Math 21a Exams, Harvard 2005-2017

Exams are in the first part, solutions in the second. **Acknowledgments:** while as a course head (CH) I prepared the exams, there was always a team involved in writing and proofreading. The entire team should are coauthors. The current version shows the last team I taught with on the front page. Thanks also to the countless students reporting typos over the years. Remaining errors and deficiencies are mine.

- Fall 2017: Oliver Knill (CH), Jameel Al-Aidroos, Dennis Tseng, Yu-Wei Fan, Koji Shimizu, Chenglong Yu, Stepan Paul, Matt Demers Jun-Hou Fung, Peter Smillie, Aukosh Jagannath, Sebastian Vasey
- Fall 2016: Oliver Knill (CH), Koji Shimizu, Can Kozcaz, Yifei Zhao, Bena Tshishiku, Jun-Hou Fung, Chenglong Yu, Jameel Al-Aidroos, Ziliang Che, George Melvin, Jake Marcinek, George Melvin.
- Fall 2015: Oliver Knill (CH), Jun-Hou Fung, Koji Shimizu, Matt Demers, Dusty Grundmeier, Erick Knight, Kate Penner, Yusheng Luo, YongSuk Moon, Will Boney, Peter Smillie, Chenglong Yu, Lukas Brantner, Yu-Wen Hsu.
- Fall 2014: Oliver Knill (CH), Chao Li, Gijs Heuts, Yu-Wen Hsu, Yong-Suk Moon, Rosalie Belanger-Rioux, Gijs Heuts, Siu-Cheong Lau, Erick Knight, Kate Penner, Peter Smillie, Jeff Kuan, Yi Xie, Jeff Kuan, Jameel Al-Aidroos.
- Fall 2013: Oliver Knill (CH), Chao Li, Gijs Heuts, Adrian Zahariuc, Yihang Zhu, Peter Garfield, Matthew Woolf, Charmaine Sia, Steve Wang, Mike Hopkins, Francesco Cavazzani, Kate Penner.
- Fall 2012: Oliver Knill (CH), Hansheng Diao, Joe Rabinoff, John Hall, Meredith Hegg, Charmaine Sia, Bence Beky, Gijs Heuts, Francesco Cavazzani, Andrew Cotton-Clay.
- Fall 2011: Oliver Knill (CH), Chao Li, Thanos Papaïoannou, Emily Riehl, Jameel Al-Aidroos, Tatyana Kobylyatskaya, Yu-Jong Tzeng, Junecue Suh, Pei-Yu Tsai, Paul Bourgade.
- Fall 2009: Oliver Knill (CH), Jameel Al-Aidroos, Andrew Cotton-Clay, HT Yau, Ana Caraiani, Chris Phillips, Ethan Street, Toby Gee, Xinwen Zhu, Jack Huizenga, Fred van der Wyck, Ming-Tao Chuan.
- Fall 2008: Oliver Knill (CH), Chung-Jun John, Ivana Bozic, Peter Garfield, Stefan Hornet, Aleksander Subotic, Ana Caraiani, Toby Gee, Valentino Tosatti, Ming-Tao Chuan, Valentino Tosatti.
- Fall 2007: Oliver Knill (CH), Chen-Yu Chi, Corina Tarnita, Veronique Godin, Stefan Hornet, Jay Pottharst, Chen-Yu Chi, Ming-Tao Chuan, Thomas Barnet-Lamb, Rehana Patel, Thomas Lam.
- Fall 2006: Oliver Knill (CH), Chen-Yu Chi, Janet Chen, Sug Woo, Jay Pottharst, Kai-Wen Lan, Valentino Tosatti, Gerald Sacks, Ilia Zharkov, David Harvey.
- Spring 2006: Multivariable Calculus Oliver Knill (CH), Samik Basu, Joachim Krieger, Matt Leingang, Veronique Godin, Thomas Lam.
- Fall 2005: Oliver Knill (CH), Ivan Petrakiev, Thomas Lam, Michael Schein, Teru Yoshida, Andrew Dittmer, Chen-Yu Chi, Kathy Paur, Valentino Tosatti, Kai-Wen Lan, Jeng-Daw Yu.

Oliver Knill, Department of Mathematics, Harvard University, Cambridge, MA, 02138

9/27/2017 FIRST HOURLY **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
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TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
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- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 and 9, we need to see **details** of your computation.
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The velocity vector of $\vec{r}(t) = \langle t, t, t\rangle$ at time $t = 2$ is twice the velocity vector at time $t = 1$.

2) T F The curvature of the curve $\vec{r}(t) = \langle 2t, 2t, 2t\rangle$ is always $1/2$.

3) T F If $\vec{u} \times \vec{v} = \vec{0}$, then $\text{Proj}_{\vec{u}}(\vec{v}) \times \text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

4) T F If $\vec{u} \times \vec{v} = \vec{v} \times \vec{w}$, then $\vec{v} \cdot (\vec{u} \times \vec{w}) = 0$.

5) T F There is a point not at the origin with Cartesian coordinates $(x, y, z) = (a, b, c)$ and spherical coordinates $(\rho, \theta, \phi) = (a, b, c)$.

6) T F The two planes $2x + 2y - z = 4$ and $-4x - 4y + 2z = 3$ intersect in a line.

7) T F If the distance between two points P and Q is zero, then $P = Q$.

8) T F If the distance between two lines L and M is zero, then $L = M$.

9) T F The arc length of a circle with constant curvature κ is $2\pi\kappa$.

10) T F The surface $x^2 + y^2 + 4y = -z^2$ is a two-sheeted hyperboloid.

11) T F There are two vectors $\vec{v}, \vec{w}$ in $\mathbf{R}^3$ of length 1 for which the dot product is 2.

12) T F If the acceleration of a curve $\vec{r}(t)$ is zero at all times and the velocity is non-zero at time $t = 0$, then the curve is a line.

13) T F The lines $\vec{r}(t) = \langle 3t, 4t, 5t\rangle$ and $\vec{s}(t) = \langle -4t, 3t, 0\rangle$ intersect perpendicularly.

14) T F The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axis.

15) T F Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|\vec{u} \cdot \vec{v}||\vec{w}| = |\vec{u}||\vec{v} \cdot \vec{w}|$.

16) T F The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a cylinder.

17) T F The arc length of the curve $\langle \sin(t), \cos(t), t\rangle$ from $t = 0$ to $t = 2\pi$ is larger than 2π.

18) T F The surface parametrized by $\vec{r}(u, v) = \langle v \sin(u), v \cos(u), 0\rangle$ with $0 \leq u < 2\pi$, $v \geq 0$ is a plane.

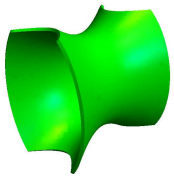
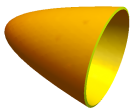
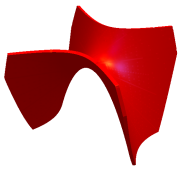
19) T F It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

20) T F For any two points P, Q and vectors $\vec{v}, \vec{w}$, the mid point $M = (P + Q)/2$ has the same distance to the two lines $\vec{r}_1(t) = P + t\vec{v}$ and $\vec{r}_2(t) = Q + t\vec{w}$.

[ ] Total

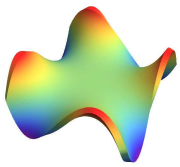
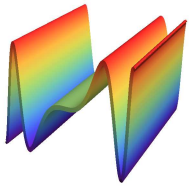
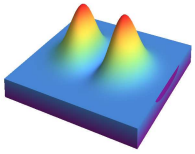
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the surfaces with their equations $g(x, y, z) = 1$. Enter O, if there is no match.

I  II  III 

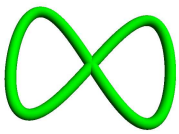
Function $g(x, y, z) =$	Enter O,I,II or III
$2x - y^2 - z^2$	
$2x^2 - y^2 + z^2$	
$2x - y$	
$2x^2 - y^2 - z$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

I  II  III 

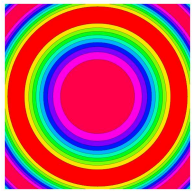
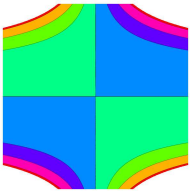
Function $f(x, y) =$	Enter O,I,II or III
$xy(x^2 - y^2)$	
$\sin(x^3)$	
$\sin(y^4)$	
$x^2 \exp(-x^2 - y^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

I  II  III 

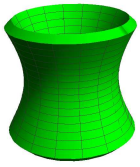
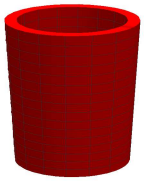
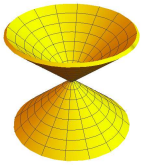
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, \sin(4t), \cos(4t)\rangle$	
$\langle \cos(t), \cos(t), \sin(2t)\rangle$	
$\langle 3t, 1 - t, 5t\rangle$	
$\langle t\sin(t), t\cos(t), t\rangle$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Function $g(x, y) =$	Enter O, I,II or III
$\sin(x^2 + y^2)$	
$\sin(x) - y$	
$\lvert x\rvert + y$	
xy^2	

e) (2 points) Match the quadrics. Enter O if there is no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2 + y^2 - z^2 = 1$	
$x^2 + y^2 + z^2 = 1$	
$x^2 + y^2 = 1$	
$x^2 + y^2 = z^2$	

Problem 3) (10 points)	(Only answers are needed)

a) (3 points) Write the equations of a surface in Cartesian, Cylindrical and Spherical coordinates. The first row gives an example:

Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
$z = 1$	$z = 1$	$\rho\cos(\phi) = 1$
		$\rho\sin(\phi) = 1$
	$r\cos(\theta) = 1$	
$x^2 + y^2 + z^2 = 1$		

b) (3 points) Assume $\vec{u}, \vec{v}$ are unit vectors which are perpendicular. Check one box in each row:

The value	is larger than 0	is smaller than 0	is equal to 0
$\vec{u} \cdot \vec{v}$			
$\lvert\vec{u} \times \vec{v}\rvert$			
$\vec{u} \cdot (\vec{v} \times \vec{u})$			

c) (2 points) Complete the following table which uses the vectors $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle, \vec{k} = \langle 0, 0, 1\rangle, -\vec{i}, -\vec{j}, -\vec{k}$ or $\vec{0}$ in one of the first 3 boxes. Enter an scalar in each of the 3 boxes at the bottom.

$\vec{i} \times \vec{i} =$		$\vec{i} \times \vec{j} =$		$\vec{k} \times \vec{j} =$	
$\vec{j} \cdot \vec{i} =$		$\vec{j} \cdot \vec{j} =$		$\vec{j} \cdot \vec{k} =$	

d) (2 points) Complete the following table about the **TNB frame**. In each case, enter either $\vec{T}, \vec{N}, \vec{B}$ or $\vec{0}$ in each of the 6 boxes. Every correct row gives a point:

$\text{Proj}_{\vec{T}}(\vec{T}) =$		$\text{Proj}_{\vec{T}}(\vec{N}) =$		$\text{Proj}_{\vec{B}}(\vec{T}) =$	
$\text{Proj}_{\vec{N}}(\vec{T}) =$		$\text{Proj}_{\vec{N}}(\vec{N}) =$		$\text{Proj}_{\vec{N}}(\vec{B}) =$	

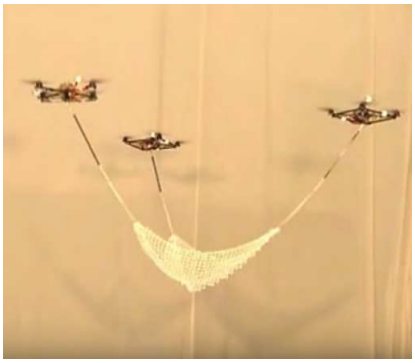
Problem 4) (10 points)

In a TED talk of 2013, **Raffaello D'Andrea** and a team demonstrated **"quadrotor athletes"**.

Assume the three robots are at positions

$$A = (2, 3, 1), B = (2, 3, 3) \text{ and } C = (5, 4, 2).$$

What is the area of the triangle they span?



A figure from an article by J. Sidman and A. St John in the Notices of the AMS, October 2017.

Problem 5) (10 points)

The six-legged **Gough-Stewart platform** has applications in flight simulators, robotics, crane technology, underwater research, telescopes and orthopedic surgery. The bottom positions of the legs are

$$A_1 = (5, -2, 0), A_2 = (5, 1, 0), A_3 = (-3, 5, 0),$$

$$A_4 = (-5, 3, 0), A_5 = (-5, -3, 0), A_6 = (-3, -5, 0).$$

The top positions of the legs are

$$B_1 = (-5, 2, 6), B_2 = (-5, -1, 6), B_3 = (3, -5, 6),$$

$$B_4 = (5, -3, 6), B_5 = (5, 3, 6), B_6 = (3, 5, 6).$$

a) (5 points) What is the distance between B_1 and the plane containing $A_1, \ldots, A_6$?
b) (5 points) What is the distance between B_1 and the line through A_1 and A_2?



Picture: Jessica Sidman and Audry St. John in the Notices of the AMS

Problem 6) (10 points)

Even though Saturn is much larger than the Earth, its gravitational force is only 7 percent larger than here on Earth. When **Cassini** plunged into Saturn, it felt an acceleration

$$\vec{r}\,''(t) = \langle \pi \sin(\pi t), 0, -10 - 2t \rangle \ .$$

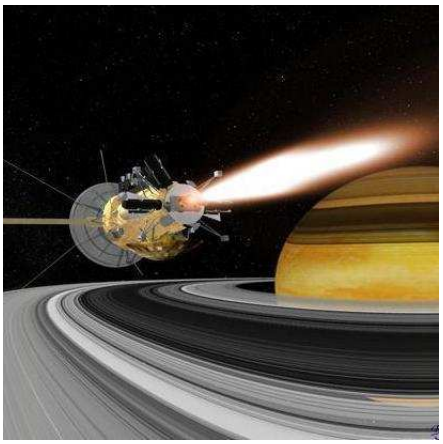
We know the initial velocity

$$\vec{r}\,'(0) = \langle 2, 5, 1 \rangle$$

and initial position

$$\vec{r}(0) = \langle 0, 0, 1000 \rangle \ .$$

Where is the spacecraft at $t = 1$?



Picture: NASA

Problem 7) (10 points)

Let's look at the two planes

$$x + y + z = 1$$

and

$$x + y - z = 1 \ .$$

a) (4 points) Find the plane $ax+by+cz = d$ through $P = (1, 0, 0)$ which is perpendicular to both.

b) (2 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in both planes.

c) (4 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in the first plane but not the second.

Problem 8) (10 points)

The world was supposed to end on September 23 due to the mysterious planetary system HD 7924. But here you sit and have to take the first hourly. A moon on HD 7924 moves on an epicycle

$$\vec{r}(t) = \langle 10\cos(t), 10\sin(t), 0\rangle + \langle 2\cos(5t), 2\sin(5t), 0\rangle \ .$$

a) (2 points) Find the velocity $\vec{r}\,'(0)$ at $t = 0$ and the speed $|\vec{r}\,'(0)|$ at $t = 0$.

b) (2 points) Find the acceleration $\vec{r}\,''(0)$ at $t = 0$.

c) (3 points) Find $\kappa(0) = |\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$.

d) (3 points) Inhabitants from HD 7924 beam you the hint $|\vec{r}\,'(t)|^2 = 400\cos^2(2t)$. Use this to find the arc length from $t = 0$ to $t = 2\pi$.

Will the world end on September 23?

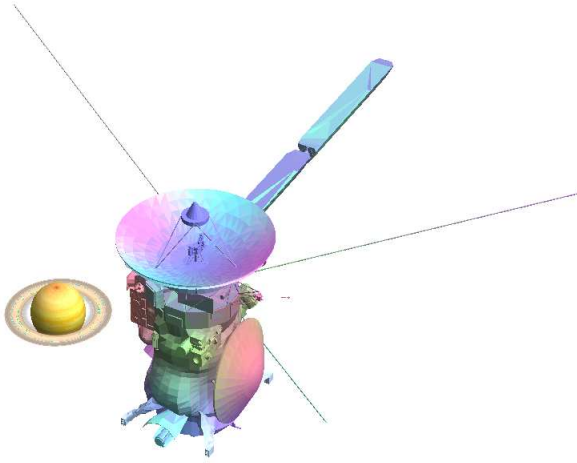
30 Updated on September 20, 2017 at 12:37 PM, Posted on September 20, 2017 at 11:03 AM



Viral videos and theories claim an apocalypse will begin on Sept. 23. This photo is an artist's impression of a view from the HD 7924 planetary system looking back toward our sun. (Karen Teramura; BJ Fulton, University of Hawaii, Institute for Astronomy.)

Fun fact: the guy who came up with the date September 23 has revised his estimate to October 15. There is still hope to avoid the second midterm ...

Problem 9) (10 points)



Two weeks ago, in a grand finale, the Cassini space craft plunged into the atmosphere of **Saturn**. To build a model of the situation we have to parametrize various parts on the probe which were used both for measurement and communication.
You don't have to specify the parameter bounds but give the parametrizations for each of the 5 objects:

Picture: by Mathematica using a printable

3D STL models provided by NASA

a) (2 points) Saturn is a sphere $(x-1)^2 + y^2 + z^2 = 16$.

$\vec{r}(\theta, \phi) = \langle \boxed{}, \boxed{}, \boxed{} \rangle.$

b) (2 points) The rings are given by $z = 0, r^2 = x^2 + y^2 \leq 25$.

$\vec{r}(r, \theta) = \langle \boxed{}, \boxed{}, \boxed{} \rangle.$

c) (2 points) The satellite dish $(x - 50)^2 + (y - 70)^2 = z$ beams pictures back to earth.

$\vec{r}(x, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle.$

d) (2 points) There is also a satellite antenna of the form $(x - 50)^2 + z^2 = 1/100$.

$\vec{r}(\theta, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle.$

e) (2 points) There is also a device of the form $(x - 50)^2 + z^2 - (y - 70)^2 = 1$.

$\vec{r}(\theta, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle.$

9/27/2017 FIRST HOURLY PRACTICE 1 **Math 21a, Fall 2017**

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] If $|\vec{r}\,'(t)| = 1$ for all t, then the curvature satisfies $\kappa(t) = |\vec{r}\,''(t)|$.

2) [T] [F] Let $\vec{r}(t)$ be a parametric curve. Suppose that at the point $\vec{r}(t)$ the unit tangent vector is $\langle 0, 1, 0\rangle$ and the binormal vector is $\langle 0, 0, 1\rangle$. Then the unit normal vector at the point is $\langle 1, 0, 0\rangle$.

3) [T] [F] If $|\vec{v} + \vec{w}|^2 = 2\vec{v} \cdot \vec{w}$, then $\vec{v} = 2\vec{w}$.

4) [T] [F] The distance between two points P and Q is smaller than or equal to $|\vec{OP}| + |\vec{OQ}|$, where $O = (0, 0, 0)$ is the origin.

5) [T] [F] The line $\vec{r}(t) = \langle 5 + 2t, 2 + t, 3 + t\rangle$ is located on the plane $x - y - z = 0$.

6) [T] [F] The arc length of a circle with constant curvature 20 is $\pi/10$.

7) [T] [F] The surface $x^2 - y^2 + 4y = z^2 + 2z$ is an elliptic paraboloid.

8) [T] [F] For any two vectors, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| + |\vec{w}|$.

9) [T] [F] The acceleration of $\vec{r}(t) = \langle t, t, t\rangle$ is $\langle 0, 0, 0\rangle$ everywhere.

10) [T] [F] If $\vec{v} = \vec{PQ} = \langle 2, 1, 1\rangle$ then $|\vec{v}|$ is larger than the distance between P and Q.

11) [T] [F] If $\vec{v} \cdot \vec{w} > 0$, then the angle between $\vec{v}$ and $\vec{w}$ is larger than $90°$.

12) [T] [F] If $\langle 5, 6, 4\rangle \times \vec{x} = \vec{x}$ then $\vec{x} = \langle 0, 0, 0\rangle$.

13) [T] [F] The curvature of $\vec{r}(t) = \langle t - 1, 1 - t, t\rangle$ is 0 at the point $t = 1$.

14) [T] [F] The line $\vec{r}(t) = t\langle 7, 7, 1\rangle$ hits the plane $-x - y = 7z$ in a right angle.

15) [T] [F] The surface given in spherical coordinates as $\sin(\phi) = 1/\rho$ is a cylinder.

16) [T] [F] Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \cdot \vec{v})(\vec{w} \cdot \vec{u})| \leq |\vec{u}|^2|\vec{v}||\vec{w}|$.

17) [T] [F] The surface given in spherical coordinates as $\rho^2 = 1$ is a sphere.

18) [T] [F] The arc length of the curve $\langle 5\sin(t), 1, 5\cos(t)\rangle$ from $t = 0$ to $t = 2\pi$ is equal to 10π.

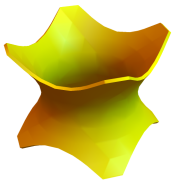
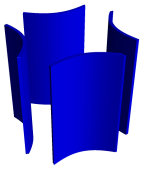
19) [T] [F] The surface parametrized by $\vec{r}(u, v) = \langle u^5 - v^5, u^5 + v^5, u^5\rangle$ is a plane.

20) [T] [F] If the non-zero cross product of a vector $\vec{v}$ with a vector $\vec{w}$ is parallel to $\vec{v}$, then the dot product between $\vec{v}$ and $\vec{w}$ is zero.

[ ] Total

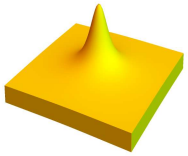
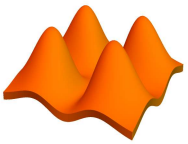
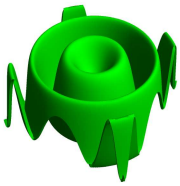
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.

I  II  III 

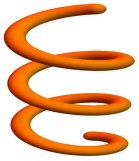
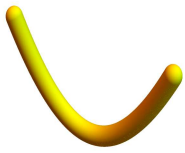
Function $g(x, y, z) = 1$	Enter O,I,II or III
$x^2 + y^2 - z^2$	
x^2y^2	
$x - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

I  II  III 

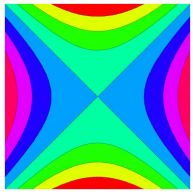
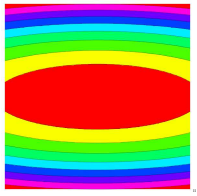
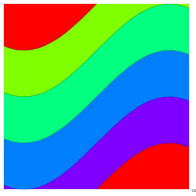
Function $f(x, y) =$	Enter O,I,II or III
$1/(1 + x^2 + y^2)$	
$x^2y^2e^{-x^2-y^2}$	
$\sin(x^2 + y^2)$	
$\cos(y)$	

c) (2 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

I  II  III 

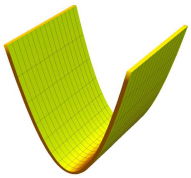
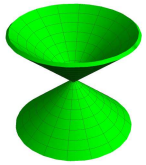
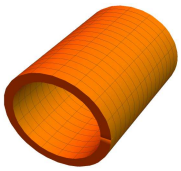
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, t\sin(3t), t\cos(3t)\rangle$	
$\langle \cos(t), \sin(t), \cos(t)\rangle$	
$\langle \sin(t), \cos(t), t\rangle$	
$\langle t, t, t^2\rangle$	

d) (2 points) Match functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Equation	Enter O, I,II or III
$\sin(x) - y$	
$x^2 - y^2$	
$10x^2 + y^2$	
$x^2 + 10y^2$	

e) (2 points) Match the quadrics. Enter O if no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2 = z$	
$x^2 + y^2 = 1$	
$xy = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

Problem 3) (10 points)	(Only answers are needed)

a) (4 points) Mark what applies for any two vectors $\vec{v}$ and $\vec{w}$ in space.

Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0 \rangle$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				
$(\vec{v} \times \vec{w}) \cdot \vec{v}$				
$(\text{proj}_{\vec{v}}\vec{w}) \times \vec{w}$				
$(\text{proj}_{\vec{w}}\vec{v}) \cdot \vec{w}$				

b) (3 points) **Conic sections** (parabolas, ellipses or hyperbolas) can be seen as intersections of a two dimensional cone

$$x^2 + y^2 = z^2$$

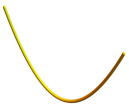
with a 2D plane. Identify the quadrics in the following three cases:

Intersect with plane	Enter A-D
$z = 1$	
$z = \sqrt{2}x$	
$x = y + 1$	

A 

B 

C 

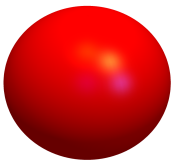
D 

c) (3 points) Three dimensional cone is given by the equation

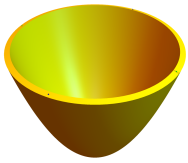
$$x^2 + y^2 + z^2 = w^2$$

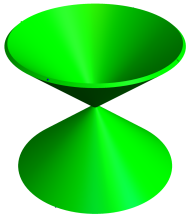
in four dimensional space. If we intersect it with a three dimensional space, we get quadrics. We want you to identify a few quadrics. In the pictures the quadrics might be turned or scaled. You get a point for every right answer, meaning that you can miss one and still have full credit.

Intersect with the 3D plane	Enter A-D or O
$w = 0$	
$w = z + 1$	
$y = z$	
$w = 1$	

A 

B 

C 

D 

Problem 4) (10 points)

We are going into the **furniture design business**. Our first task is to construct a chair. Find the distance between the lines spanned by the parallel arm rests AB and CD, where

$$A = (1, 0, 2), B = (4, 1, 2)$$

and

$$C = (3, 3, 3), D = (6, 4, 3) \ .$$

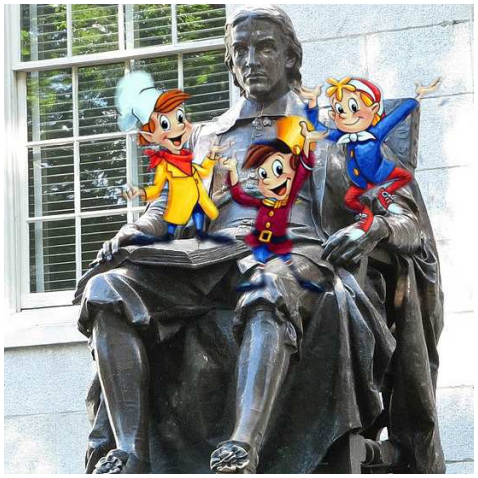


Problem 5) (10 points)

a) (6 points) The first derivative of parametrized curve is called **"velocity"**. You have also learned the terms **"acceleration"** and maybe **"jerk"** for the second and third derivative. Less well known are **"snap"**, **"crackle"**, **"pop"** for the fourth, fifth and sixth derivatives. Since we could not yet find the seventh derivative named, let's call it the **"Harvard"**. Compute the **"Harvard"** of the curve

$$\vec{r}(t) = \langle \cos(2t) + t, \sin(2t), t^2 \rangle$$

at time $t = 0$.



b) (4 points) The parametrization $\vec{v}(t) = \vec{r}\,'(t)$ defines a new curve. It is located on a surface. Which of the following surface is it?

	Check one
cylinder	A
cone	B
plane	C
ellipsoid	D
paraboloid	E

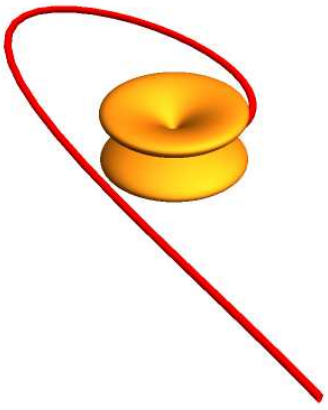
Problem 6) (10 points)

A kid plays with a **Yo-Yo**. It is accelerated periodically with $\vec{r}\,''(t) = \langle \sin(t), 0, \cos(t) - 10 \rangle$. Find the position of the Yo-Yo at time $t = 2\pi$ if the initial position is

$$\vec{r}(0) = \langle 5, 5, 0 \rangle$$

and the initial velocity is

$$\vec{r}\,'(0) = \langle 1, 1, 1 \rangle \ .$$



Problem 7) (10 points)

a) (5 points) Compute the arc length of the curve

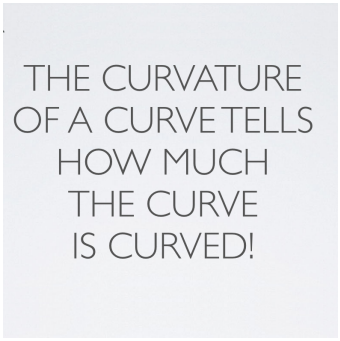
$$\vec{r}(t) = \langle t, t^2, 2t^3/3 \rangle$$

if $0 \leq t \leq 4$.

b) (5 points) What is the curvature

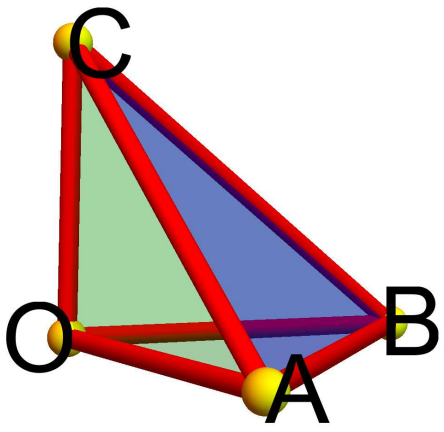
$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

of this curve at $t = 0$?



Problem 8) (10 points)

The triangle ABC is obtained by slicing a corner ABC off from a cube. One obtains a so called **trirectangular tetrahedron**.



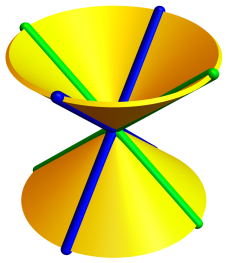
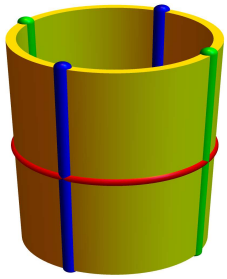
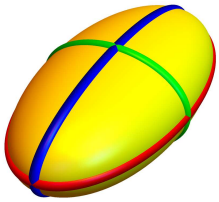
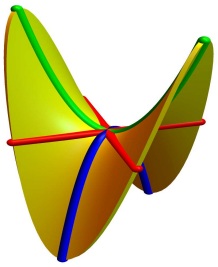
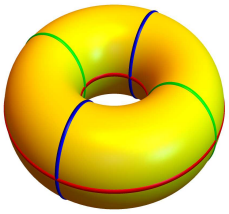
a) (5 points) Find the square of the area of the triangle ABC, where $A = (3, 0, 0), B = (0, 6, 0), C = (0, 0, 8)$.

b) (5 points) Compute also the sum of the squares of the areas of the triangles OAB, OBC and OCA, where $O = (0, 0, 0)$ is the origin. The sum should get the same value you got in a).

P.S. the same computation can be repeated for arbitrary points $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$. It proves a not so well known theorem telling that the sum of the squares of the side wall areas is the square of the face area. It is a 3 dimensional version of Pythagoras and also goes by the name **de Gua theorem** or **Faulhaber theorem**.

Problem 9) (10 points)

The 3D printing venture "Math-Candy" (math-candy.com) asks you to do some product development. In each of the 5 following parametrizations, two entries are still missing, each entry being worth one candy (1 point).

a)		The surface $\boxed{x^2+y^2=z^2}$ is parametrized by $\vec{r}(\theta,z)=\langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z\rangle$
b)		The surface $\boxed{x^2+y^2=1}$ is parametrized by $\vec{r}(\theta,z)=\langle \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, z\rangle$
c)		The surface $\boxed{2(x-1)^2+(y-5)^2+4z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=\langle \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, \cos(\phi)/2\rangle$
d)		The surface $\boxed{x^2-y^2=z}$ is parametrized by $\vec{r}(x,y)=\langle \ldots\ldots\ldots, \ldots\ldots\ldots, x^2-y^2\rangle$
e)		The surface $\boxed{(\sqrt{x^2+y^2}-2)^2+z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=\langle (2+\cos(\phi))\cos(\theta), \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\rangle$

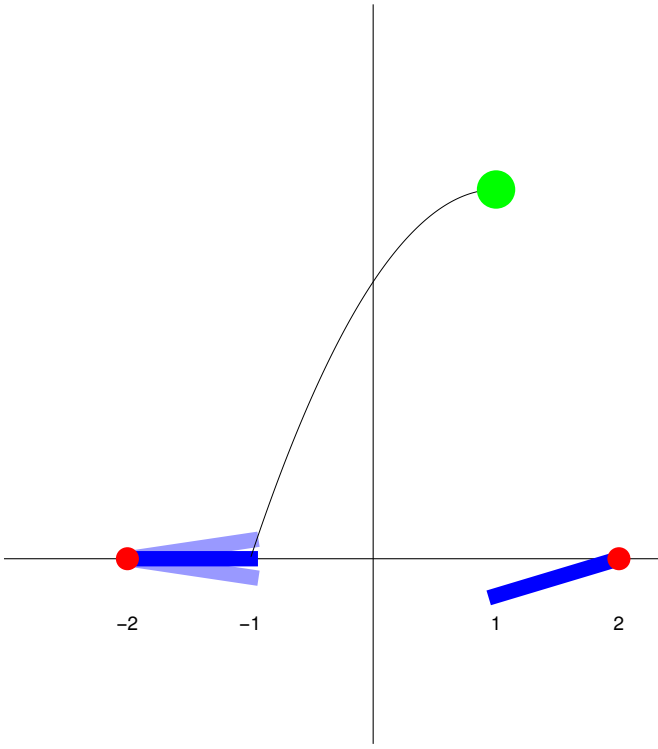
Problem 10) (10 points)

A **pinball machine** is tilted in such a way that a ball in the xy-plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}\,'(0) = \langle 1/2, 5 \rangle$.

a) (4 points) Find the velocity $\vec{r}\,'(t)$.

b) (4 points) What trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ does the ball follow?

c) (2 points) As the ball hits the line $y = 0$, is it reachable by the player? In other words, does it hit $y = 0$ within the interval $x \in [1, 2]$?



9/27/2017 FIRST HOURLY PRACTICE 2 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) T F The vector $\langle 2, 3, 6\rangle$ has a length which is an integer.

2) T F The surface $x^2 + y^2 + z^2 - 2x = 3$ is a sphere.

3) T F If $\vec{v} \cdot \vec{w}$ is negative, then the angle between $\vec{v}$ and $\vec{w}$ is acute (= smaller than $\pi/2$).

4) T F The level curves $f(x, y) = 1$ and $f(x, y) = 0$ do not intersect for $f(x, y) = (xy + \cos(x))^6$.

5) T F For any nonzero $\vec{a}$, the equation $\vec{a} \times \vec{x} = \vec{b}$ always has a solution $\vec{x}$.

6) T F For two non-parallel $\vec{a}, \vec{b}$, the equation $(\langle x, y, z\rangle \times \vec{a}) \cdot \vec{b} = 1$ defines a plane.

7) T F The curvature of $\vec{r}(t) = \langle t^3, 1 - t^3, t^3\rangle$ is 0 if $t = 1$.

8) T F If $\vec{N}$ is the normal vector and $\vec{T}$ the unit tangent vector to a curve $\vec{r}(t)$ then the vector projection of $\vec{N}(t)$ onto $\vec{T}(t)$ is zero.

9) T F There exist non-parallel vectors $\vec{v}, \vec{w}$ such that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

10) T F The point given in spherical coordinates as $\rho = 3, \phi = \pi/2, \theta = \pi$ is on the x-axes.

11) T F The parametrized curve $\vec{r}(t) = \langle 5\cos(3t), 3\sin(3t), 0\rangle$ is an ellipse.

12) T F If the vector projection of $\vec{v}$ onto $\vec{w}$ is $\vec{w}$ then $\vec{v} = \vec{w}$.

13) T F Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \times \vec{v}) \times \vec{w}| \leq |\vec{u}||\vec{v}||\vec{w}|$.

14) T F The surface $y^2 + z = x^2$ is a hyperbolic paraboloid.

15) T F The curvature of a curve $\vec{r}(t)$ at time $t = 0$ is the same as the curvature of $\vec{r}(\sin(t))$ at time $t = 0$.

16) T F The arc length of the curve $\langle \sin(t), 0, \cos(t)\rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

17) T F The curve $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t) + \sin(t)\rangle$ is on the intersection of $x^2 + y^2 = 1$ and $x + y - z = 0$.

18) T F Using $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle$, the identity $(\vec{i} \times \vec{j}) \times \vec{j} = \vec{i} \times (\vec{j} \times \vec{j})$ holds.

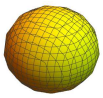
19) T F Using the same notation, the identity $(\vec{i} \cdot \vec{j})\vec{j} = \vec{i}(\vec{j} \cdot \vec{j})$ holds.

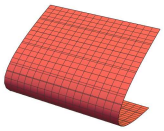
20) T F $\langle \cos t, \sin t, t\rangle$, $0 \leq t \leq 2$ and $\langle \cos(t^3), \sin(t^3), t^3\rangle$, $0 \leq t \leq 2$ have the same arc length.

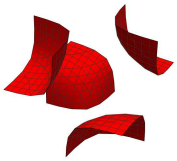
Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contours $g(x, y, z) = 1$. Enter O, if there is no match.

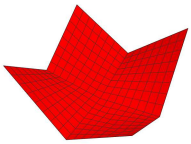

I

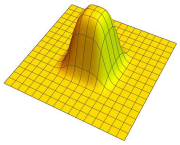

II

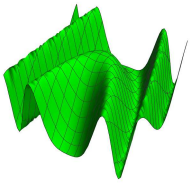

III

Function $g(x, y, z) = 1$	Enter O,I,II or III
xyz	
$x^2 + y^2 + z^2$	
$z^2 - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

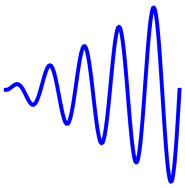

I

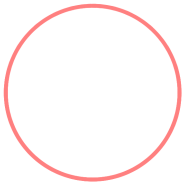

II

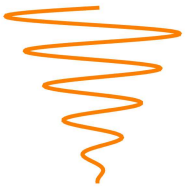

III

Function $f(x, y) =$	Enter O,I,II or III
$\cos(x^2 + y)$	
$\|x + y\| + \|xy\|$	
$\exp(-x^4 - y^4)$	
x^3	

c) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

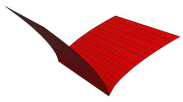

I

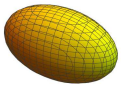

II

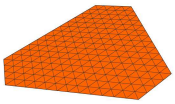

III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = \langle t, t\sin(5t)\rangle$	
$\vec{r}(t) = \langle t\sin(5t), t\rangle$	
$\vec{r}(t) = \langle \sin(5t), \cos(5t)\rangle$	
$\vec{r}(t) = \langle \cos(5t), \cos(5t)\rangle$	

d) (2 points) Match functions g with level surface $g(x, y, z) = 1$. Enter O, if there is no match.

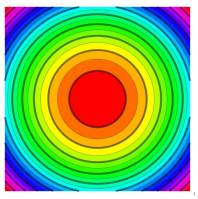

I

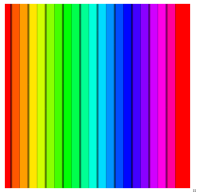

II

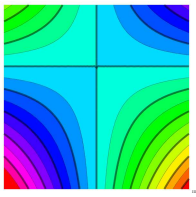

III

Function $g(x, y, z) = 1$	Enter O, I,II or III
$x^2 - y^2 + z^2 = 1$	
$x - y - z = 1$	
$y^3 = z^2$	
$x^2/4 + y^2 + z^2/2 = 1$	

e) (2 points) Match the contour maps to a function $f(x, y)$. Enter O if no match.


I

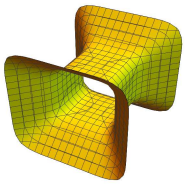

II


III

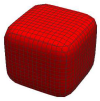
$f(x, y) =$	Enter O,I,II or III
$x^2 - y^4$	
$xy - x$	
x	
y	
$x^2 + y^2$	

Problem 3) (10 points) (Only answers are needed)

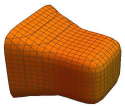
a) (4 points) The following contour surfaces were deformed by setting $X = x^3, Y = y^3, Z = z^3$. Can you label the original quadrics from which it was deformed?



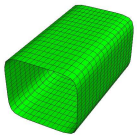
I



II



III



IV

Surface	I-IV	name A-D
$X^2 + Y^2 + Z^2 = 1$		
$X + Y^2 + Z^2 = 1$		
$X^2 - Y^2 + Z^2 = 1$		
$X^2 + Z^2 = 1$		

Fill in:
A) Ellipsoid'esque
B) Paraboloid'esque
C) Hyperboloid'esque
D) Cylinder'esque

b) Help Larry the physicist in the movie "A serious man", to compute some quantities:

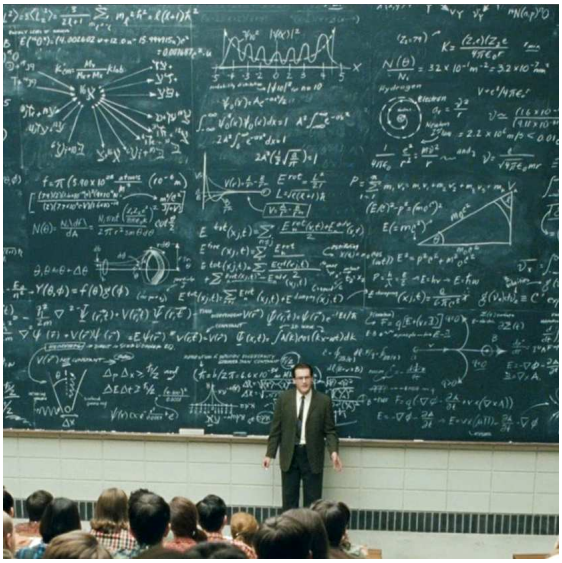
$\vec{v} = \langle 1, 2, 3\rangle$ represent the velocity
$\vec{w} = \langle 0, 1, 1\rangle$ represent angular velocity
$\vec{B} = \langle 1, 0, 1\rangle$ represent magnetic field
$\vec{r} = \langle 0, 0, 1\rangle$ represent position. Compute:

(i) (2 points) Coriolis force $\vec{v} \times \vec{w}$.

(ii) (2 points) Lorentz force $\vec{v} \times \vec{B}$.

(iii) (1 point) Kinetic energy $(\vec{v} \cdot \vec{v})/2$.

(iv) (1 point) Magnetic energy $\vec{B} \cdot \vec{B}/2$.



Larry: "I mean - even I don't understand the dead cat. The math is how it really works."

(i)

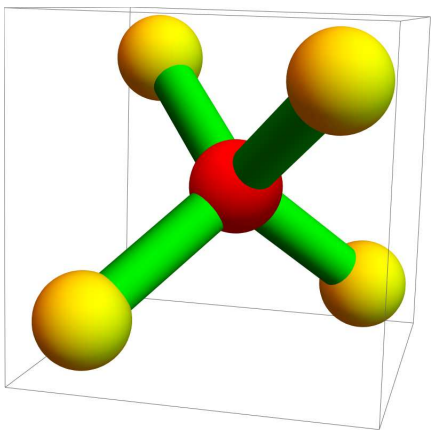
ii)

(iii)

iv)

Problem 4) (10 points)

Methane CH_4 is the number two greenhouse gas emitted by human activity in the US. The four hydrogen atoms of **methane** are located at the vertices $P = (2,2,2), Q = (2,0,0), R = (0,2,0), S = (0,0,2)$ and form a regular tetrahedron, while C is the central carbon atom located at $(1,1,1)$.



a) (2 points) Find one bond distance $|CP|$ and the distance $|PQ|$

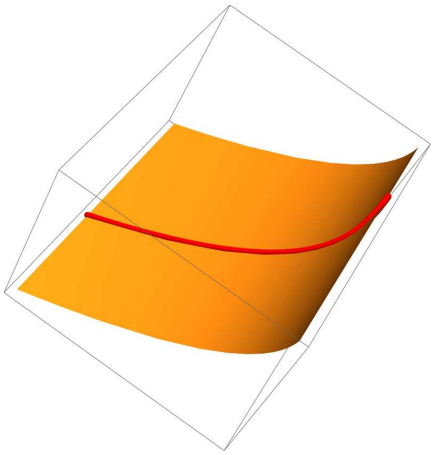
b) (4 points) Find the cosine of the bond angle between $\overrightarrow{PC}$ and $\overrightarrow{PQ}$.

c) (4 points) What is volume of the parallelepiped spanned by $\overrightarrow{PC}, \overrightarrow{QC}, \overrightarrow{RC}$?

Problem 5) (10 points)

Consider the curve

$$\vec{r}(t) = \langle 2e^t, t, e^{2t} \rangle \ .$$



a) (3 points) Compute the speed $|\vec{r}\,'(0)|$.

b) (5 points) Find the arc length from $t = -2$ to $t = 1$.

c) (2 points) There exists a constant a such that the curve lies on the cylindrical paraboloid $x^2 = az$. Which a does apply?

Problem 6) (10 points)

The highest **bungee jump** ever recorded was done from the 233 meter high Macau Tower. Assume the rope pulls back with a force $2t$ so that the acceleration is

$$\vec{r}\,''(t) = \langle 0, 0, 2t - 10 \rangle \ .$$



Assume the initial velocity is $\langle 1, 0, 0 \rangle$ and that the daredevil jumps from $\vec{r}(0) = \langle 0, 0, 233 \rangle$:

a) (5 points) Find $\vec{r}\,'(t)$ and determine t_0 for which the third component $z'(t_0) = 0$. This is the time of the lowest point.

b) (5 points) Find $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $\vec{r}(t_0)$. Did the jumper hit the ground $z = 0$?

