Math464/564 O. Knill, **February 1997**

First Midterm

Time: Thursday, February 20, 1997, 14:00-15:15

Topics: Probability spaces (chapter 1), Combinatorics (chapter 2), Discrete random variables (chapter 3).

<u>**Material**</u>: you can use all your notes (as many as you have), your homework, my distributed text. During this midterm, we don't consult the text book nor any other book.

<u>**Points**</u>: There are maximal 100 points. Points for each questions and sub-question are indicated.

<u>**Form**</u>: The answer to question 1 has to be answered on this page. Use seperate pages for the rest of the questions. Please sign these additional pages and indicate the problem number on each of the seperate pages.

NAME:

1) (20 points) Random (true/false) questions which are not necessarily sorted according to topics. A correct answer gives 2 points. Check the true boxes with a cross.

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01) ☐ In a σ-algebra, the union of an arbitrary number of sets is also in the σ-algebra.

02) ☐ The set of all subsets of a set is a σ-algebra.

03) ☐ The probability of a union of two events is always the sum of the probabilies of the single events.

04) ☐ $\mathrm{P}[A \Delta B] = \mathrm{P}[A \cup B] - \mathrm{P}[A \cap B]$.

05) ☐ If $A \subset B$, then $\mathrm{P}[A] \leq \mathrm{P}[B]$.

06) ☐ Five people can sit in 32 different ways on five chairs.

07) ☐ Throw a dime 5 times. There are 10 different possibilities to have excatly 2 times head.

08) ☐ A discrete random variable takes only finitely many values.

09) ☐ It is always true that $\mathrm{P}[A|B] \leq \mathrm{P}[A]$ if $\mathrm{P}[B] > 0$.

10) ☐ The distribution function F_X of a random variable is monotone: that is it satisfies $F_X(x) \leq F_X(y)$ if $x \leq y$.

2) (10 points) Consider $\Omega = \{0, 1, 2, \dots\}$, $\mathcal{A} = \{A \subset \Omega\}$ and $\mathrm{P}[\{n\}] = e^{-\beta n}(1 - e^{-\beta})$, where β is a real constant.

a) (5 points) Check the axioms of Kolmogorov to verify that $(\Omega, \mathcal{A}, \mathrm{P})$ is a probability space.
b) (5 points) What is the probability of the event $A = \{2, 4, 6, \dots\}$ of all even numbers?

("This probability space is important for Planck's law of black body radiation which was crucial for the development of quantum mechanics.")

3) (10 points) Topic: Conditional probability. In the book "uniformity in the world", the german philosopher Marbe expresses the believe that in a repeated toss of a fair coins, a run of heads makes a tail more likely in the next toss. He thinks that this is implied by the "law of averages".
For four tosses of a fair coin, find the conditional probability that the fourth toss comes up is tail, given that the first three tosses give heads. Solve this problem carefully along the following steps:

a) (3 points) Construct the probability space $(\Omega, \mathcal{A}, \mathrm{P})$.
b) (3 points) Find the event B that the first three tosses give head and the event A that the fourth toss gives tail.
c) (4 points) Determine the conditional probability using the definition.

4) (10 points) Topic: Probability spaces, independence of events.
a) (5 points) Prove that two events A and B with $\mathrm{P}[B] > 0$ are independent if and only if $\mathrm{P}[A|B] = \mathrm{P}[A]$.
b) (5 points) You know that $\mathrm{P}[A|B] = 0.1$ and $\mathrm{P}[B|A] = 0.3$. What is $\mathrm{P}[A]/\mathrm{P}[B]$?

5) (10 points) Topic: Combinatorics. In the "California super lotto", there is this February a jackpot of 33 Millions. In this game, one chooses 6 numbers out of 51 numbers (The Arizona lotto analyzed in class has a smaller Jackpot however better chances to win the jackpot).

a) (5 points) What is the probability to have 6 right in this game?
b) (5 points) What is the probability to have 2 right?

6) (10 points) Topic: Combinatorics. A standard piano has 88 keys. A common figure for a pianist is to play two accords in sequence where the first accord consists of three tunes (the pianist presses three different keys simultaneously) and the second accord of two tunes (the pianist presses two different keys simultanously). The two accords are played after each other and do not need to be different. A jazz pianist wonders how many such figures can be played. (Actually, the pianist will need both hands and a nose to play some of those accords, but as mathematicians we do not care).

7) (10 points) Topic: Independence. The chances of having rain during a day in the summer in Tucson is $1/100$. Assume the events to have rain at different mondays are independent.

a) (5 points) What is the probabiliy to have no rain during 15 mondays in summer?
b) (5 points) What is the probabilty to have no rain during 13 of 15 mondays in summer?

8) (10 points) Topic: Discrete random variables. The desert laboratory (located on the nearby hill with the little telescope and the antennas in Tucson at a 10 minute drive from the university) measures that the number X of Saguaros in one acre is Poisson distributed with parameter $\lambda = 2$. Assume, you want to buy an acre of beautyful land on a nearby hill. What is the probability to become with this buy the owner of maximal 5 Saguaros (that is you will either have 0,1,2,3,4 or 5 of these cacti on your land)?