Math464/564 O. Knill, **May 1997**

Final

Time: Thursday, May 15, 1997, 14:00-16:00

Topics: The material of the whole course

Material: your notes (as much you have), homework, distributed text, no book.

Points: There are maximal 100 points. You can choose 5 questions of 20 points from the 6 questions (or (if time permits) solve all 6 questions and have the minimal points deleted).

Form: The first question has to be answered on this page. You can solve all the problems on these pages and add if necessary separate pages. Please sign those additional pages and indicate the problem number on each of them.

NAME:

1) (20 points) Not necessarily sorted (True/False) questions. Each correct answer gives two points.

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01) ☐ $\mathrm{Var}[X+Y+Z] = \mathrm{Var}[X] + \mathrm{Var}[Y] + \mathrm{Var}[Z]$ if X, Y, Z are independent random variables.

02) ☐ $\mathrm{P}[X \geq \epsilon] \leq M_X(t)/e^{t\epsilon}$ holds, where $M_X(t) = E[e^{tX}]$ is the moment generating function of X.

03) ☐ The characteristic function of a random variable is a periodic function if X takes only finitely many integer values.

04) ☐ If the covariance of two random variables is different from zero, then they are not independent.

05) ☐ The sum of two independent Gaussian distributed random variables is Gaussian distributed.

06) ☐ If X takes the value 2 with probability 1/2 and the value -2 with probability 1/2, then $\mathrm{Var}[X] = 4$.

07) ☐ If N_B is a random variable counting the number of points in a set B in the Poisson process with mean measure μ, then $\mathrm{P}[N_B = 0] = e^{-\mu(B)}$.

08) ☐ The random walk in four dimensions is transient.

09) ☐ Given a sequence of random variables with mean zero and variance 1, then $(X_1 + X_2 + \ldots + X_n)/\sqrt{n}$ converges in distribution.

10) ☐ If X_n is a standard random walk, then $V_n = X_{n-1}$ is a previsible process and $E[S_n^V] = E[\sum_{k=1}^n V_k X_k] = 0$.

2) (20 points) Topic: **Combinatorics**

a) (10 points) In the Arizona "Fantasy 5" lotto game, one has to choose 5 numbers out of 35 numbers. What is the probability to have 4 numbers right?(if this happens, you win 500 dollars).

b) (10 points) During the second world war, the Germans used the Enigma machine for encrypting messages. In such a machine, rotors were used which were used to permute the 26 letters. Depending on the position of the rotor, different permutation could be performed. More precicely, a rotor could be in 26 different positions and so perform 26 different permutations. How many different rotors could be built in principle?

3) (20 points) Topic: **Discrete distributions**

Assume X and Y are independent random variables both having the negative binomial distribution with parameter n and p. That is:

$$\mathrm{P}[X = k] = \binom{n+k-1}{k} p^n q^k ,$$

where $q = 1 - p$, and $k = 0, 1, \ldots$.

a) (10 points) You see in a table or book that the probability generating function of X is $p^n/(1-qt)^n$. Verify this.

Hint. You know (and can use without proof) that

$$(1-q)^{-n} = p^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} q^k$$

because $\sum_{k=0}^{\infty} \mathrm{P}[X = k] = 1$.

b) (10 points) Verify that $X + Y$ is negative binomial distributed if both X and Y are independent negative binomial distributed with parameter n and p. Determine the parameters for $X + Y$.

4) (20 points) Topic: **Joint distribution of a random vector**

a) (10 points) Determine the joint density of the random vector

$$(U, V) = \phi(X, Y) = (X^3 - Y, X) ,$$

where (X, Y) is standard normal distributed with joint density

$$f(x, y) = (2\pi)^{-1} e^{-\frac{x^2+y^2}{2}} .$$

b) (10 points) Determine the density of the random variable U. (You have not to evaluate the integral).

5) (20 points) Topic: **Independence, random walks, limit theorems**

Let X_n be IID random variables taking values in $\{-1, 1\}$ and satisfying $\mathrm{P}[X_n = 1] = \mathrm{P}[X_n = -1] = p \in (0, 1)$.

a) (10 points) What is the expectation $\mathrm{E}[Z_n]$ and variance $\mathrm{Var}[Z_n]$ of the random variable of $Z_n = \sum_{k=1}^{n} X_{2k} X_{2k+1}$?

b) (10 points) In which sense does $Z_n^* = (Z_n - \mathrm{E}[Z_n])/\sigma(Z_n)$ converges to a random variable Z? What is the distribution of Z?

6) (20 points) Topic: **Characteristic functions**

a) (10 points) Determine the characteristic function of the discrete random variable $X+Y+Z$, where X, Y, Z are independent, X is λ-Poisson distributed, Y is μ-geometric distributed and Z is (m, σ^2)-normal distributed.

b) (10 points) Compute the expectation (5 points) and variance (5 points) of $X + Y + Z$.