Problem 7) (10 points)

The **logarithmic spiral** is parametrized by $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), 0 \rangle$.



a) (5 points) Find the angle between $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at time $t = 0$.

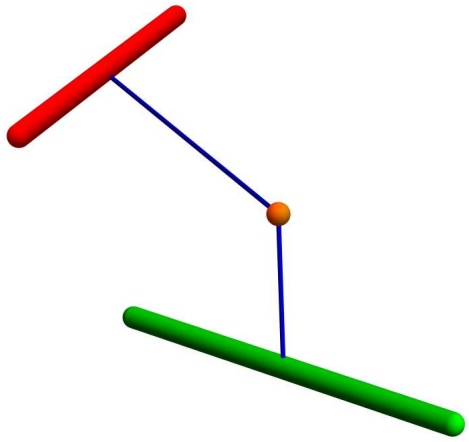
b) (5 points) Compute the curvature at $t = 0$.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \ .$$

One miracle about the spiral is that arc length from 0 to t multiplied with curvature at t is constant. Jacob Bernoulli called it the curve the "Spira mirabilis" which means "miraculous spiral".

Problem 8) (10 points)

Given a line $\vec{r}(t) = \langle 1 + t, t, t \rangle$ and a line $\vec{s}(t) = \langle 1 - t, 1 + t, 1 - t \rangle$.



a) (6 points) Find the sum of the distances of the point $(0, 0, 0)$ to the two lines.

b) (4 points) Find the distance between the two lines.

Problem 9) (10 points)

Parametrize the following surfaces in space. As usual r, θ, z are cylindrical and ρ, θ, ϕ are spherical coordinate variables. You do not need to give bounds on the parameters.

a) (2 points) Parametrize $y = \cos(3x) - \sin(3z)$ as

$\vec{r}(x, z) =$

b) (2 points) Parametrize $\rho = 2 + \cos(8\theta + 5\phi)$ as

$\vec{r}(\theta, \phi) =$

c) (2 points) Parametrize $r^2 - z^2 = 1$ as

$\vec{r}(\theta, z) =$

d) (2 points) Parametrize $x = 0$ as

$\vec{r}(y, z) =$

e) Decide whether none, one, or both of the grid curves u = 1, v = 1 is a circle, if

$$\vec{r}(u, v) = \langle (3u + u\cos(v))\cos(2u), (3u + u\cos(v))\sin(2u), (3u + u\sin(v)) \rangle$$

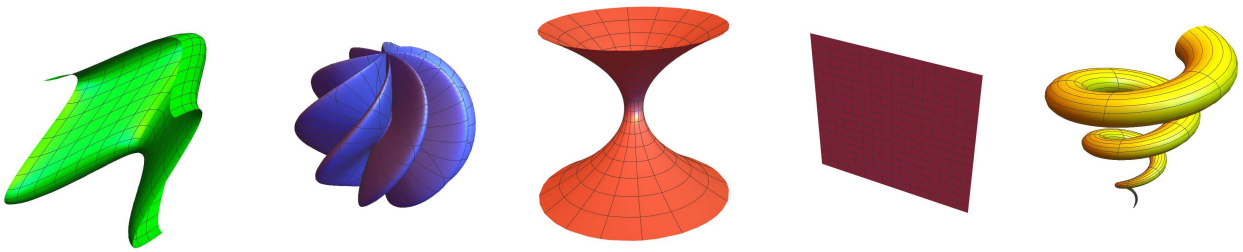
(1 point) Is the curve $u = 1$ a circle?

Yes or No

(1 point) Is the curve $v = 1$ a circle?

Yes or No

Illustrations:



a) b) c) d) e)

Problem 10) (10 points)

We enjoy the fall sun outside and sit in a local restaurant for a refreshment. In each of the ordered items, give a **surface parametrization** of the form

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \ .$$

As indicated, we can use also other variables. Your task is to fill in the three parametrization functions in each case, using the variables provided.

a) (2 points) Parametrize the **lemonade glass** $x^2 + y^2 = 1$.

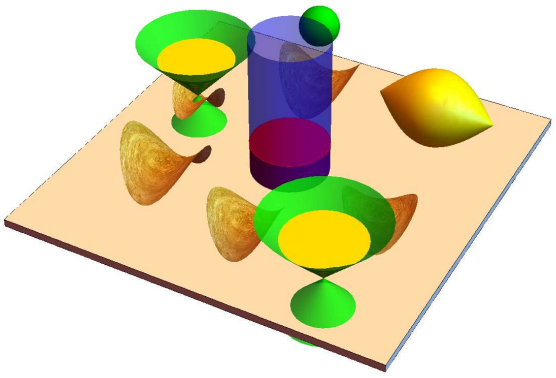
$\vec{r}(\theta, z) = \langle$ [ ] , [ ] , [ ] $\rangle$.

b) (2 points) Parametrize the **sorbet glass** $x^2+y^2 = z^2$.

$\vec{r}(\theta, z) = \langle$ [ ] , [ ] , [ ] $\rangle$.

c) (2 points) Parametrize the **lemon** surface $x^2 + y^2 = \sin(z)$.

$\vec{r}(\theta, z) = \langle$ [ ] , [ ] , [ ] $\rangle$.



d) (2 points) Parametrize one of the **chips** $z = x^2 - y^2$.

$\vec{r}(x, y) = \langle$ [ ] , [ ] , [ ] $\rangle$.

e) (2 points) Parametrize the **lime** $x^2+y^2+(z-3)^2 = 1$.

$\vec{r}(\theta, \phi) = \langle$ [ ] , [ ] , [ ] $\rangle$.

9/27/2017 FIRST HOURLY PRACTICE 3 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 or problem 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		11
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The surface $-x^2 + y^2 + z^2 = -1$ is a one-sheeted hyperboloid.

2) [T] [F] The equation $y = 3x + 2$ in space defines a plane.

3) [T] [F] Whenever $|\vec{r}\,'(t)| = 1$ then $|\vec{T}\,'(t)| = 1$.

4) [T] [F] The length of the vector projection $\text{Proj}_{\vec{v}}(\vec{w})$ is smaller than or equal to the length of $\vec{w}$.

5) [T] [F] The velocity vector of $\vec{r}(t) = \langle t, t, t\rangle$ at time $t = 2$ is the same as the velocity vector at $t = 1$.

6) [T] [F] If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$ then $\vec{v}, \vec{w}$ are parallel.

7) [T] [F] The vector $\langle -2, 1, 0\rangle$ is perpendicular to the line $\langle 1 + t, 2t, 3t\rangle$.

8) [T] [F] The point given in spherical coordinates as $\rho = 3, \phi = 0, \theta = \pi$ is the same point as the point $\rho = 3, \phi = 0, \theta = 0$.

9) [T] [F] The parametrized curve $\vec{r}(t) = \langle 0, 3\cos(t), 5\sin(t)\rangle$ is an ellipse.

10) [T] [F] The curvature of the line $\vec{r}(t) = \langle t, t, t\rangle$ is $\sqrt{3}$ everywhere.

11) [T] [F] If $|\vec{v} \times \vec{w}| = \vec{v} \cdot \vec{w}$ then either $\vec{v}$ is parallel to $\vec{w}$ or perpendicular to $\vec{w}$.

12) [T] [F] If the dot product between two unit vectors $\vec{v}, \vec{w}$ is -1, then $\vec{v} = -\vec{w}$.

13) [T] [F] Writing $\vec{k} = \langle 0, 0, 1\rangle$, we have $|(\vec{k} \times \vec{v}) \times \vec{w}| \leq |\vec{v}||\vec{w}|$ for all vectors $\vec{v}, \vec{w}$.

14) [T] [F] The curvature of a curve $\vec{r}(t)$ is given by $\kappa(t) = |\vec{T}\,'(t)|/|\vec{r}\,'(t)|$. If $|\vec{r}\,'(t)| = 1$ for all times, then $\kappa(t) = |\vec{r}\,''(t)|$.

15) [T] [F] The arc length of the curve $\langle \sin(t/2), 0, \cos(t/2)\rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

16) [T] [F] If L, K are skew lines in space, there is a unique plane which is equidistant from L, K.

17) [T] [F] The curve $\vec{r}(t) = \langle t, t^2, 1 - t\rangle$ is the intersection curve of a plane $x + z = 1$ and $y = x^2$

18) [T] [F] The lines $\vec{r}_1(t) = \langle 5 + t, 3 - t, 2 - t\rangle$ and $\vec{r}_2(t) = \langle 6 - t, 2 + t, 1 - 2t\rangle$ intersect at $(6, 2, 1)$ perpendicularly.

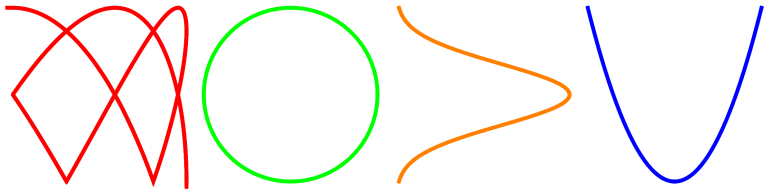
19) [T] [F] The vector $\langle 3/13, 12/13, 4/13\rangle$ is a unit vector.

20) [T] [F] $\vec{v} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

[ ] Total

Problem 2) (10 points) No justifications are needed in this problem.

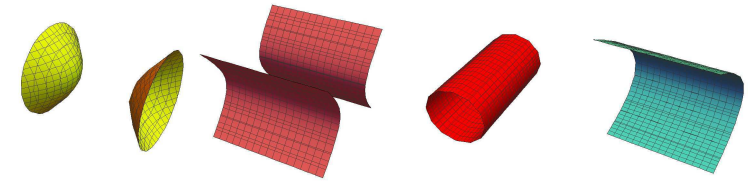
a) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match. In each of the problems a) - e), every of the entries O, I, II, III, IV appears exactly once.



I II III IV

$\vec{r}(t) =$	Enter O-IV
$\langle \exp(-t^2), t \rangle$	
$\langle \cos(t), \sin(t) \rangle$	
$\langle \lvert\cos(3t)\rvert, \lvert\sin(5t)\rvert \rangle$	
$\langle 2t, 3t \rangle$	
$\langle t, t^2 + 1 \rangle$	

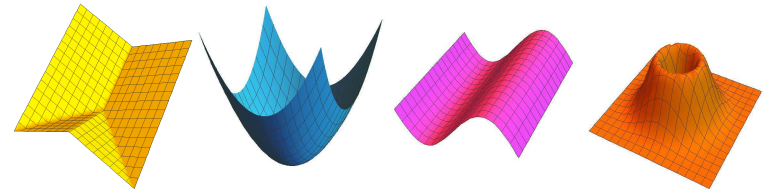
b) (2 points) Match the contour surfaces. Enter O, if there is no match.



I II III IV

$g(x, y, z) = 1$	Enter O-IV
$x + 2y = 1$	
$x^2 - y^2 - z^2 = 1$	
$z^2 + 2y = 1$	
$y^2 - z^2 = 1$	
$x^2 + z^2 = 1$	

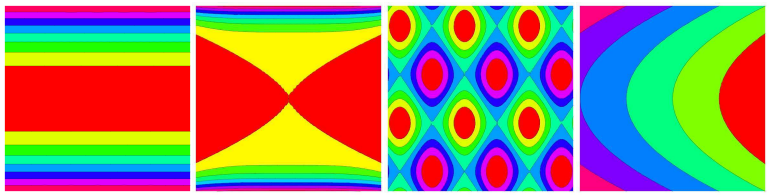
c) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.



I II III IV

$f(x, y) =$	Enter O-IV
$\exp(-x^2 - y^2)(x^2 + y^2)$	
$\sin(x)$	
$\exp(-x^2 - y^2)\sin(x^2)$	
$\lvert\lvert x\rvert + y\rvert$	
$x^2 + y^2$	

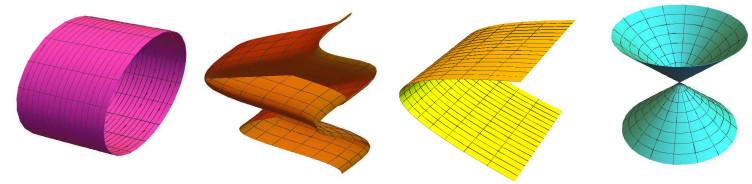
d) (2 points) Match functions $g(x, y)$ with contour maps. Enter O, if no match.



I II III IV

$g(x, y) =$	Enter O-IV
$\sin(3x) + \sin(2y)$	
$y^2 + x^2$	
$y^2 - 2x$	
y^2	
$y^6 - x^4$	

e) (2 points) Match the surface parametrization. Enter O, where is no match.



I II III IV

$\vec{r}(u, v) =$	Enter O,I-IV
$\langle u^2, v, u \rangle$	
$\langle u\cos(v), u\sin(v), u \rangle$	
$\langle \cos(u), \sin(v), u + v \rangle$	
$\langle v, \cos(u), \sin(u) \rangle$	
$\langle v, u, v \rangle$	

Problem 3) (10 points) No justifications are needed

In this problem $\vec{v}, \vec{w}$ are arbitrary vectors in space, $\vec{r}(t)$ is an arbitrary space curve.
The vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ are assumed to be nonzero where $\vec{N}$ is the normal vector and $\vec{B}$ the bi-normal vector.
All these vectors $\vec{r}, \vec{T}, \vec{B}, \vec{N}$ and its derivatives are evaluated at the fixed time $t = 0$.

first vector	second vector	always parallel	always perpendicular	depends
$\vec{r}\,'$	$\vec{r}\,''$			
$\vec{B}$	$\vec{N}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{v}$			
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			

Problem 4) (10 points)

A parallelepiped has vertices at $A = (0, 0, 0), B = (1, 1, 1), C = (2, 3, 4)$ and $A' = (3, 4, 8)$ and contains the sides AB, AC and AA'.

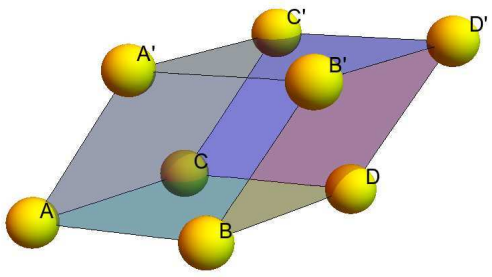
a) (2 points) Find a fourth point D so that A, B, C, D is a parallelogram.

b) (2 points) What is the area of that parallelogram $ABCD$?

c) (2 points) What is the volume of the parallelepiped?

d) (2 points) Find the height of the parallelepiped with floor $ABCD$ and roof A', B', C', D'.

e) (2 points) Find the distance between the face diagonals AD and $B'C'$.

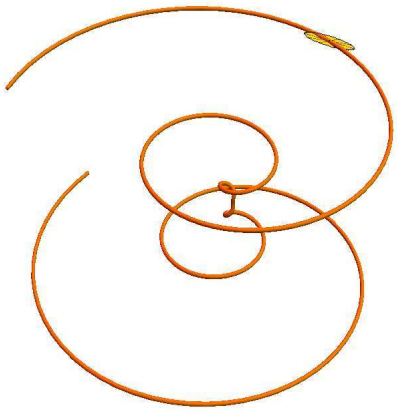


Problem 5) (10 points)

Autumn is here. A leaf tumbles down along the curve

$$\vec{r}(t) = \langle t^2 \cos(t), t^2 \sin(t), 16 - 2t \rangle$$

in space.



a) (3 points) What is the speed of the leaf at $t = \pi$?

b) (7 points) Find the arc length of the curve traced in the time interval $-8 \leq t \leq 8$.

Problem 6) (10 points)

On September 21, 2014, SpaceX launched a Dragon capsule with tons of supplies and experiments including a 3D printer to the space station. Assume the rocket experiences an acceleration

$$\vec{r}\,''(t) = \langle 2t, 0, 3t^2 - 5t^4 \rangle$$



starts at Cape Canaveral Air force station $\vec{r}(0) = \langle 2, 3, 0 \rangle$ with zero velocity $\vec{r}\,'(0) = \langle 0, 0, 0 \rangle$.

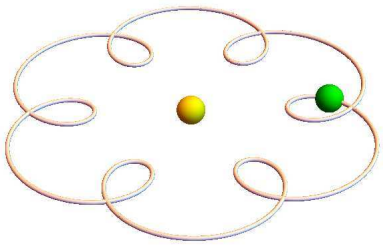
a) (5 points) Where is the capsule at time $t = 1$?

b) (5 points) What is the curvature of the path at $t = 1$? You can use the formula $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$.

Problem 7) (10 points)

Before Kepler and Newton clarified planetary motion, there was the **Ptolemaic universe** which was based on the idea that planets move on epicycles like

$$\vec{r}(t) = \langle 3\cos(t) + \cos(7t), \sin(t) + \sin(7t), 3 \rangle \ .$$



a) (2 points) What is the velocity $\vec{v} = \vec{r}\,'(t)$ at $t = \pi$?

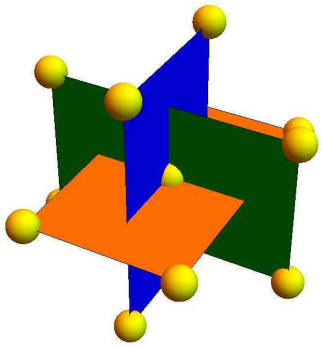
b) (2 points) What is the velocity $\vec{w} = \vec{r}\,'(t)$ at $t = \pi/2$?

c) (2 points) Yes or no? Is $\vec{v} \times \vec{w}$ parallel to the binormal vector $\vec{B}(t)$ for all times t?

d) (4 points) Parametrize the line tangent to the curve at the point $A = \vec{r}(\pi)$.

Problem 8) (10 points)

In this problem, the symbol φ is used to represent the golden ratio $\varphi = (\sqrt{5}+1)/2 \sim 1.618$ which satisfies the equation $\varphi^2 = \varphi + 1$.
The centers of four unit spheres are placed in the xy-plane at $A = (1, \varphi, 0), C = (-1, \varphi, 0), B = (1, -\varphi, 0)$ and $D = (-1, -\varphi, 0)$. 8 further points are located in the same way in the yz and xz plane so that we get 12 points which form the vertices of an **icosahedron** and surround a unit sphere centered at $(0, 0, 0)$.



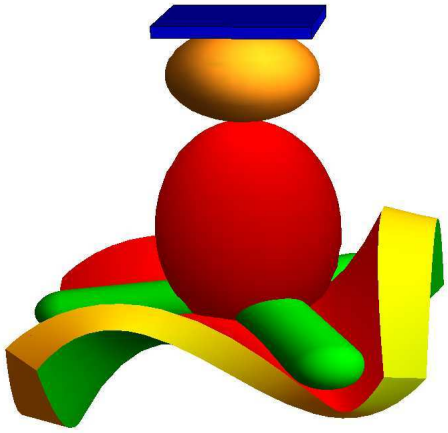
a) (3 points) Consider the distances between the points A and B. Verify that the unit spheres centered at A and B do not intersect. Likewise, verify that the unit spheres centered at A and C do just intersect in a point.
b) (2 points) Using the concept of an icosahedron, explain why all 12 spheres either pairwise do not intersect or intersect in a point.

c) (3 points) The centers of all spheres have equal distance d from $(0, 0, 0)$. What is d in terms of φ?

d) (2 points) Why does the central unit sphere intersect all other unit spheres?

Isaac Newton and **James Gregory** argued whether 13 unit spheres can be placed around a central unit sphere just "kissing the central sphere". They knew that 12 work. Newton believed 13 is impossible, but it was only proven in 1954 that the **kissing number** is 12. Here we have seen how to place 12 spheres: by pushing the 12 spheres constructed here a bit so that they just touch the central sphere, you showed that they have positive distance from each other and solve the 12 sphere kissing problem. It is known since 2003 that the kissing number in 4 dimensions is 24 but nobody has any clue what the kissing number in 5 dimensions is! It is only known that the answer is between 40 and 44.

Problem 9) (10 points)



As a souvenir for this exam, we build a Monkey riding a "Monkey saddle" and 3D print it. No explanations are necessary.

a) (2 points) Parametrize the hat $z = 5$.

b) (2 points) Parametrize the saddle $z = yx^2 - x^3$.

c) (2 points) Parametrize the torso $x^2 + y^2 + \frac{(z-1)^4}{4} = 1$.

d) (2 points) Parametrize the head $4x^2+y^2+(z-4)^2 = 1$.

e) (2 points) Parametrize the monkey tail $x^2 + z^2 = \frac{1}{4}$.

Problem 10) (10 points)



This June 2014, the Swiss extreme sports women **Géraldine Fasnacht** jumped with a wing-suit from the Matterhorn (a mountain in Switzerland). When flying with a wingsuit, there is the gravitational force, the force from the wind and a force from the wing. Assume

$$\vec{r}\,''(t) = \langle 1, t, \exp(-t) - 10\rangle$$

and $\vec{r}(0) = \langle 0, 0, 4500\rangle$ and $\vec{r}\,'(0) = \langle 0, 2, 0\rangle$. Find the path $\vec{r}(t)$.

9/27/2017 FIRST HOURLY PRACTICE 4 **Math 21a, Fall 2017**

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MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) T F The vector $\langle 0, 6/10, 8/10\rangle$ is a direction = unit vector.

2) T F Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v} \times \vec{w} = \vec{0}$.

3) T F For any vectors $\vec{u}$ and $\vec{v}$, we must have $\vec{v} \cdot \text{Proj}_{\vec{u}}\vec{v} = \vec{u} \cdot \text{Proj}_{\vec{v}}\vec{u}$.

4) T F The plane parametrized by $\vec{r}(t, s) = \langle t, s, 1\rangle$ is the same as $z = 1$.

5) T F The surface $x^2 + y^2 - 2y - z^2 = 0$ is a cone.

6) T F The curvature of the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ at any t is less than 1.

7) T F The volume of a parallelepiped generated by the vectors $\vec{u}, \vec{v}, \vec{w}$ is equal to $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

8) T F If a curve in space is parametrized by $\vec{r}(t)$ with $0 \leq t \leq 1$, then the same curve in the opposite direction can be parametrized by $\vec{r}(1 - t)$ with $0 \leq t \leq 1$.

9) T F The two-sheeted hyperboloid $x^2+y^2-z^2 = -1$ separates space into regions. The points $(3, 4, 6)$ and $(5, 12, -14)$ lie in the same region.

10) T F Given two vectors $\vec{u}$ and $\vec{v}$ which are perpendicular. Then $\text{Proj}_{\vec{u}}(\text{Proj}_{\vec{v}}\vec{w}) = \vec{0}$ for any vector $\vec{w}$.

11) T F The velocity vector $\vec{r}\,'(t)$ is always perpendicular to the curve.

12) T F If a point P with cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) has the property that $r = \rho$, then it must be on the xy plane.

13) T F The curvature of a circle of radius 3 is $1/3$.

