is $\phi_X(t) = (t^4 + t + 1)/3$.

e) (5 points)

Surely, everybody of you has seen the **Hale-Bopp comet** which is now nicely visible even in a lighted street near campus. There is an understandable concern of humans to calculate the risk that a comet of this size would hit the earth (remember the dino-doom 65 million years ago). One knows of about 100 ring structures which earth which come from meteorites. An example is the Arizona crater which was created 50'000 years ago. One thinks that impact of such a 60 meter meteorite occurs in average every 500-1000 years. One believes that impacts with considerable change of the earth climate can occur several times per million years. Even so this sounds pretty scary, no human in the past 1000 years is known to have been killed by a meteorite or by the effects of an impact. (For more information on comets or meteorites surf the site $http://www.jpl.nasa.gov/$.)

Problem. Assume, the number of years one has to wait until a $\sim 60$ meter meteorite impacts the earth is a Poisson distributed random variable, with an expected value of 700 years. What is the probability to get a hit in the next 2 years? (Hint: the probability in the answer is smaller than $1/(1gogool)$.)