14) T F The triple scalar product satisfies $\vec{u} \cdot (\vec{v} \times \vec{w}) \leq |\vec{u}||\vec{v}||\vec{w}|$.

15) T F If the dot product between two vectors is positive, then the two vectors form an acute angle.

16) T F The surface given in cylindrical coordinates as $z^2 + r^2 = 1$ is a sphere.

17) T F The arc length of the curve $\langle \sin(t), \cos(t)\rangle$ from $t = 0$ to $t = 1$ is equal to 1.

18) T F The curve $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ hits the plane $z = 0$ at a right angle.

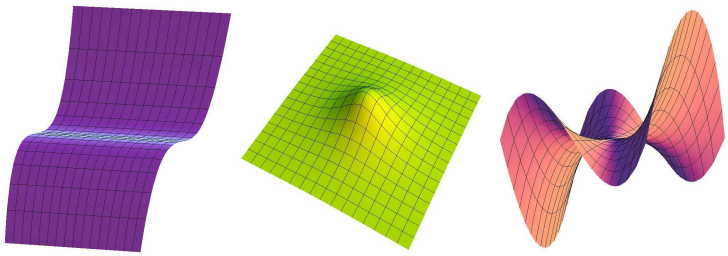
19) T F The parametrized curve $\langle 0, 7\cos(1 + t), 3\sin(1 + t)\rangle$ is an ellipse.

20) T F $\vec{u} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Total

Problem 2) (10 points) No justifications are needed here.

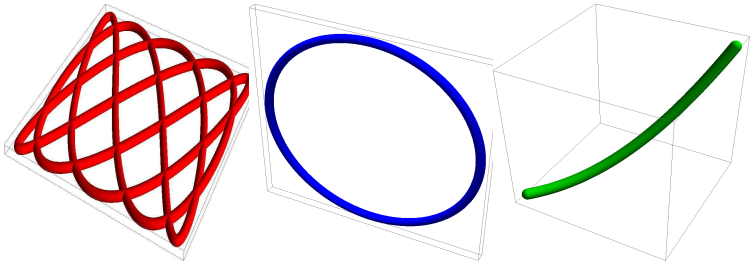
a) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.



I II III

Function $f(x,y) =$	Enter O,I,II or III
$x^3 - xy^2$	
y^3	
$1/(1+x^2+y^2)$	
$x^4 + y^4$	

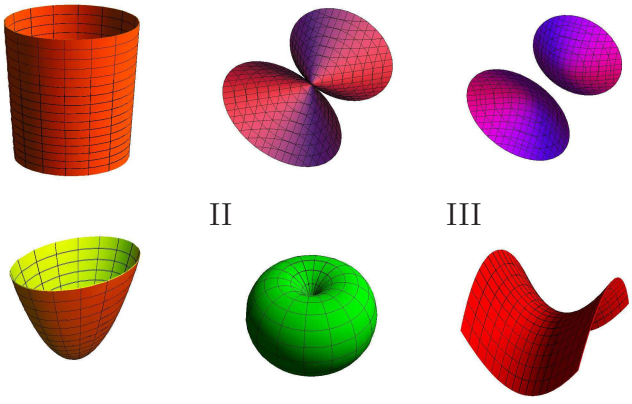
b) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\vec{r}(t) = \langle \cos(3t), \sin(5t), 0 \rangle$	
$\vec{r}(t) = \langle t, t, t^2 \rangle$	
$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$	
$\vec{r}(t) = \langle \sin(t), \sin(t), \sin(t) \rangle$	

c) (4 points) Match the surfaces to the pictures. There is an exact match here.

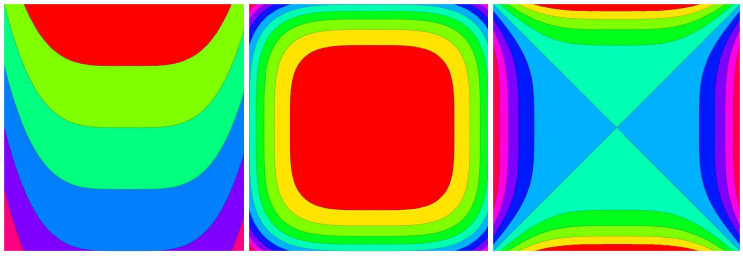


I II III

IV V VI

Description	I,II,III,IV,V,VI
$\langle 2u\cos(v), 4u\sin(v), u^2 \rangle$	
$\langle u^3, v^3, u^6 - v^6 \rangle$	
$\rho = \sin(\phi)$	
$r = 1$	
$x^2 - y^2 + z^2 = -1$	
$x^2 = y^2 - z^2$	

d) (2 points) Match the contour maps for $f(x,y)$. Enter O if no match.



I II III

function $f(x,y) =$	O,I,II,III
$f(x,y) = x^4 + y^4$	
$f(x,y) = x^4 - y^4$	
$f(x,y) = x - y$	
$f(x,y) = x^4 - y$	

Problem 3) (10 points)



The front roof line of the "spider" on the Harvard lecture halls forms a line

$$\vec{r}(t) = \langle 1 + t, 2 + t, 1 \rangle \ .$$

On top of the telescope sits a fly at the point $P = (0, 0, 10)$. Find the distance of P to the line.

Problem 4) (10 points)

The kinect sensor can be used to scan objects. An infrared laser is used to measure distances in the horizontal plane.



a) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 5 from the sensor $(0, -1)$.

b) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 4 from the sensor $(0, 1)$.

c) (6 points) Assume we know that P has distance 5 from $(0, -1)$ and distance 4 from $(0, 1)$. Where is this point (x, y) if we assume that it has a positive x-coordinate?

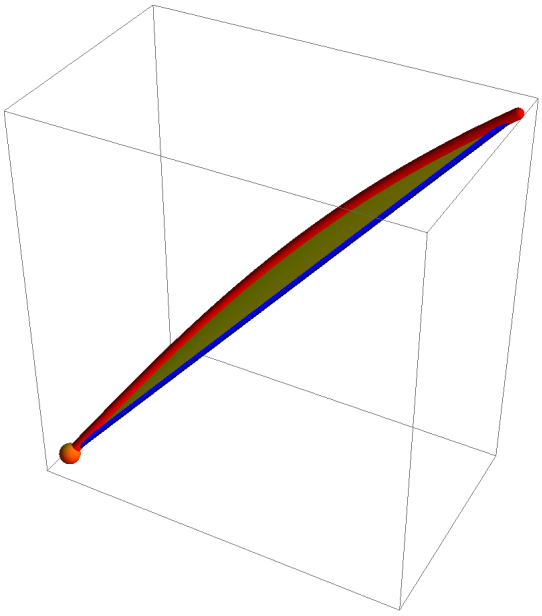
Problem 5) (10 points)

a) (6 points) Given $\vec{r}(t) = \langle t + t^3/3, \arctan(t), \sqrt{2}t \rangle$. Find the arc length from $t = 0$ to $t = 1$.

b) (4 points) Compute the vector integral

$$\int_0^1 \vec{r}\,'(t)\, dt$$

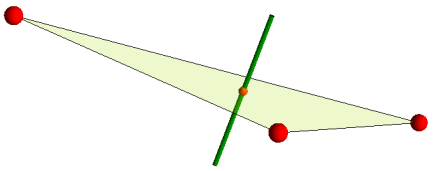
by integrating coordinate by coordinate. Verify that the length of this vector agrees with the arc length of the straight line connecting $\vec{r}(0)$ with $\vec{r}(1)$.



Problem 6) (10 points)

Given four points $A = (1, 2, 1), B = (1, 0, 1), C = (0, 1, 1), D = (1, 1, 2)$.

a) (4 points) Find an equation $ax + by + cz = d$ for the plane which contains A, B, C.
b) (3 points) Parametrize the line L which passes through D perpendicular to the plane ABC.
c) (3 points) Where does L hit the plane through A, B, C?



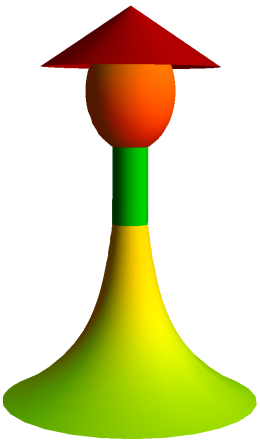
Problem 7) (10 points)

British stuntman **Gary Connery** made aviation history last year by becoming the first skydiver to land without parachute. He landed in 18000 boxes. Assume he started with an initial velocity $\langle 0, 100, 0 \rangle$ from the initial point $\langle 0, 0, 800 \rangle$. He was exposed to an acceleration $\vec{r}\,''(t) = \langle 0, 0, -10 + t \rangle$. Where is his location at time t=6?



Problem 8) (10 points)

We parametrize the queen in a fancy chess set. It consists of 5 surfaces. Parametrize them. You do not have to give bounds for the parameters. In each case, just give an answer of the form $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ without further explanations.



a) (2 points) "hat" Cone $x^2 + y^2 = (1-z)^2$.

b) (2 points) "head" Sphere $x^2 + y^2 + (z+1/2)^2 = 1$.

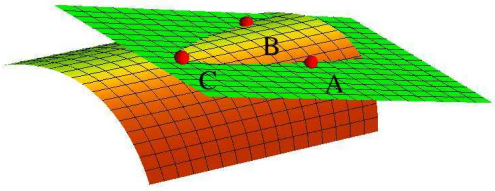
c) (2 points) "neck" Cylinder $x^2 + y^2 = 1/4$.

d) (2 points) "robe" Hyperboloid $x^2 + y^2 - (z+4)^2 = 1$.

e) (2 points) "floor" Plane $z = -8$

Problem 9) (10 points)

We are given a surface parametrized as $\vec{r}(u,v) = \langle u+v, u^2, v\rangle$.



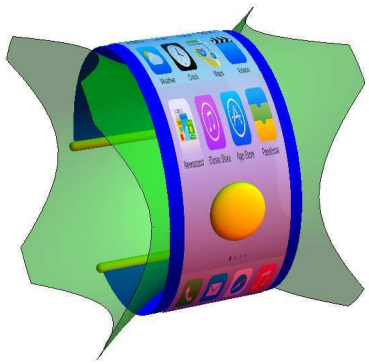
a) (2 points) Locate the points $A = \vec{r}(1,2), B = \vec{r}(-1,2)$ and $C = \vec{r}(0,0)$.

b) (4 points) Parametrize the plane through A, B, C.

c) (4 points) Find the area of the triangle with vertices A, B, C.

Problem 10) (10 points)

The reason for the name AppleWatch is that we can still hope for an **iWatch** which does not need pairing with a phone and which is waterproof. Simplicity is the rule: it consists of a band and "home button". As for fancy packaging, we strap it around a one sheeted hyperboloid. For each of the following surfaces, find a parametrization of the form



$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \ .$$

a) (3 points) "Band" cylinder

$$x^2 + y^2 = 9 \ .$$

b) (4 points) "Button" ellipsoid

$$10(x-3)^2 + y^2 + z^2 = 1/4 \ .$$

c) (3 points) "Package" 1-sheeted hyperboloid

$$x^2 + y^2 - z^2 - 1/2 = 0 \ .$$

9/27/2017 FIRST HOURLY PRACTICE 5 **Math 21a, Fall 2017**

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MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
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TTH 10 Peter Smillie
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- No notes, books, calculators, computers, or other electronic aids can be allowed.
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F For

2) T F There are unit vectors $\vec{v}$ and $\vec{w}$ in space for which $|\vec{v} \times \vec{w}| = 2$.

3) T F The vector $\langle 4, 5, 0 \rangle$ is perpendicular to the plane $-5x + 4y + z = 2$.

4) T F The distance between the cylinders $x^2 + z^2 = 1$ and $x^2 + (z-3)^2 = 1$ is 3.

5) T F The vector projection of $\langle 2, 3, 1 \rangle$ onto $\langle 1, 1, 1 \rangle$ is parallel to $\langle 1, 1, 1 \rangle$.

6) T F The equation $\rho \sin(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

7) T F There is a planar curve for which the arc length is 2π and the curvature is constant 1.

8) T F If we know the intersection of a graph $z = f(x, y)$ with the coordinate planes $x = 0, y = 0$ and $z = 0$, the function f is determined uniquely.

9) T F The surface given in cylindrical coordinates as $r = z^2$ is a paraboloid.

10) T F If the curvature of a space curve is constant 2 everywhere along the curve then the curve is a circle.

11) T F If $\vec{u}, \vec{v}$, and $\vec{w}$ are unit vectors then the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$, and $\vec{w}$ is largest when the parallelepiped is a cube.

12) T F If a point is moving along a straight line parametrized by $\vec{r}(t)$ then the velocity $\vec{r}\,'(t)$ vector and acceleration vector $\vec{r}\,''(t)$ must be parallel.

13) T F The parametrization $\vec{r}(u, v) = \langle v\cos(u), v\sin(u), v \rangle$ with $0 \leq u < 2\pi$ and $v \in \mathbf{R}$ is a cylinder.

14) T F If two lines in space are not parallel, then they must intersect.

15) T F If two planes do not intersect, then their normal vectors are parallel.

16) T F $(\vec{i} \times \vec{j})$ and $(\vec{i} \times (\vec{i} \times (\vec{i} \times \vec{j})))$ are parallel.

17) T F The surface parametrized by $\vec{r}(u, v) = \langle \sin(u)\sin(v), \sin(u)\cos(v), \cos(u) \rangle$ with $0 \leq v < 2\pi, 0 \leq u \leq \pi$ is a sphere.

18) T F The unit tangent vector $\vec{T}$ to a curve at a given point is independent of the parametrization up to a factor of -1.

19) T F $z^2 = r^2(\cos^2(\theta) - \sin^2(\theta)) + 1$ is a one-sheeted hyperboloid.

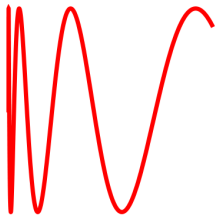
20) T F If $\vec{a} \cdot \vec{b} > 0$ and $\vec{b} \cdot \vec{c} > 0$, then $\vec{a} \cdot \vec{c} > 0$.

Total

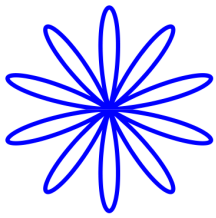
Problem 2) (10 points)

No explanations needed. I,II,III,O appear all once in each box.

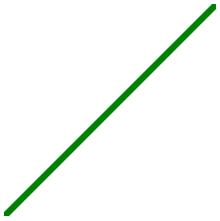
a) (2 points) Match curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I



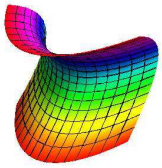
II



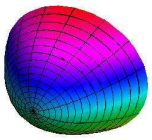
III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\sin^2(5t)\langle\cos(t), \sin(t)\rangle$	
$\langle t^3, \sin(7t)\rangle$	
$\langle t^5, 1+t^5\rangle$	
$\langle\sin(t), \cos(t)\rangle$	

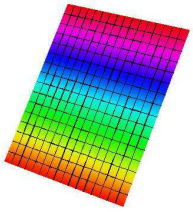
b) (2 points) Match the parametrization. Enter O, where no match.



I



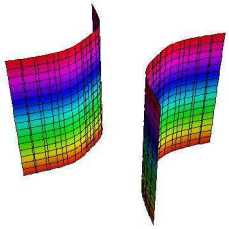
II



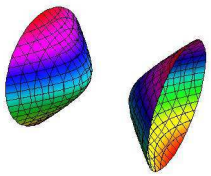
III

$\vec{r}(s,t)$	O,I,II,III
$\langle 1-t, 1+s, 2+s\rangle$	
$\langle s, t^2-s^2, t\rangle$	
$\langle t\cos(s), t\sin(s), s\rangle$	
$\langle s\cos(t), s^2, s\sin(t)\rangle$	

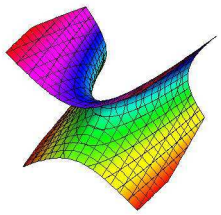
c) (2 points) The pictures show contour surfaces. Enter O, where no match.



I



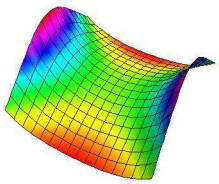
II



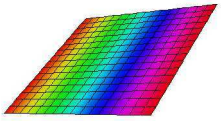
III

$g(x,y,z) =$	O, I,II,III
$x^2-y^2+z^2=-1$	
$x^2-y^2=1$	
$x^4+z=1$	
$x^2+y-z^2=1$	

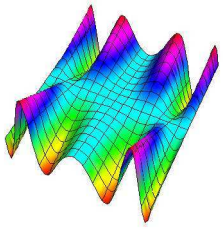
d) (2 points) Match the graphs $z=f(x,y)$ with the functions. Enter O, where no match.



I



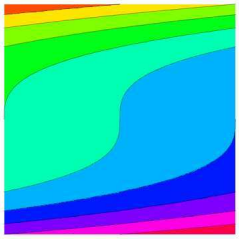
II



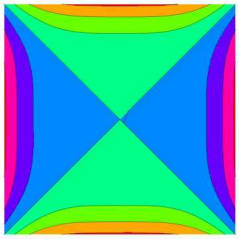
III

function $f(x,y) =$	O,I,II,III
$2x$	
$e^{-2x^2-2y^2}$	
$e^{x^2-y^2}$	
$y\sin(x^2)$	

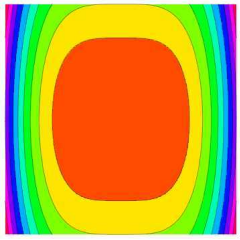
e) (2 points) Match the family of level curves with $f(x,y)$. Enter O, where no match.



I



II



III

Function $f(x,y) =$	O,I,II,III
x^4+y^2	
x^4-y^4	
$x-y$	
$x-y^3$	

Problem 3) (10 points)

No explanations needed. In 3a), in each row check only one box.

a) (4 points) The intersection of a plane with a cone $S : x^2 + y^2 - z^2 = 0$ is called a **conic section**. What curve do we get?

Intersect S with	hyperbola	parabola	circle	line
$z = 1$ gives a				
$z = x$ gives a				
$z = x + 1$ gives a				
$x = 1$ gives a				

b) (3 points) By intersecting the upper half sphere $x^2 + y^2 + z^2 = 5, z > 0$ with the hyperboloid $x^2 + y^2 - z^2 = -3$ we get a curve. Which one? Check exactly one box.

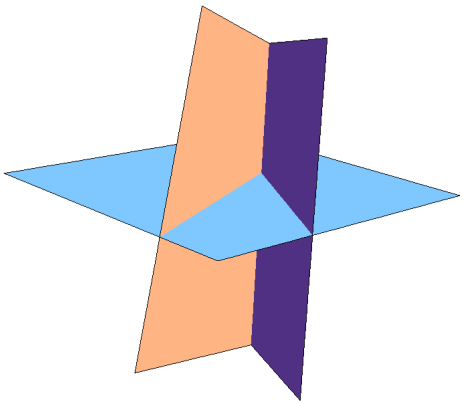
$\vec{r}(t) = \langle \cos(t), \sin(t), 2 \rangle$	
$\vec{r}(t) = \langle 0, 0, t \rangle$	
$\vec{r}(t) = \langle \cos(t), \sin(t), 2t \rangle$	

c) (3 points) Which of the following surface parametrizations gives a one sheeted hyperboloid? Check exactly one box.

$\vec{r}(t, s) = \langle s, t, s^2 - t^2 \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 + s^2} \cos(t), \sqrt{1 + s^2} \sin(t), s \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 - s^2} \cos(t), \sqrt{1 - s^2} \sin(t), s \rangle$	

Problem 4) (10 points)

We are given two planes $x+y+z=1$ and $x-y-z=2$. Find a third plane which contains the point $(1,0,0)$ and which is perpendicular to both.

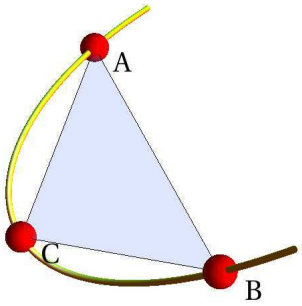


Problem 5) (10 points)

We are given a curve $\vec{r}(t) = \langle 1+t, t^2, t^3 \rangle$.

a) (5 points) Find the area of the triangle with vertices $A = \vec{r}(-1), B = \vec{r}(1)$ and $C = \vec{r}(0)$.

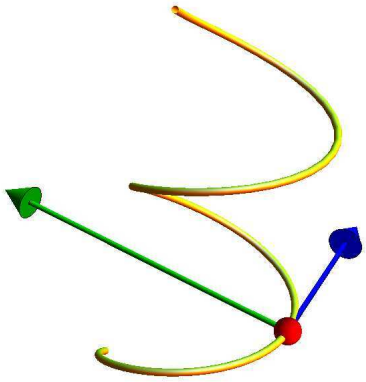
b) (5 points) Find an equation $ax + by + cz = d$ for the plane through A, B, C.



Problem 6) (10 points)

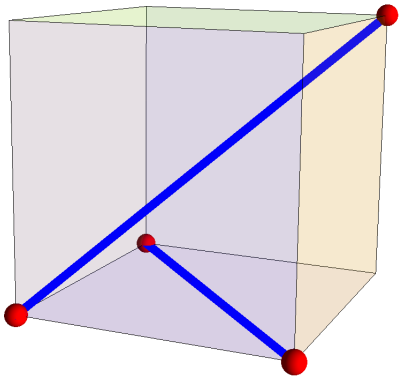
a) (3 points) Find the unit tangent vector $\vec{T}(t)$ of the curve $\vec{r}(t) = \langle t^2, \cos(t^2\pi), \sin(t^2\pi)\rangle$ at $t = 1$.
b) (3 points) What is the acceleration vector $\vec{r}\,''(t)$ at $t = 1$?
c) (4 points) Find the curvature at the time $t = 1$. You may use the formula

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \ .$$



Problem 7) (10 points)

What's the closest that the long diagonal of the unit cube connecting the corners $(0, 0, 0)$ to $(1, 1, 1)$, comes to the diagonal of a face connecting the corners $(1, 0, 0)$ and $(0, 1, 0)$?



Problem 8) (10 points)

In a parallel universe of ours, the inhabitants live under a "Newton's law" of gravity in which the "jerk" $\vec{r}\,'''(t)$ rather than the acceleration is constant. Suppose that $\vec{r}\,'''(t) = \langle 0, 0, -10\rangle$ for all t.

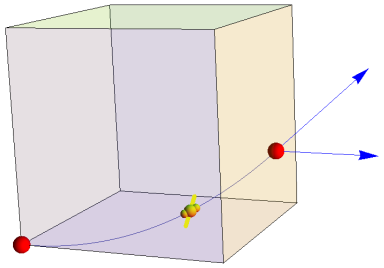


a) (3 points) Find $\vec{r}\,''(t)$ if you know $\vec{r}\,''(0) = \langle 0, 0, 0\rangle$.
b) (3 points) Now find $\vec{r}\,'(t)$ if we know also $\vec{r}\,'(0) = \langle 1, 0, 0\rangle$.
c) (4 points) Finally find $\vec{r}(t)$ if we know additionally $\vec{r}\,(0) = \langle 0, 0, 10\rangle$.

Problem 9) (10 points)

a) (2 points) A fly is trapped inside a unit cubicle made of planar glass panes. It flies, starting at $t = 0$ at the origin $(0, 0, 0)$ along the curve

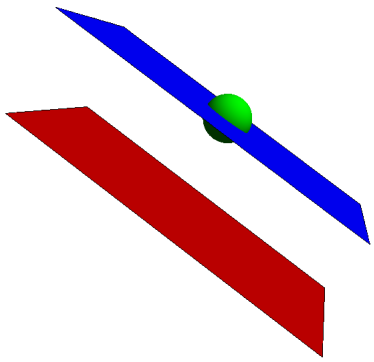
$$\vec{r}(t) = \langle t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3} \rangle \ .$$



At what time does it bump into the glass wall $x = 1$?
b) (4 points) Find the impact angle (= the angle between the normal vector of the plane and the velocity vector).
c) (4 points) How long is the path it has followed from $t = 0$ to the impact point?

Problem 10) (10 points)

When two uncharged metallic parallel plates are put close together, there is an attractive force between them which can be explained by quantum field theory only. In May 14, 2013, an article suggested to use this Casimir effect for microchip designs. (Source Nature: http://www.nature.com/ncomms/journal/v4/n5/full/nc



a) (3 points) Locate a point P on the plane $x + 2y + 2z = 4$.

b) (7 points) Find the distance d between the plane $x + 2y + 2z = 1$ and plane $x + 2y + 2z = 4$.

9/27/2017 FIRST HOURLY PRACTICE 6 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] If t
$a =$

2) [T] [F] The point $(x, y, z) = (1, 1, \sqrt{2})$ has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$.

3) [T] [F] Every point on the parametric curve $\vec{r}(t) = \langle t, t^2, -t \rangle$ lies on the surface $xz + y = 0$.

4) [T] [F] The two surfaces $f(x, y, z) = 3$ and $f(x, y, z) = 5$ of the function $f(x, y, z) = 2x^2 + y^3 + z^4$ do not intersect at any point in space.

5) [T] [F] $\vec{u} \times \vec{i}$ and $\vec{u} \times \vec{j}$ are perpendicular for all vectors $\vec{u}$.

6) [T] [F] If $\vec{u}$ and $\vec{v}$ are parallel (remember that this means $\vec{u} = \lambda \vec{v}$ for some real λ) then $\vec{u} \cdot \vec{v} \geq |\vec{u} \times \vec{v}|$.

7) [T] [F] If a surface has the property that all intersections with the planes $y =$ *constant* are straight lines, then the surface is a plane.

8) [T] [F] For any non-zero vectors $\vec{u}$ and $\vec{w}$, we must have $\text{Proj}_{\vec{u}}\vec{w} = -\text{Proj}_{\vec{w}}\vec{u}$.

9) [T] [F] In the parametric surface $\vec{r}(s, t) = \langle \sqrt{1 + e^t}\cos(s)\,,\, \sqrt{1 + e^t}\sin(s)\,,\, t \rangle$ the grid curves with constant s are ellipses.

10) [T] [F] There is a vector $\vec{v}$ with the property that $\vec{v} \times \langle 1, 1, 1 \rangle = \langle 0, 0, 1 \rangle$.

11) [T] [F] We can assign a value $f(0, 0)$ such that the function $f(x, y) = (x^3 + y^3)/(x^2 + y^2)$ is continuous at $(0, 0)$.

12) [T] [F] The curvature of a curves $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same at $t = 1$.

13) [T] [F] The curve given in spherical coordinates as $\phi = \pi/2, \rho = \pi/2$ is a circle.

14) [T] [F] Two nonparallel planes with normal vectors $\vec{n}, \vec{m}$ intersect in a line parallel to $\vec{n} \times \vec{m}$.

15) [T] [F] If $f(x, y) = x^3/3 - y^2$, then the graph of the function $f(x, y)$ is called an elliptic paraboloid.

16) [T] [F] The equation $\rho\cos(\theta)\sin(\phi) = 2$ in spherical coordinates defines a plane.

17) [T] [F] The vector $\langle 3, -2 \rangle$ in the two dimensional plane is perpendicular to the line $3x - 2y = 7$.

18) [T] [F] The volume of the parallelepiped spanned by the vectors $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle$ and $\langle 1, 1, 1 \rangle$ is 2.

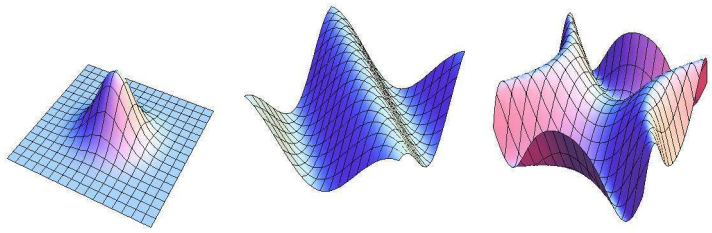
19) [T] [F] If $\vec{r}(t)$ is a curve and $|\vec{r}\,'(t)| > 0$ and $|\vec{T}'| > 0$, we have $\vec{T}(t) \cdot (\vec{N}(t) \times \vec{B}(t)) = 1$.

20) [T] [F] The arc lengths of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same for $0 \leq t \leq 1$.

[ ] Total

Problem 2) (10 points)

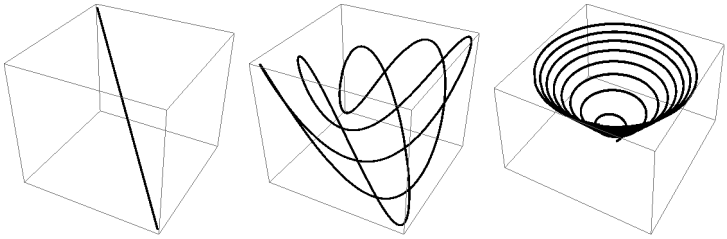
a) (2 points) Match the graphs $z = f(x, y)$ with the functions. Enter O, if there is no match. In each of the problems a) - d), each entry O,I,II,III appears exactly once.



I II III

Function $f(x, y) =$	O,I,II or III
$e^{-x^2-y^2}$	
$\cos(x + y)$	
$\sin(x^2 - y^2)$	
$x^4 + y^4$	

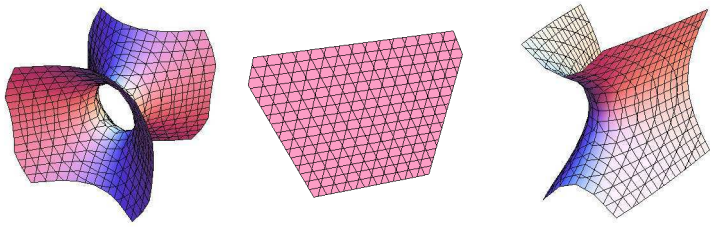
b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\langle 1 + t, 1 - t, t\rangle$	
$\langle t\cos(t^2), t\sin(t^2), t\rangle$	
$\langle t, t, \sin(t^3)\rangle$	
$\langle \cos(3t), \sin(2t), \sin(5t)\rangle$	

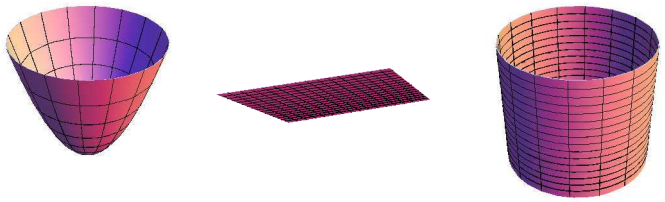
c) (2 points) Match the functions g with the level surface $g(x, y, z) = 1$. Enter O, where no match.



I II III

$g(x, y, z) =$	O, I,II,III
$(x - 1)^2 - y^2 + z^2 = 1$	
$(x - 1)^2 + y + z^2 = 1$	
$(x - 1) + y + z = 1$	
$(x - 1)^2 - y - z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.

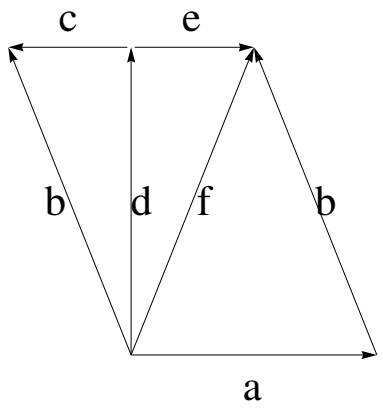


I II III

Parametriztion $\vec{r}(s, t) =$	O,I,II,III
$\langle s\cos(t), s\sin(t), s^2\rangle$	
$\langle t - 1, s, s + t\rangle$	
$\langle \cos(t), \sin(t), s\rangle$	
$\langle s\cos(t), s\sin(t), s^2\sin(t)\rangle$	

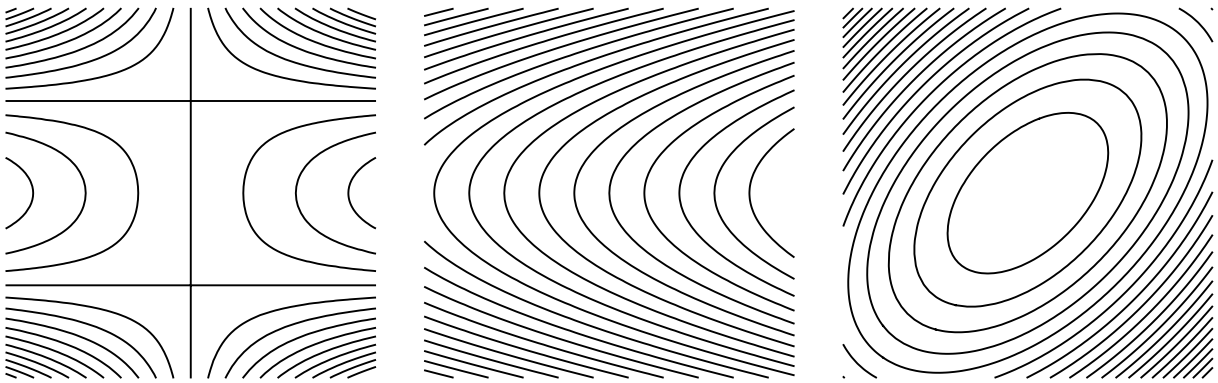
Problem 3) (10 points)

a) (7 points) Each of the vectors $a, b, c, d, e, f, 0$ (written without arrows for clarity) will appear in the blanks exactly once. As the picture indicates, you know $d \cdot e = d \cdot c = 0$.



the vector	is equal to
$\text{Proj}_d f$	
$f - d$	
$-2c$	
$d - c$	
$-e$	
$\text{Proj}_d e$	
$d + c$	

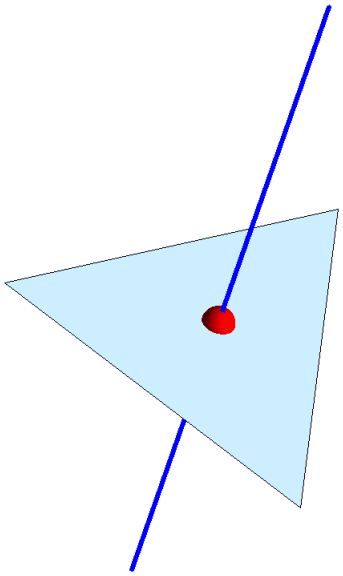
b) (3 points) Match the contour maps with the functions



I II III

Function $f(x, y) =$	Enter O,I,II or III
$y - x$	
$(y^2 - 1)x$	
$y^2 + x^2 - xy$	
$y^2 - x$	

Problem 4) (10 points)



a) (4 points) The **center** of the triangle $A = (3,2,1), B = (1,1,1), C = (2,0,4)$ is the point $P = (A+B+C)/3 = (2,1,2)$. Find the line L perpendicular to the plane which contains A, B, C and which goes through P.

b) (3 points) Find the equation of the plane through A, B, C.

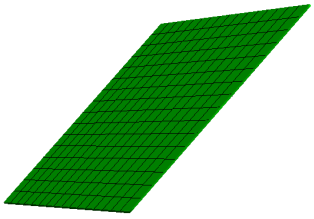
c) (3 points) Find the area of the triangle ABC.

Problem 5) (10 points)

Complete the parametrizations:

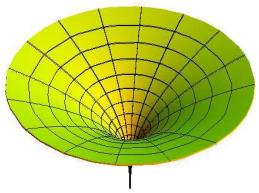


a) (3 points) $\vec{r}(u,v) = \langle 2 + 3\cos(u)\sin(v), 3 + \sin(u)\sin(v), \boxed{} \rangle$ parametrizes the ellipsoid $(x-2)^2/9 + (y-3)^2 + (z-5)^2/16 = 1$.

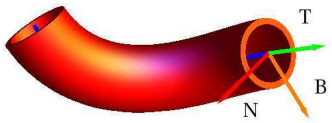


b) (2 points) $\vec{r}(u,v) = \langle u, v, \boxed{} \rangle$ parametrizes the plane $x + y + z = 1$.

c) (3 points) $\vec{r}(u,v) = \langle v^3\cos(u), \boxed{}, v\rangle$ parametrizes the surface of revolution $x^2+y^2=z^6$.



d) (2 points) $\vec{r}(u,v) = \vec{r}(v) + \cos(u)\vec{N}(v) + \sin(u)\boxed{}$ parametrizes a tube around a curve $\vec{r}(v)$ which has unit tangent vector $\vec{T}(v)$, normal vector $\vec{N}(v)$ and binormal vector $\vec{B}(v)$.

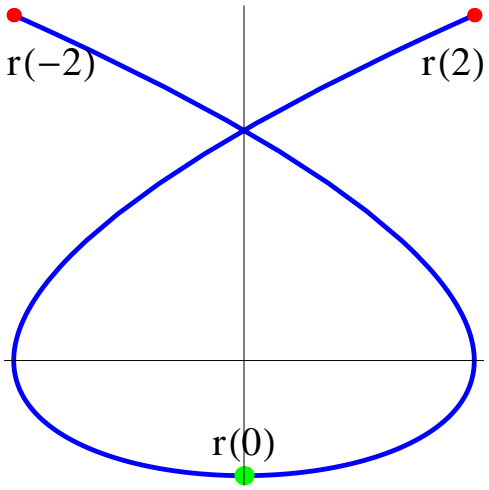


Problem 6) (10 points)

We look at the parametrized curve

$$\vec{r}(t) = \langle \frac{t^3}{3} - t, t^2 - 1, 0\rangle$$

whose image you see in the picture showing it in the xy plane for $-2 \le t \le 2$.



a) (3 points) Find the velocity $\vec{r}\,'(t)$, the acceleration $\vec{r}\,''(t)$ and speed $|\vec{r}\,'(t)|$.

b) (2 points) Evaluate this at $t = 0$ to get $\vec{r}\,'(0), \vec{r}\,''(0)$ and $|\vec{r}\,'(0)|$.

c) (2 points) Find the curvature $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$ at $(0,-1,0)$.

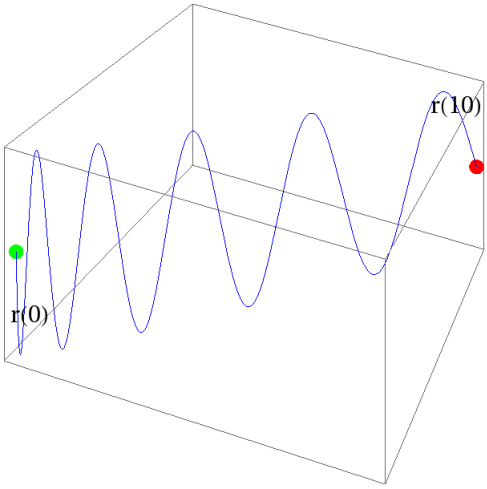
d) (3 points) Find the arc length of the curve $\vec{r}(t)$ from $-2 \le t \le 2$.

Problem 7) (10 points)

a) (4 points) We know $\vec{r}\,''(t) = \langle 1, 2, \pi^2 \sin(\pi t)\rangle$ and the initial velocity $\vec{r}\,'(0) = \langle 1, 0, -\pi\rangle$. Find $\vec{r}\,'(t)$.

b) (3 points) Assume we know also $\vec{r}(0) = \langle 0, 0, 10\rangle$. Find $\vec{r}(10)$.

c) (3 points) What is the projection of $\vec{r}\,'(10)$ onto $\langle 1, 1, 0\rangle$?

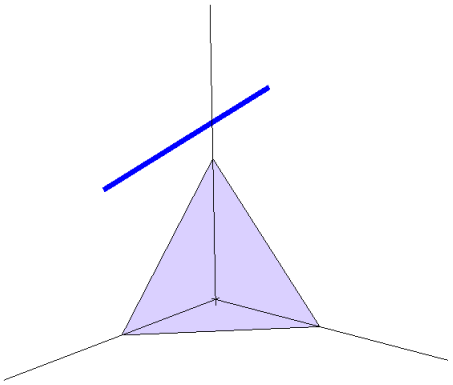


Problem 8) (10 points)

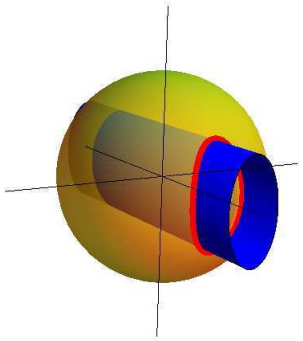
a) (5 points) Find the distance between the plane $x + y + z = 1$ and the line

$$x - 1 = \frac{(y-1)}{-2} = z - 1$$

which is parallel to the plane.
(You do not have to check that it is parallel).



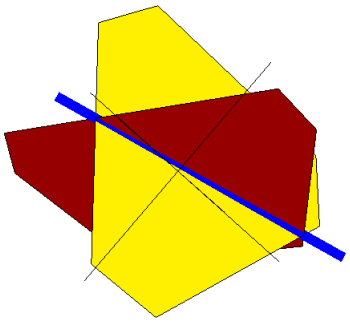
b) (5 points) The intersection of the cylinder $4x^2+z^2 = 1$ with the sphere centered at $(0, 0, 0)$ with radius $\rho = \sqrt{2}$ cuts out two curves. Parametrize the curve which contains the point $(0, 1, 1)$.



Problem 9) (10 points)

a) (5 points) Find a parametrization of the intersection line L of the two planes

$$\begin{aligned} 2x - 2y + z &= 1\,, \\ x + y + z &= 1\,. \end{aligned}$$

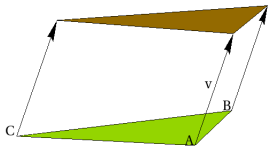


b) (5 points) Find a parametrization for the line M parallel to the line L computed in a) which passes through $(1, 2, 3)$.

Problem 10) (10 points)

a) (5 points) What is the area of the triangle through the points $A = (1, 1, 1)$ and $B = (0, 1, 0)$ and $C = (1, 2, 4)$.

b) (5 points) Find the volume of the prism which has the triangle T as base as well as a by $\vec{v} = \langle 0, 1, 1 \rangle$ translated triangle as roof.



9/27/2017 FIRST HOURLY PRACTICE 7 **Math 21a, Fall 2017**

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MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

2) [T] [F] Any three distinct points A, B, C in space determine a unique plane which passes through these points.

3) [T] [F] For any two non-intersecting lines L, K, there are two parallel planes Σ, Δ whose distance $d(\Sigma, \Delta)$ is equal to the distance $d(L, K)$ such that L is in Σ and K is in Δ.

4) [T] [F] If $z - f(x, y) = g(x, y, z)$ then the graph of $f(x, y)$ is a level surface $g(x, y, z) = c$ of $g(x, y, z)$.

5) [T] [F] The graph of the function $f(x, y) = x^2 + y$ is called an elliptic paraboloid.

6) [T] [F] The equation $\rho \sin(\phi) \sin(\theta) = 1$ in spherical coordinates defines a plane.

7) [T] [F] The vector $\langle 1, 2, 3 \rangle$ is parallel to the plane $2x + 4y + 6z = 4$.

8) [T] [F] The cross product between $\langle 2, 3, 1 \rangle$ and $\langle 1, 1, 1 \rangle$ is 6.

9) [T] [F] The curve $\vec{r}(t) = \langle \cos(t), t^2, \sin(t) \rangle, 1 \leq t \leq 9$ and the curve $\vec{r}(t) = \langle \cos(t^2), t^4, \sin(t^2) \rangle, 1 \leq t \leq 3$ have the same length.

10) [T] [F] The point $(1, -1, 1)$ has the spherical coordinates of the form $(\rho, \theta, \phi) = (\sqrt{3}, \pi/4, \pi/4)$.

11) [T] [F] The distance between two parallel lines in space is the distance of any point on one line to the other line.

12) [T] [F] For two nonzero vectors $\vec{v}$ and $\vec{w}$, the identity $\text{Proj}_{\vec{w}}(\vec{v} \times \vec{w}) = \vec{0}$ holds.

13) [T] [F] The vector projection of $\langle 2, 3, 4 \rangle$ onto $\langle 1, 0, 0 \rangle$ is $\langle 2, 0, 0 \rangle$.

14) [T] [F] The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ between three vectors $\vec{u}, \vec{v}, \vec{w}$ is zero if and only if two or more of the 3 vectors are parallel.

15) [T] [F] There are two vectors $\vec{v}$ and $\vec{w}$ so that the dot product $\vec{v} \cdot \vec{w}$ is equal to the length of the cross product $|\vec{v} \times \vec{w}|$.

16) [T] [F] The distance between two spheres of radius 2 whose centers have distance 8 is 4.

17) [T] [F] If two vectors $\vec{v}$ and $\vec{w}$ are both parallel and perpendicular, then at least one of the vectors must be the zero vector.

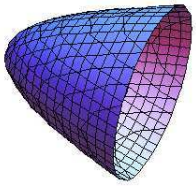
18) [T] [F] The curvature $\kappa(\vec{r}(t))$ is always smaller than or equal to the length $|\vec{r}\,''(t)|$ of the acceleration vector $\vec{r}\,''(t)$.

19) [T] [F] The curve $\vec{r}(t) = \langle \cos(t^2) \sin(t^2), \sin(t^2) \sin(t^2), \cos(t^2) \rangle$ is located on a sphere.

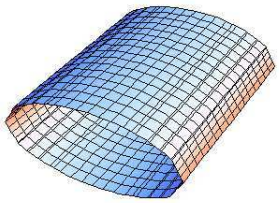
20) [T] [F] The surface $x^2 + y^2 + z^2 = 2z$ is a sphere.

Problem 2) (10 points)

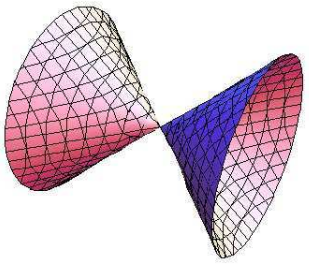
a) (6 points) Match the surfaces the equations $g(x, y, z) = 0$.



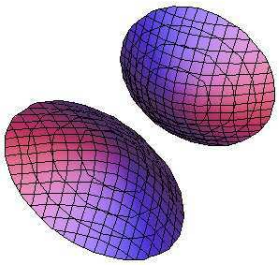
I



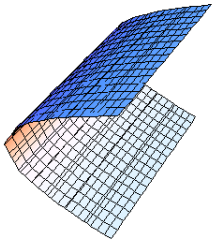
II



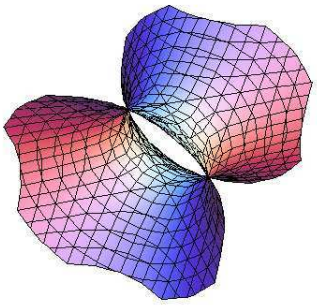
III



IV



V



VI

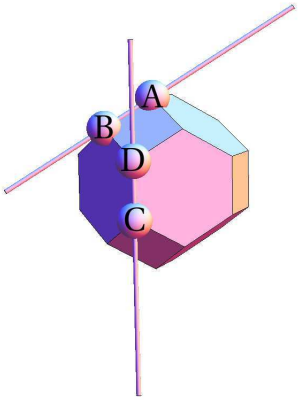
Function $g(x, y, z) = 0$	Enter I,II,III,IV,V,VI	Function $g(x, y, z) =$	Enter I,II,III,IV,V,VI
$y^2 + z^2 - x$		$x^2 - y^2 - z^2 + 1$	
$y^2/4 + z^2/4 - 1$		$x - z^2$	
$x^2 - y^2 - z^2 - 1$		$y^2 - z^2 + x^2$	

b) (4 points) Match the surfaces given in cylindrical and spherical coordinates with the surfaces given in Cartesian coordinates:

	surface
A	$r = 1$
B	$\sin(\theta) = 0$
C	$\cos(2\phi) = 0$
D	$\rho = 1$

Enter A-D	surface
	$x^2 + y^2 = 1$
	$x^2 + y^2 = z^2$
	$x^2 + y^2 + z^2 = 1$
	$y = 0$

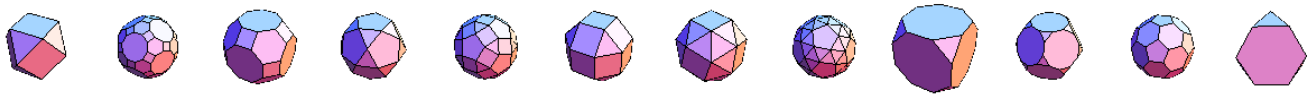
Problem 3) (10 points)



A **truncated octahedron** has an edge connecting the vertices $A = (-1, 3, 0), B = (-1, 1, -1)$ and an edge connecting the vertices $C = (-3, -1, 0), D = (-3, 1, 0)$.

a) (5 points) Find the distance of C to the line through A, B.

b) (5 points) Find the distance between the line L through A, B and the line K through C, D.



Here are the remaining 12 **Archimedean solids**. These are polyhedra bound by different types of regular polygons but for which each vertex of the polyhedron looks the same. There are 13 such semiregular polyhedra. Archimedes studied them first in 287BC. Kepler was the first to describe the complete set of 13 in his work **"Harmonices Mundi"** of 1619.

Problem 4) (10 points)

a) (3 points) Give a parametrization $\vec{r}(\theta, z) = \langle x(\theta, z), y(\theta, z), z(\theta, z)\rangle$ of the surface which is in cylindrical coordinates given by

$$r = z^4 \ .$$

b) (2 points) Find a parametrization $\vec{r}(u, v)$ of the graph $z = \sin(xy)$.

c) (2 points) Find a parametrization $\vec{r}(u, v)$ of the yz-plane $x = 0$.

d) (3 points) Give a parametrization $\vec{r}(\phi, \theta)$ of the surface which is in spherical coordinates given by

$$\rho = 2 + \cos(13\phi) \ .$$

Problem 5) (10 points)

a) (7 points) Find the arc length of the curve

$$\vec{r}(t) = \langle \cos(t^2/2), \sin(t^2/2), (1/3)(1 - t^2)^{3/2}\rangle$$

from $0 \leq t \leq 1$.

b) (3 points) Decide whether the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous.

Problem 6) (10 points)



Wall-e explores a planet, where a strong solar wind produces a time-dependent magnetic field and where the combined force of gravity and magnetic lift produces a time dependent vertical acceleration

$$\vec{r}\,''(t) = \langle 0, 0, 10\cos(t) \rangle \ .$$

a) (6 points) Wall-e knows that he is at time $t = 0$ at $\vec{r}(0) = \langle 1, 2, 3 \rangle$ with velocity $\langle 0, 1, 2 \rangle$. Where is he at time $t = \pi$?

b) (4 points) What speed does he have at time $t = \pi$?

Problem 7) (10 points)



Potter plays Quidditch. At time $t = 0$ he is at $P = (1, 3, 5)$. At time $t = 1$ he is at $Q = (0, 1, 3)$. Harry is spell-bound and can not change direction, nor change speed and crashes into a tilted side wall of the stadium crushing his knee (*).

a) (3 points) If Potter flies on a straight line through PQ, find a parametrization for that line.

b) (4 points) Where and when does he hit the tilted side wall $x + y + z = 1$ of the stadium?

c) (3 points) What is the angle between Harry's velocity vector and the upwards pointing normal vector of the side wall?

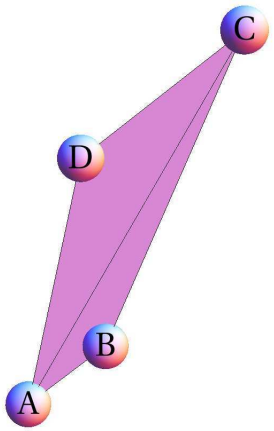
(*) Don't worry, Madam Pomfrey will fix it.

Problem 8) (10 points)

No justifications are needed in this problem. All vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ can be assumed to be nonzero. The vector $\vec{N} = \vec{T}'/|\vec{T}'|$ is the normal vector and $\vec{B}$ is the binormal vector. Recall that two vectors are perpendicular, if and only if their dot product is zero and that two vectors are parallel if and only if their cross product is the zero vector.

first vector	second vector	always parallel	always perpendicular	neither
$\vec{T}'(0)$	$\vec{r}\,''(0)$			
$\vec{w}$	$(\vec{v} \times \vec{w}) \times \vec{v}$			
$\vec{v}$	$\vec{v}/\|\vec{v}\|$			
$\vec{v} = \langle a, b, c \rangle$	normal to $ax + by + cz = 4$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{B}$	$\vec{N}$			

Problem 9) (10 points)



The four points $A = (0, 0, 5), B = (1, 1, 6), C = (2, 4, 11), D = (0, 2, 9)$ are in a plane.

a) (5 points) Find the equation $ax + by + cz = d$ for this plane.

b) (5 points) The quadrilateral $ABCD$ is the union of two triangles ABC and ACD. Find the area of the quadrilateral.

Problem 10) (10 points)

In this problem we find some parametrizations of surfaces which is of the form

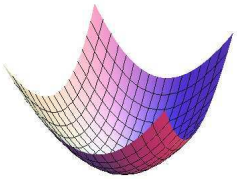
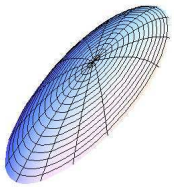
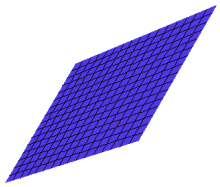
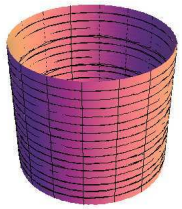
$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \ .$$

a) (2 points) Parametrize the **paraboloid** $z = x^2 - y^2$.

b) (3 points) Parametrize (the entire!) **ellipsoid** $(x-1)^2 + \frac{(y-2)^2}{4} + z^2 = 1$.

c) (2 points) Parametrize the **plane** $x + y + z = 3$.

d) (3 points) Parametrize the **cylinder** $x^2 + z^2 = 1$.

9/27/2017 FIRST HOURLY PRACTICE 8 **Math 21a, Fall 2017**

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MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The

2) | T | F | The length of the vector $\langle 1, 2, 2\rangle$ is an integer.

3) | T | F | The vector $\langle 3, 4\rangle$ appears as a velocity vector of the curve $\vec{r}(t) = \langle \cos(5t), \sin(5t)\rangle$. Namely, there is a t such that $\vec{r}\,'(t) = \langle 3, 4\rangle$.

4) | T | F | If $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the unit normal vector, and $\vec{B}$ is the binormal vector, then $\vec{B} \times \vec{N} = \vec{T}$.

5) | T | F | The curvature of a larger circle $r = 2$ is greater than the curvature of a smaller circle $r = 1/2$.

6) | T | F | The surface $x^2 - y^2 - z^2 - 1 = 0$ is a one sheeted hyperboloid.

7) | T | F | The function $f(x, y) = y^2 - x^2$ has a graph that is an elliptic paraboloid.

8) | T | F | Let $\vec{r}(t)$ be a parametrization of a curve. If $\vec{r}(t)$ is always parallel to the tangent vector $\vec{r}\,'(t)$, then the curve is part of a line through the origin.

9) | T | F | If $\mathrm{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\vec{u}$ is the zero vector.

10) | T | F | If $\mathrm{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\mathrm{Proj}_{\vec{k}}(\vec{u})$ is the zero vector.

11) | T | F | If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

12) | T | F | There are two vectors $\vec{a}$ and $\vec{b}$ such that the scalar projection of $\vec{a}$ onto $\vec{b}$ is 100 times the magnitude of $\vec{b}$.

13) | T | F | The curve $\vec{r}(t) = \langle \cos(t), e^t + 10, t^2\rangle, 2 \le t \le 6$ and the curve $\vec{r}(t) = \langle \cos(2t), e^{2t}, 4t^2\rangle, 1 \le t \le 3$ have the same length.

14) | T | F | The equation $\rho \sin(\phi) - 2\sin(\theta) = 0$ in spherical coordinates defines a two sheeted hyperboloid.

15) | T | F | If triple scalar product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is larger than $|\vec{u} \times \vec{v}|$ then $|\vec{w}| > 1$.

16) | T | F | The distance between the x-axis and the line $x = y = 1$ is $\sqrt{2}$.

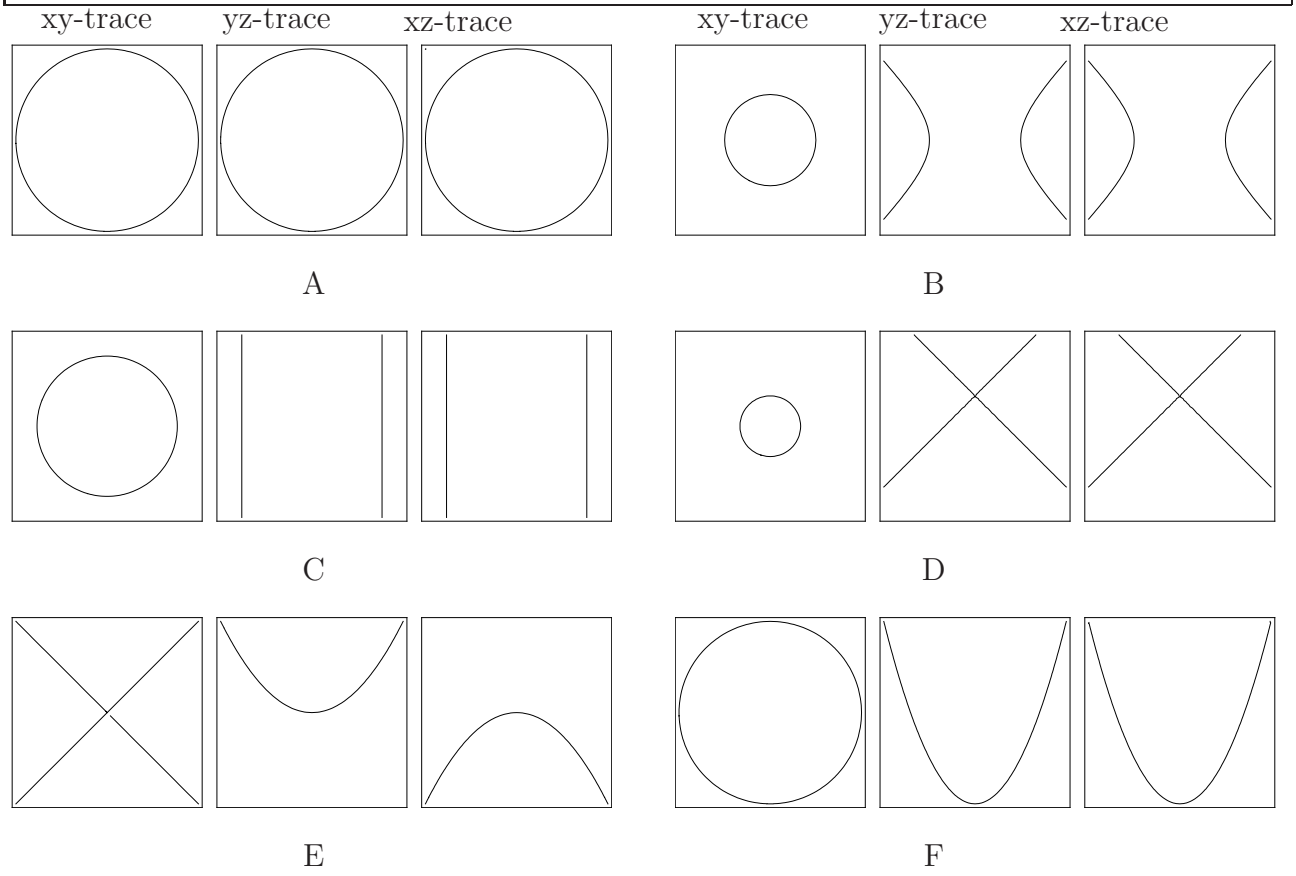
17) | T | F | The vector $\langle -1, 2, 3\rangle$ is perpendicular to the plane $x - 2y - 3z = 9$.

18) | T | F | The curve $\vec{r}(t) = t^3\langle 1, 2, 3\rangle$ is a line.

19) | T | F | The point $(1, 1, -\sqrt{3})$ is in spherical coordinates given by $(\rho, \theta, \phi) = (\sqrt{5}, \pi/4, 2\pi/3)$.

20) | T | F | If the cross product satisfies $(\vec{v} \times \vec{w}) \times \vec{v} = \vec{0}$ then $\vec{v}$ and $\vec{w}$ are orthogonal.

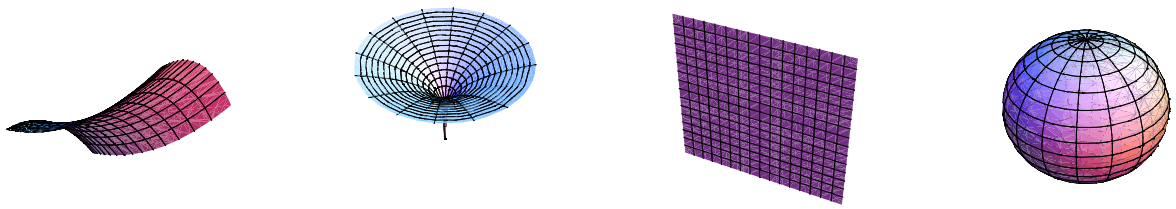
Problem 2a) (6 points)



The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2 + y^2 - (z - 1/3)^2 = 0$
	$x^2 + y^2 + z^2 - 1 = 0$
	$x^2 - y^2 - z = 0$
	$x^2 + y^2 - 1 = 0$
	$x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + y^2 - z = 1$

Problem 2b) (4 points)



I II III IV

Match the parametric surfaces with their parameterization. No justifications are needed.

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u, v) = \langle u^2, v^2, u^4 - v^4 \rangle$
	$\vec{r}(u, v) = \langle \cos(u)\sin(v), 1 + \sin(u)\sin(v), \cos(v) \rangle$
	$\vec{r}(u, v) = \langle v\cos(u), v\sin(u), v^{1/4} \rangle$
	$\vec{r}(u, v) = \langle u, 3, v \rangle$

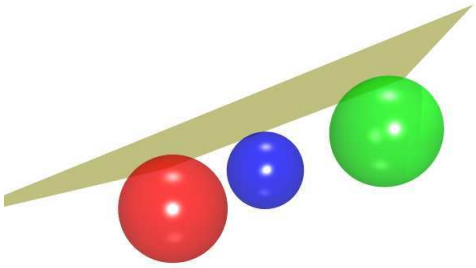
Problem 3) (10 points)

Find the distance of the point $P = (3, 4, 5)$ to the line

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} .$$

Problem 4) (10 points)

Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches each of three spheres from the same side.



Problem 5) (10 points)

Find the arc length of the curve

$$\vec{r}(t) = \langle t^3/3, t^4/2, 2t^5/5 \rangle$$

from $0 \leq t \leq 1$.

Problem 6) (10 points)

An apple at position $(0, 0, 20)$ rests 20 meters above Newton's head, the tip of whose nose is at $(1, 0, 0)$. The apple falls with constant acceleration $\vec{r}\,''(t) = \langle a, 0, -10\rangle$ (where $\langle 0, 0, -10\rangle$ is caused by gravity and $\langle a, 0, 0\rangle$ by the wind) precisely onto the nose of Newton. Find the wind force $\langle a, 0, 0\rangle$ which achieves this. Give a parametrization for the path along which the apple falls.



Problem 7) (10 points)



a) (5 points) A red maple leaf falls to the ground $z = 0$. It falls along the curve $\vec{r}(t) = \langle 3\sqrt{3}\cos(t), 3\sqrt{3}\sin(t), 5-t-4t^2\rangle$. At which angle does it hit the xy-plane?

b) (5 points) Find the tangent line to the curve at the impact point.

Problem 8) (10 points)

a) (5 points) The surface

$$\vec{r}(t, s) = \langle 1 + t + s, 1 - t - 2s, 1 + t - s\rangle$$

with $0 \leq t \leq 1, 0 \leq s \leq 1$ is a parallelogram in space. Find the area of this parallelogram.

b) (5 points) Another surface is given in spherical coordinates by $\rho = 2\sin(\phi)\cos(\theta)$. Write down the equation of this surface in rectangular coordinates as well as in cylindrical coordinates.

Problem 9) (10 points)

a) (5 points) Parametrize the curve obtained by intersecting the surface $z - x^2 + y^3 = 0$ with the cylindrical surface $x^2/4 + 9y^2 = 1$.

b) (5 points) Find the unit tangent vector $\vec{T}$ and the normal vector $\vec{N}(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$ to the curve

$$\vec{r}(t) = \langle 3, t^2, t \rangle$$

at the point $(3, 0, 0)$. What is the binormal vector $\vec{B} = \vec{T} \times \vec{N}$?

Problem 10) (10 points)

a) (4 points) Give a parametrization of the hyperboloid $x^2 + y^2 = z^2 + 1$.

b) (3 points) Give a parametrization of the plane $x + y = 1$.

c) (3 points) Give a parametrization of the ellipsoid $x^2 + y^2 + z^2/4 = 1$.

9/27/2017 FIRST HOURLY PRACTICE 9 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
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5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) T F The length of the sum of two vectors is always the sum of the length of the vectors.

2) T F For any three vectors, $\vec{v} \times (\vec{w} + \vec{u}) = \vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.

3) T F The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

4) T F The surface $\vec{r}(u, v) = \langle \cos(u^2)\sin(v^2), \sin(u^2)\sin(v^2), \cos(v^2) \rangle$ with $0 \leq u < \sqrt{2\pi}, 0 \leq v \leq \sqrt{\pi}$ is a sphere.

5) T F If P, Q, R are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing P, Q, R.

6) T F The line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ hits the plane $2x + 3y + 4z = 9$ at a right angle.

7) T F The function $f(x, y) = \sin(xy)/y$ is continuous everywhere.

8) T F For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

9) T F If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.

10) T F If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

11) T F Every vector contained in the plane $x + y + z = 1$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

12) T F The sphere can in cylindrical coordinates described as $r^2 = 1 - z^2$.

13) T F The curvature of the curve $2\vec{r}(4t)$ at $t = 0$ is twice the curvature of the curve $\vec{r}(t)$ at $t = 0$.

14) T F The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.

15) T F If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.

16) T F Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

17) T F Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

18) T F The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is always in the same plane together with $\vec{v}$ and $\vec{w}$.

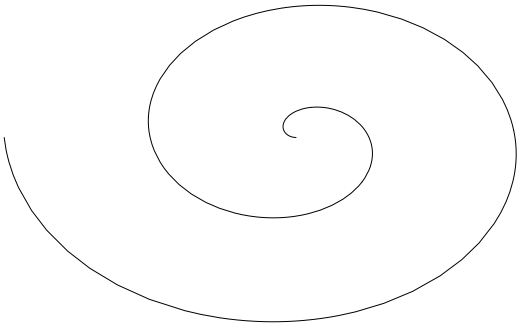
19) T F The line $\vec{r}(t) = \langle 1 + 2t, 1 + 2t, 1 - 4t \rangle$ hits the plane $x + y + z = 9$ at a right angle.

20) T F The intersection of the ellipsoid $x^2/3 + y^2/4 + z^2/3 = 1$ with the plane $y = 1$ is a circle.

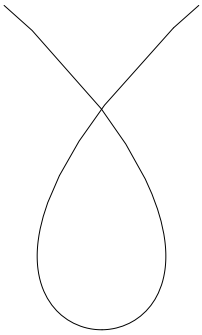
Problem 2a) (3 points)

Match the curves with their parametric definitions.

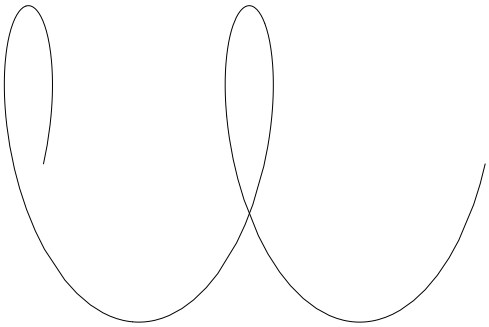
I



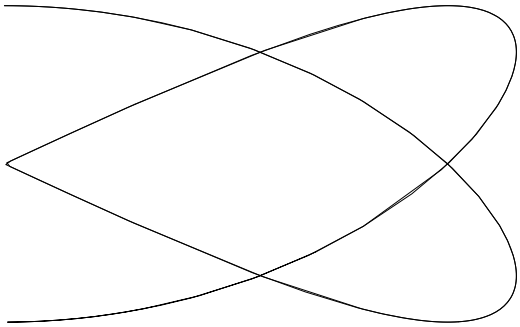
II



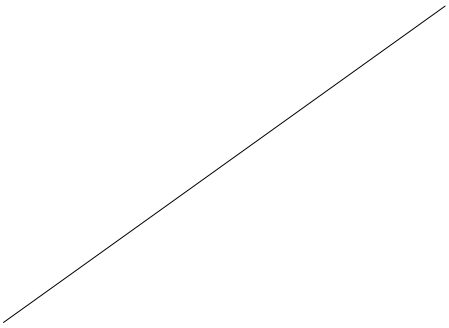
III



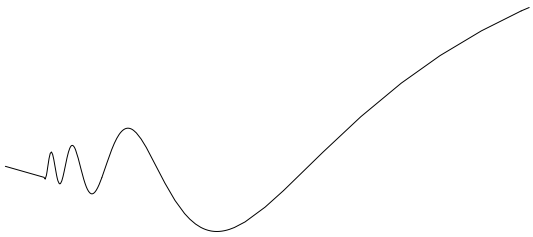
IV



V



VI

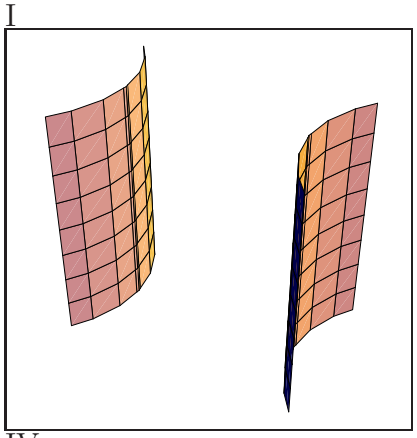


Enter I,II,III,IV,V or VI here	Parametric equation for the curve
	$\vec{r}(t) = \langle t, \sin(1/t)t \rangle$
	$\vec{r}(t) = \langle t^3 - t, t^2 \rangle$
	$\vec{r}(t) = \langle t + \cos(2t), \sin(2t) \rangle$
	$\vec{r}(t) = \langle \lvert \sin(2t) \rvert, \cos(3t) \rangle$
	$\vec{r}(t) = \langle 1 + t, 5 + 3t \rangle$
	$\vec{r}(t) = \langle -t\cos(t), 2t\sin(t) \rangle$

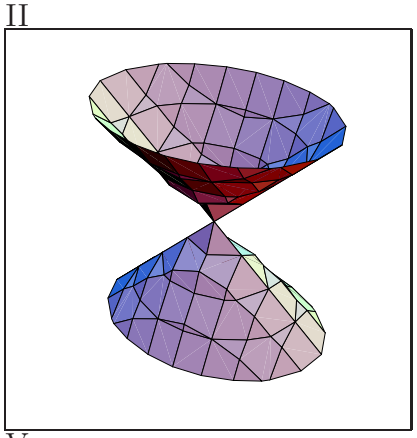
Problem 2b) (3 points)

Match the equations with the surfaces.

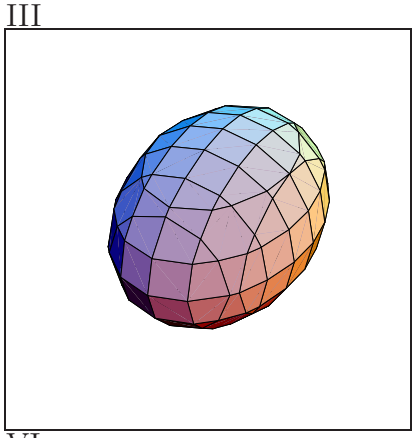
I



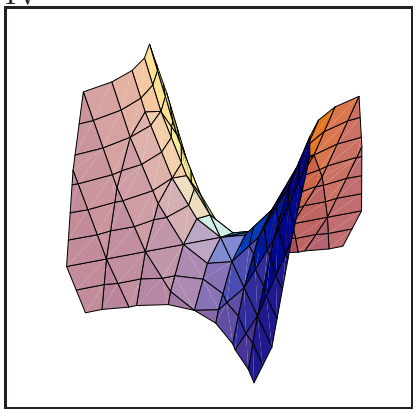
II



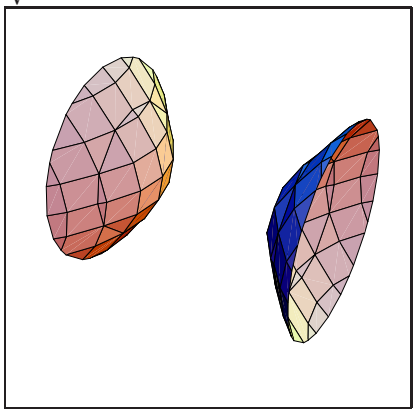
III



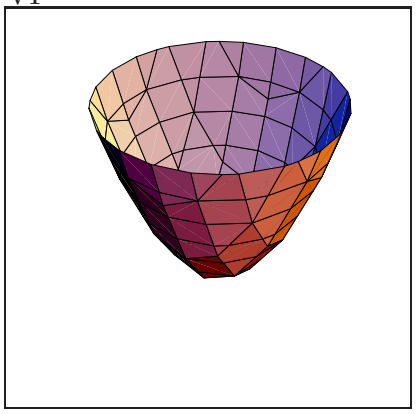
IV



V



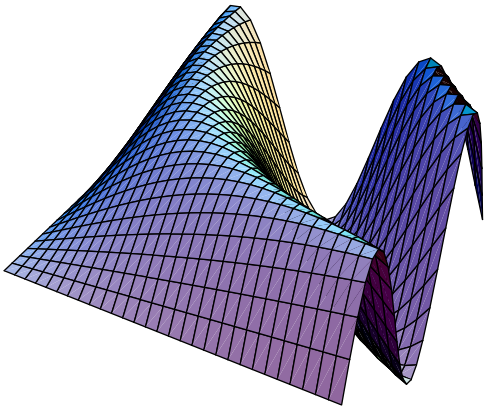
VI



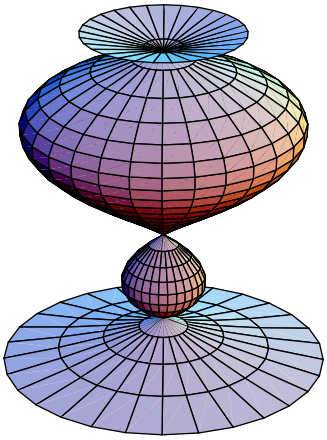
Enter I,II,III,IV,V,VI here	Equation
	$x^2 - y^2 - z^2 = 1$
	$x^2 + 2y^2 = z^2$
	$2x^2 + y^2 + 2z^2 = 1$
	$x^2 - y^2 = 5$
	$x^2 - y^2 - z = 1$
	$x^2 + y^2 - z = 1$

Problem 2c) (4 points)

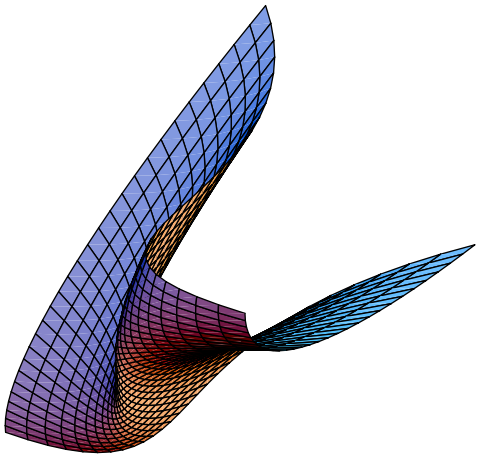
Match the parametric surfaces with their parameterization. No justification is needed.

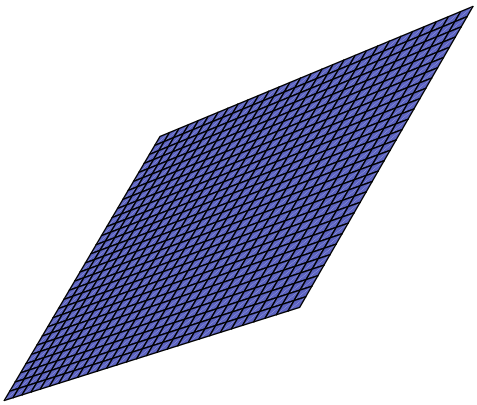


I



II





III

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u,v) = \langle u, v, u+v \rangle$
	$\vec{r}(u,v) = \langle u, v, \sin(uv)) \rangle$
	$\vec{r}(u,v) = \langle 0.2 + u(1-u^2)\cos(v), (0.2 + u(1-u^2))\sin(v), u \rangle$
	$\vec{r}(u,v) = \langle u^3, (u-v)^2, v \rangle$

Problem 3) (10 points)

a) (6 points) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.

b) (4 points) Find a plane perpendicular to that line of intersection.

Problem 4) (10 points)

a) (4 points) Find the area of the parallelogram with vertices $P = (1, 0, 0)$ $Q = (0, 2, 0)$, $R = (0, 0, 3)$ and $S = (-1, 2, 3)$.

b) (3 points) Verify that the triple scalar product has the property $[\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]$.

c) (3 points) Verify that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ has the property

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||$$

Problem 5) (10 points)

Find the distance between the two lines

$$\vec{r}_1(t) = \langle t, 2t, -t \rangle$$

and

$$\vec{r}_2(t) = \langle 1 + t, t, t \rangle \ .$$

Problem 6) (10 points)

Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

Problem 7) (10 points)

The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
c) (3 points) What is the arc length of this curve?

Problem 8) (10 points)

a) (6 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(t), \sin(t), -2t\rangle$ at the point $\vec{r}(0)$.

b) (4 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(5t), \sin(5t), -10t\rangle$ at the point $\vec{r}(0)$.

Hint. Use one of the two formulas for the curvature

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \,,$$

where $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$. The curvatures in b) can be derived from the curvature in a). There is no need to redo the calculation in b) if you give a proper justification.

Problem 9) (10 points)

For each of the following, fill in the blank with $<$ (less than), $>$ (greater than), or $=$ (equal). Justify your answer completely.

1.	The arc length of the curve parameterized by $\vec{f}(t) = \langle \cos 2t, 0, \sin 2t\rangle$, $0 \le t \le \pi$.	_____	The arc length of the curve parameterized by $\vec{g}(u) = \langle 3, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le \sqrt{\pi}$.

2. The arc length of the curve parameterized by $\vec{f}(t) = \langle t^2, 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$. _____ The arc length of the curve parameterized by $\vec{g}(u) = \langle u^4, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le 2\pi$.

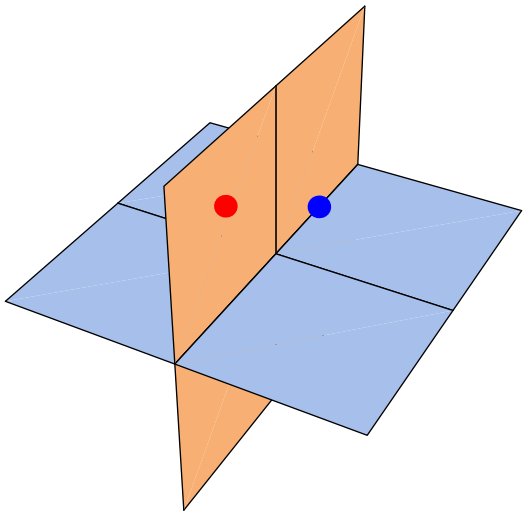
3. The arc length of the curve parameterized by $\vec{f}(t) = \langle 1 + 3t^2, 2 - t^2, 5 + 2t^2\rangle$, $0 \le t \le 1$. _____ The arc length of the curve parameterized by $\vec{g}(u) = \left\langle \frac{1}{2}u^2, u, \frac{2\sqrt{2}}{3}u^{3/2}\right\rangle$, $0 \le u \le 2$.

4. The arc length of the curve parameterized by $\vec{f}(t) = \langle \sin t, \cos t, t\rangle$, $1 \le t \le 5$. _____ The arc length of the curve parameterized by $\vec{g}(u) = \langle u\sin u, u\cos u, u\rangle$, $1 \le u \le 5$.

Problem 10) (10 points)

Given the plane $x + y + z = 6$ containing the point $P = (2, 2, 2)$. Given is also a second point $Q = (3, -2, 2)$.

Find the equation $ax + by + cz = d$ for the plane through P and Q which is perpendicular to the plane $x + y + z = 6$.



9/27/2017 FIRST HOURLY PRACTICE 10 **Math 21a, Fall 2017**

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TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

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- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] For $\vec{N}$ i $\vec{B}\cdot$

2) [T] [F] For any three points P, Q, R in space, $\vec{PQ} \times \vec{PR} = \vec{QP} \times \vec{RP}$

3) [T] [F] The triangle defined by the three points $(-1, 0, 2), (-4, 2, 1), (1, -1, 2)$ has a right angle.

4) [T] [F] The function $f(x, y, z) = x^2 + y^2 + z^2 / \sin(x^2 + y^2 + z^2)$ is continuous everywhere in space.

5) [T] [F] $\vec{u} \times \vec{u} = 0$ implies $\vec{u} = \vec{0}$.

6) [T] [F] The level curves $f(x, y) = 1$ and $f(x, y) = 2$ of a smooth function f never intersect.

7) [T] [F] For any vector $\vec{v}$, we have $\text{proj}_{\vec{i}}(\text{proj}_{\vec{j}}(\vec{v})) = \vec{0}$.

8) [T] [F] $(\vec{j} \times \vec{i}) \times \vec{i} = \vec{k} \times (\vec{i} \times \vec{k})$

9) [T] [F] If a parametrized curve $\vec{r}(t)$ lies in a plane and the velocity $\vec{r}\,'(t)$ is never zero, then the normal vector $\vec{N}(t)$ also lies in that plane.

10) [T] [F] The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

11) [T] [F] If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $proj_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

12) [T] [F] A line intersects an ellipsoid in at most 2 distinct points.

13) [T] [F] For any vectors $\vec{v}$ and $\vec{w}$, the formula $(\vec{v} - \vec{w}) \cdot \vec{P}_{\vec{w}}(\vec{v}) = 0$ holds.

14) [T] [F] Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on S. If $\vec{n} \cdot \vec{PQ} = 0$, then P and Q lie on the same side of S.

15) [T] [F] The vectors $\langle 2, 2, 1\rangle$ and $\langle 1, 1, -4\rangle$ are perpendicular.

16) [T] [F] $\|\vec{v} \times \vec{w}\| = \|v\|\|w\| \cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$.

17) [T] [F] The vector $\vec{i} \times (\vec{j} \times \vec{k})$ has length 1.

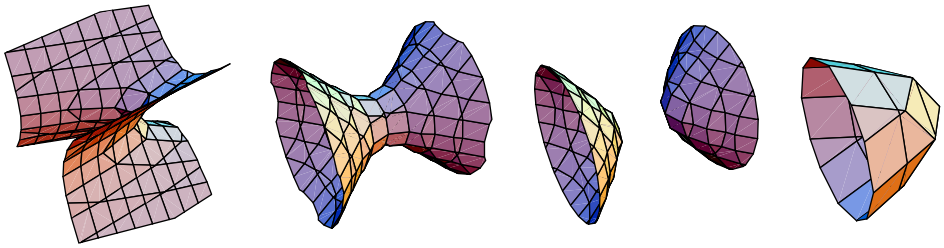
18) [T] [F] The distance between the z-axis and the line $x - 1 = y = 0$ is 1.

19) [T] [F] There is a quadric surface which both hyperbola and parabola appear as traces. Traces are intersections of the surface with the coordinate planes $x = 0$, $y = 0$, or $z = 0$.

20) [T] [F] The equation $x^2 + y^2 - z^2 = -1$ defines a one-sheeted hyperboloid.

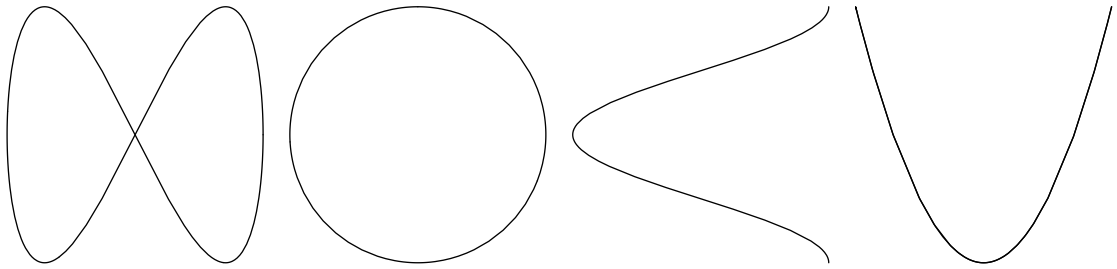
Problem 2) (10 points)

Match the equation with the pictures. No justifications are necessary in this problem.



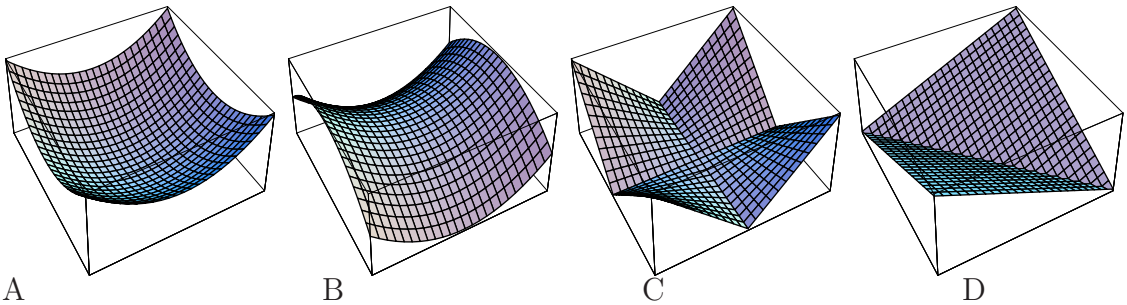
I II III IV

Enter I,II,III,IV here	Equation
	$x + y^2 - z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 + 1 = 0$
	$-x + y^2 + z^2 + 1 = 0$



1 2 3 4

Enter 1,2,3,4 here	Equation
	$\langle \cos(t), \sin(t) \rangle$
	$\langle \cos(t), t \rangle$
	$\langle \cos(t), \cos^2(t) \rangle$
	$\langle \cos(t), \sin(2t) \rangle$



A B C D

Enter A,B,C,D here	Equation
	$f(x, y) = x^2 - y^2$
	$f(x, y) = \lvert x + y \rvert$
	$f(x, y) = x^2 + y^2$
	$f(x, y) = \lvert xy \rvert$

Problem 3) (10 points)

Imagine the planet Earth as the unit sphere in 3D space centered at the origin. An asteroid is approaching from the point $P = (0, 4, 3)$ along the path

$$\vec{r}(t) = \langle (4-t)\sin(t), (4-t)\cos(t), 3-t \rangle \ .$$



a) When and where will it first hit the Earth?

b) What velocity will it have at the impact?

Problem 4) (10 points)

Find the distance between the cylinder $x^2 + y^2 = 1$ and the line

$$L : \frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2}.$$

Problem 5) (10 points)

a) Find a parametrization $\vec{r}(t)$ of the line which is the intersection of the two planes

$$4x + 6y - z = 1$$

and

$$4x + z = 0 \ .$$

b) Find the point on the line which is closest to the origin.

Problem 6) (10 points)

Consider the parameterized curve

$$\vec{r}(t) = \langle e^t + e^{-t}, 2\cos(t), 2\sin(t) \rangle \ .$$

Find the arc length of this curve from $t = 0$ to $t = 4$.

Problem 7) (10 points)

The set of points P for which the distance from P to $A = (1, 2, 3)$ is equal to the distance from P to $B = (5, 8, 5)$ forms a plane S.

a) Find the equation $ax + by + cz = d$ of the plane S.

b) Find the distance from A to S.

Problem 8) (10 points)

The Swiss tennis player Roger Federer hits the ball at the point $\vec{r}(0) = (0, 0, 3)$. The initial velocity is $\vec{r}\,'(0) = \langle 100, 10, 13\rangle$. The tennis ball experiences a constant acceleration $\vec{r}\,''(t) = \langle 2, 0, -32\rangle$ which is due to the combined force of gravity and a constant wind in the x direction.



a) Where does the tennis ball hit the ground $z = 0$?

b) What is the z-component = (projection onto z vector) $proj_{\vec{k}}(\vec{r}\,'(t))$ of the ball velocity at the impact?

Problem 9) (10 points)

a) (4 points) Parameterize the intersection of the ellipsoid

$$\frac{x^2}{4} + \frac{(y-5)^2}{4} + \frac{z^2}{9} = 2$$

with the plane $z = 3$.

b) (3 points) Parametrize the ellipsoid itself in the form

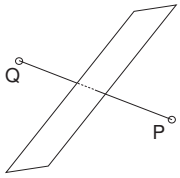
$$\vec{r}(\theta, \phi) = ...$$

c) (3 points) What is the curvature of the curve at the point $(2, 5, 3)$?

Hint. While you can use the curvature formula $\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$ you are also allowed to cite a fact which you know about the curvature.

Problem 10) (10 points)

Find an equation $ax + by + cz = d$ for the plane which has the property that $Q = (5, 4, 5)$ is the reflection of $P = (1, 2, 3)$ through that plane.



9/27/2017 FIRST HOURLY PRACTICE 11 **Math 21a, Fall 2017**

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- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The $(\rho, \theta$

2) [T] [F] If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$.

3) [T] [F] The surface $z^2 + 4y^2 = x^2 + 1$ is a two sheeted hyperboloid.

4) [T] [F] The surface $4x^2 - 4x + y^2 - 2y - 120 = -z^2$ is an ellipsoid.

5) [T] [F] The parametrized lines $\vec{u}(t) = \langle 1 + 2t, 2 - 5t, 1 + t \rangle$ and $\vec{v}(t) = \langle 3 - 4t, -3 + 10t, 2 - 2t \rangle$ are the same line.

6) [T] [F] The surface $\sin(x) = z$ contains lines which are parallel to the y-axis.

7) [T] [F] If $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{w} = 0$ and $\vec{v}$ is not the zero vector, then $\vec{u} \cdot \vec{w} = 0$.

8) [T] [F] The curvature of a curve depends upon the speed at which one travels upon it.

9) [T] [F] Two lines in space that do not intersect must be parallel.

10) [T] [F] A line in space can intersect an elliptic paraboloid in 4 points.

11) [T] [F] If $\vec{u} \times \vec{v} = 0$ and $\vec{u} \cdot \vec{v} = 0$, then one of the vectors $\vec{u}$ and $\vec{v}$ is zero.

12) [T] [F] If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel at time $t = 1$, then the curvature $\kappa(t)$ of the curve is zero at time $t = 1$.

13) [T] [F] If the speed of a parametrized curve is constant over time, then the curvature of the curve $\vec{r}(t)$ is zero.

14) [T] [F] The length of the vector projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the length of the vector projection of $\vec{w}$ onto $\vec{v}$.

15) [T] [F] A quadric $ax^2 + by^2 + cz^2 = 1$ is contained in the interior of a sphere $x^2 + y^2 + z^2 < 100$, then the constants a, b, c are all positive and the quadric is an ellipsoid.

16) [T] [F] There is a hyperboloid of the form $ax^2 + by^2 - cz^2 = 1$ which has a trace which is a parabola.

17) [T] [F] The set of points in space which have distance 1 from the line $x = y = z$ form a cylinder.

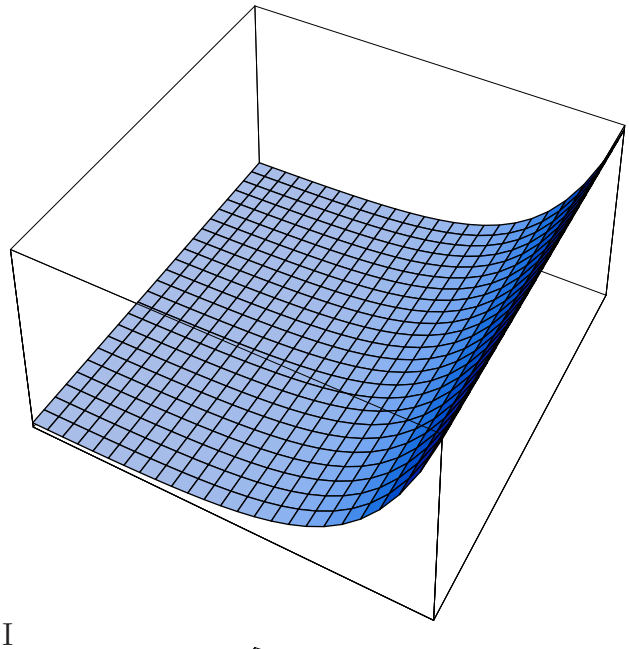
18) [T] [F] The velocity vector of a parametric curve $\vec{r}(t)$ always has constant length.

19) [T] [F] The volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u} \times \vec{v}) \times \vec{w}|$.

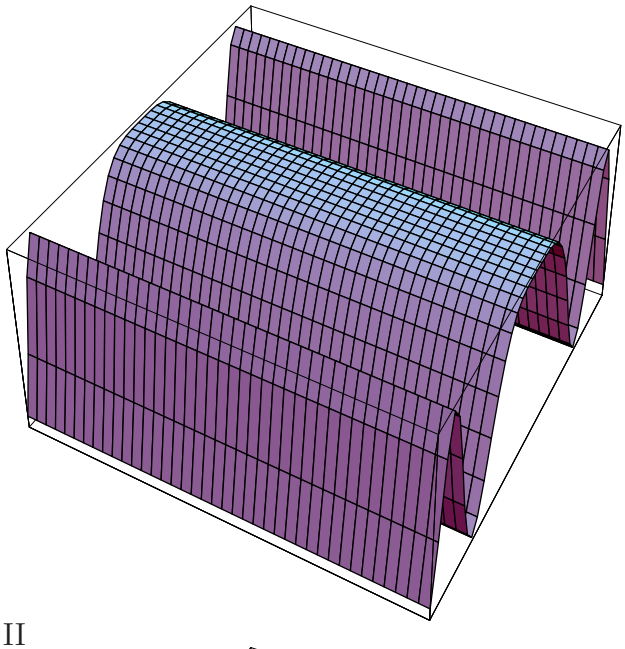
20) [T] [F] The equation $x^2 + y^2/4 = 1$ in space describes an ellipsoid.

Problem 2a) (3 points)

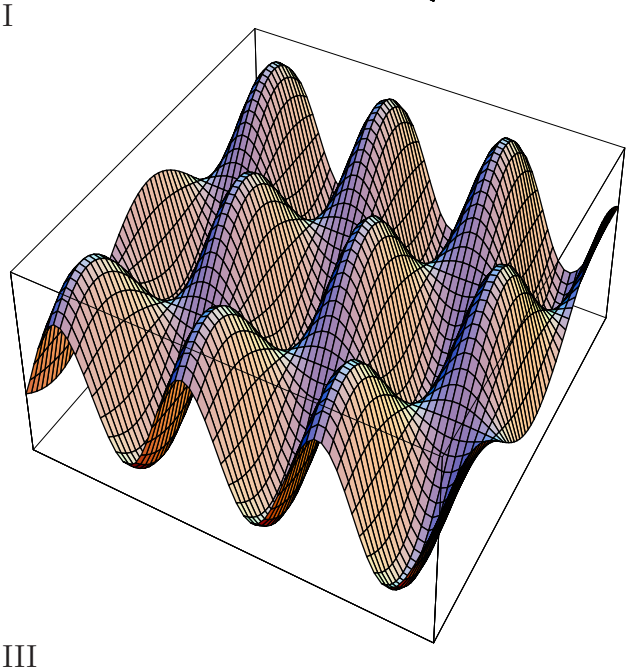
Match the equation with their graphs. No justifications are needed.



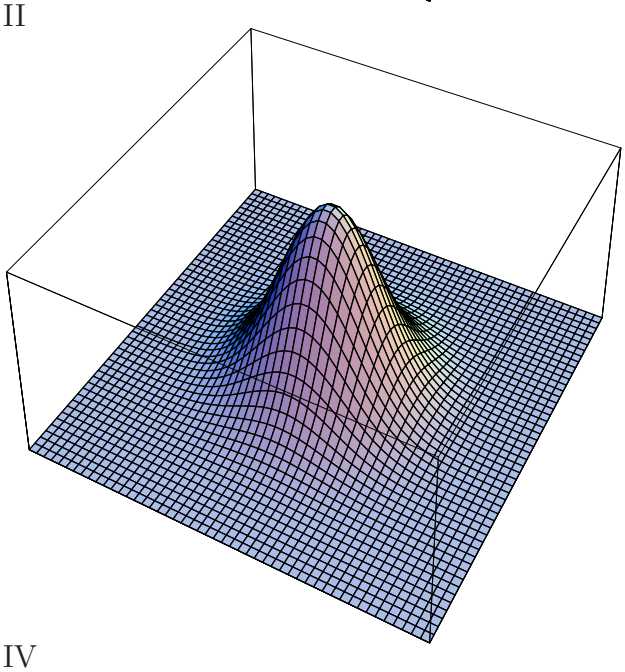
I



II



III



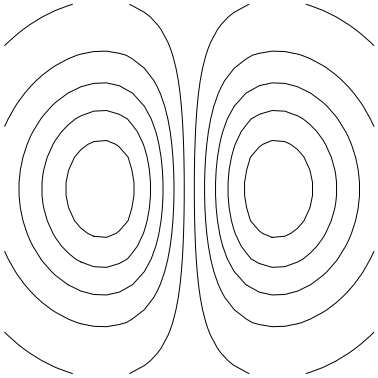
IV

Enter I,II,III,IV here	Equation
	$z = \sin(5x)\cos(2y)$
	$z = \cos(y^2)$
	$z = e^{-x^2-y^2}$
	$z = e^x$

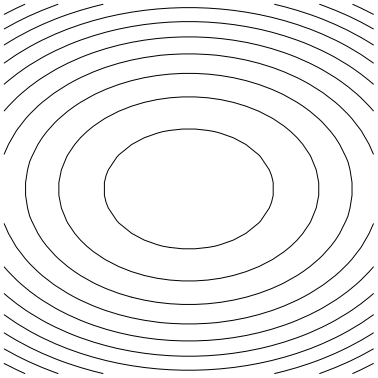
Problem 2b) (4 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

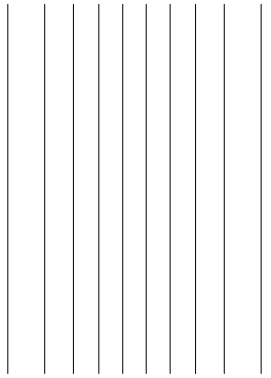
I



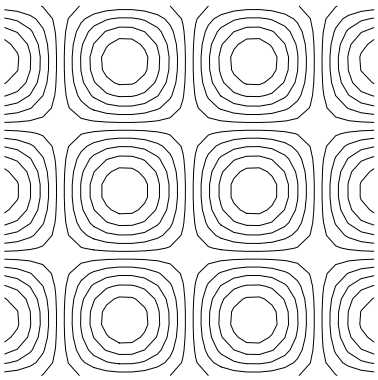
II



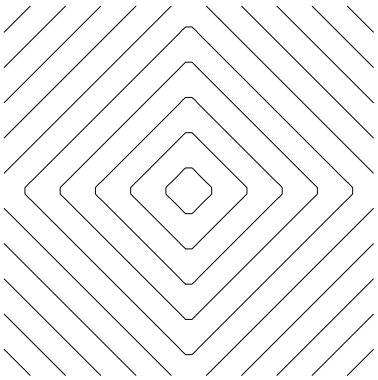
III



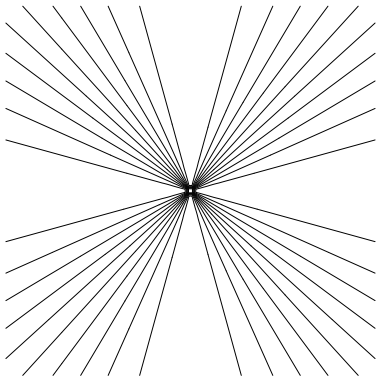
IV



V



VI



Enter I,II,III,IV,V or VI here	Function $f(x, y)$
	$f(x, y) = \sin(x)$
	$f(x, y) = x^2 + 2y^2$
	$f(x, y) = \lvert x\rvert + \lvert y\rvert$
	$f(x, y) = \sin(x)\cos(y)$
	$f(x, y) = xe^{-x^2-y^2}$
	$f(x, y) = x^2/(x^2 + y^2)$

Problem 2c) (3 points)

Match the following points in cartesian coordinates with the points in spherical coordinates:

a) $(x, y, z) = (\sqrt{2}, 0, 0)$
b) $(x, y, z) = (0, \sqrt{2}, 0)$
c) $(x, y, z) = (0, 0, \sqrt{2})$
d) $(x, y, z) = (1, 1, 0)$
e) $(x, y, z) = (1, 0, 1)$
f) $(x, y, z) = (0, 1, 1)$

1) $(\rho, \phi, \theta) = (\sqrt{2}, 0, 0)$.
2) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/4)$.
3) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, 0)$.
4) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/2)$.
5) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$.
6) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, 0)$.

Problem 3) (10 points)

a) (7 points) Find a parametric equation for the line which is the intersection of the two planes $2x - y + 3z = 9$ and $x + 2y + 3z = -7$.

b) (3 points) Find a plane perpendicular to both planes given in a) which has the additional property that it passes through the point $P = (1, 1, 1)$.

Problem 4) (10 points)

Given the vectors $\vec{v} = \langle 1, 1, 0 \rangle$ and $\vec{w} = \langle 0, 0, 1 \rangle$ and the point $P = (2, 4, -2)$. Let Σ be the plane which goes through the origin $(0, 0, 0)$ and which contains the vectors $\vec{v}$ and $\vec{w}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

a) (6 points) Compute the distance from P to the plane Σ.

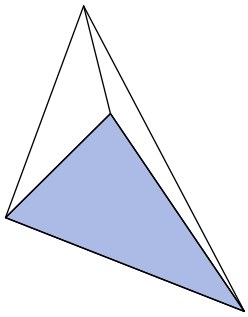
b) (4 points) Find the shortest distance from P to the sphere S.

Problem 5) (10 points)

a) (6 points) Find an equation for the plane through the points $A = (0, 1, 0), B = (1, 2, 1)$ and $C = (2, 4, 5)$.

b) (4 points) Given an additional point $P = (-1, 2, 3)$, what is the volume of the tetrahedron which has A, B, C, P among its vertices.

A useful fact which you can use without justification in b): the volume of the tetrahedron is 1/6 of the volume of the parallelepiped which has AB, AC, and AP among its edges.



Problem 6) (10 points)

The parametrized curve $\vec{u}(t) =< t, t^2, t^3 >$ (known as the "twisted cubic") intersects the parametrized line $\vec{v}(s) = \langle 1+3s, 1-s, 1+2s \rangle$ at a point P. Find the angle of intersection.

Problem 7) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (\log(t), 2t, t^2)$, where $\log(t)$ is the natural logarithm (denoted by $\ln(t)$ in some textbooks).

a) What is the velocity and what is the acceleration at time $t = 1$?

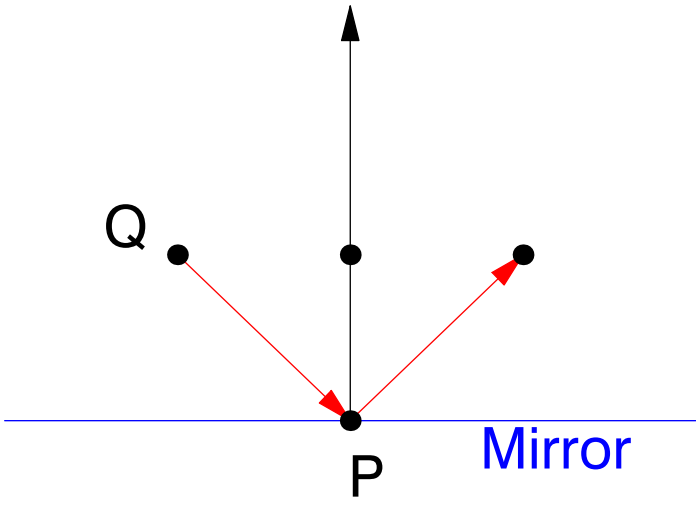
b) Find the length of the curve from $t = 1$ to $t = 2$.

Problem 8) (10 points)

A planar mirror in space contains the point $P = (4, 1, 5)$ and is perpendicular to the vector $\vec{n} = \langle 1, 2, -3 \rangle$. The light ray $\vec{QP} = \vec{v} = \langle -3, 1, -2 \rangle$ with source $Q = (7, 0, 7)$ hits the mirror plane at the point P.

a) (4 points) Compute the projection $\vec{u} = \vec{\mathrm{P}}_{\vec{n}}(\vec{v})$ of $\vec{v}$ onto $\vec{n}$.

b) (6 points) Identify $\vec{u}$ in the figure and use it to find a vector parallel to the reflected ray.



Problem 9) (10 points)

We know the acceleration $\vec{r}\,''(t) = \langle 2, 1, 3\rangle + t\langle 1, -1, 1\rangle$ and the initial position $\vec{r}(0) = \langle 0, 0, 0\rangle$ and initial velocity $\vec{r}\,'(0) = \langle 11, 7, 0\rangle$ of an unknown curve $\vec{r}(t)$. Find $\vec{r}(6)$.

Problem 10) (10 points)

Intersecting the elliptic cylinder $x^2 + y^2/4 = 1$ with the plane $z = \sqrt{3}x$ gives a curve in space.

a) (3 points) Find the parametrization of the curve.

b) (3 points) Compute the unit tangent vector $\vec{T}$ to the curve at the point $(0, 2, 0)$.

c) (4 points) Write down the arc length integral and evaluate the arc length of the curve.

9/27/2017 FIRST HOURLY PRACTICE 12 **Math 21a, Fall 2017**

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TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

2) [T] [F] The length of the unit tangent vector $\vec{T}$ for a curve $\vec{r}(t)$ is independent of t.

3) [T] [F] For all vectors $\vec{v}$ and $\vec{w}$ the vector $\vec{w} \times (\vec{w} \times \vec{v})$ is perpendicular to $\vec{v}$.

4) [T] [F] There is a point (x, y, z) in space, for which the cylindrical coordinates (r, θ, z) and spherical coorindates (ρ, θ, ϕ) satisfy $(r, \theta, z) = (\rho, \theta, \phi - \pi/2)$.

5) [T] [F] The two planes $x + y - z = 1$ and $-x - y + z = 2$ intersect in a line.

6) [T] [F] $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$ implies $|u| = |v|$.

7) [T] [F] The contour curves $\sin(x) + y = 1$ and $\sin(x) + y = 2$ do not intersect.

8) [T] [F] There is a vector $\vec{v}$ for which the vector projection $\text{proj}_{\vec{v}}(\vec{j})$ is equal to $2\vec{j}$.

9) [T] [F] $(\vec{k} \times \vec{i}) \times \vec{i} = \vec{j} \times (\vec{i} \times \vec{k})$

10) [T] [F] If a curve $\vec{r}(t)$ lies in a plane, goes through the point $(1, 1, 1)$, and has the binormal vector $\vec{B}(t) = \langle 3, 4, 5 \rangle$, then the plane is $3x + 4y + 5z = 12$.

11) [T] [F] The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

12) [T] [F] A line intersects a hyperbolic paraboloid always in 2 distinct points.

13) [T] [F] There is a quadric surface, each of whose intersections with the coordinate planes is either an ellipse or a parabola.

14) [T] [F] The equation $x^2 - y^2 - z^2 = 1$ defines a one-sheeted hyperboloid.

15) [T] [F] The function $f(x, y) = 1/(1 + x^2 + y^2)$ is continuous everywhere.

16) [T] [F] If the number $\vec{u} \cdot (\vec{v} \times \vec{w})$ is positive, then $(\vec{w} \times \vec{v}) \cdot \vec{u}$ is positive.

17) [T] [F] The number $|\vec{u} \times (\vec{v} \times \vec{w})|$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

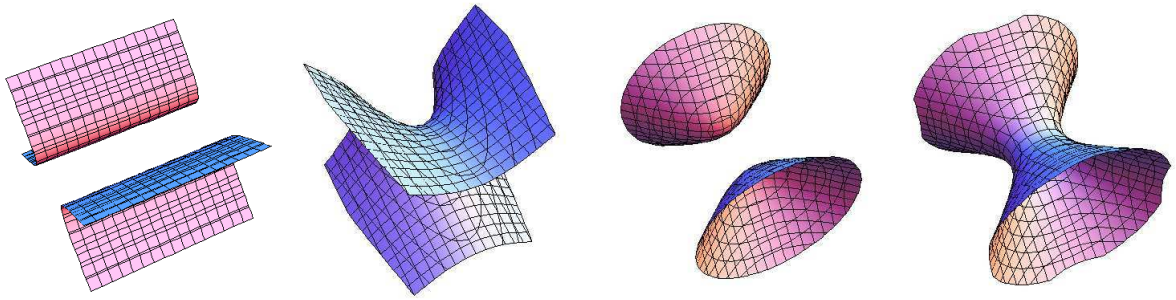
18) [T] [F] The set of points P for which the distance of P to the point $(0, 0, 0)$ is 1 less than the distance to the point $(0, 0, 2)$ is a paraboloid.

19) [T] [F] If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $\text{proj}_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

20) [T] [F] The number $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

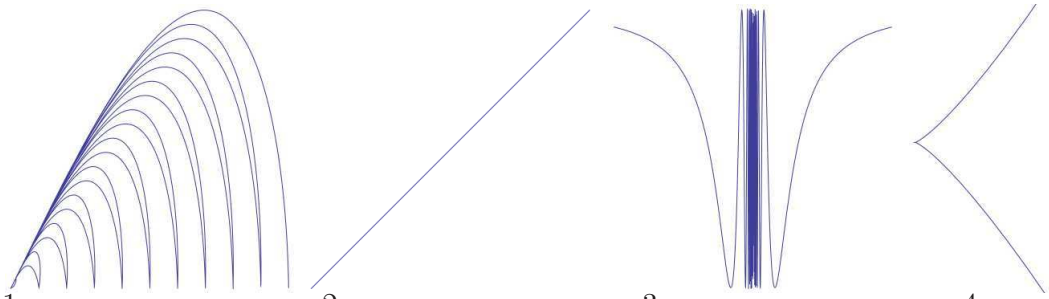
Problem 2a) (5 points)

Match the equations with the pictures. No justifications are necessary in this problem.



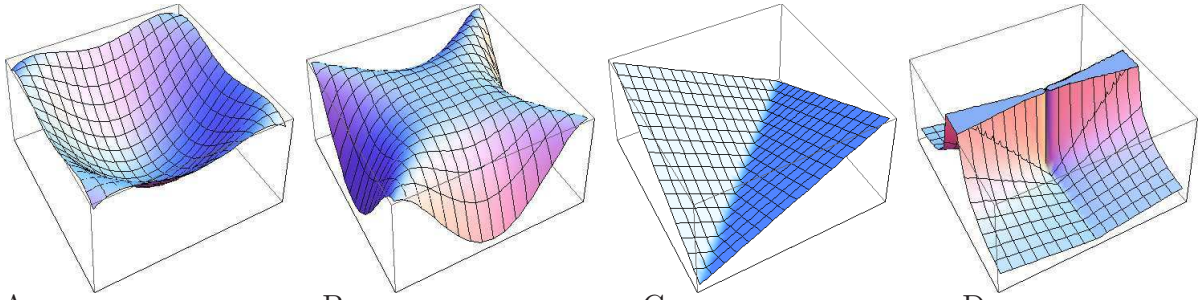
I II III IV

Enter I,II,III,IV here	Equation
	$x^2 + y - z^2 - 1 = 0$
	$y^2 - 2z^2 - 1 = 0$
	$x^2 - y^2 + z^2 + 1 = 0$
	$x^2 - y^2 + z^2 - 1 = 0$



1 2 3 4

Enter 1,2,3,4 here	Equation
	$\langle t^4, 1 + t^5 \rangle$
	$\langle t\cos(5t), t\cos(5t) \rangle$
	$\langle \lvert t\cos(5t)\rvert, \lvert t\sin(10t)\rvert \rangle$
	$\langle 3 + 2t, \cos(1/t) \rangle$

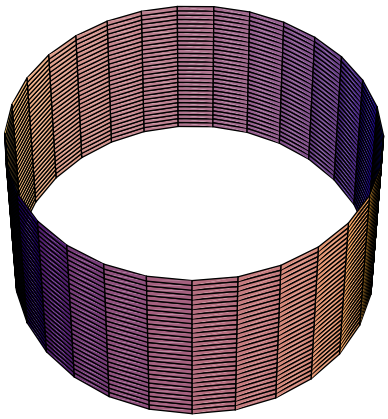


A B C D

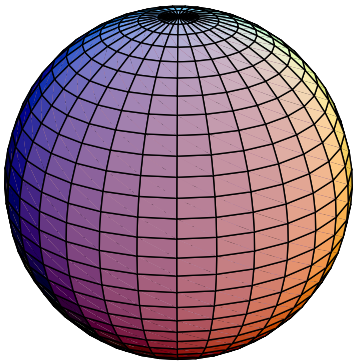
Enter A,B,C,D here	Equation
	$f(x, y) = \lvert x/y\rvert$
	$f(x, y) = \sin(x^2 + y^2)$
	$f(x, y) = \lvert x - y\rvert$
	$f(x, y) = \cos(x^2 - y^2)$

Problem 2b) (5 points)

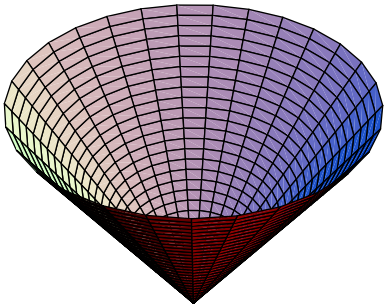
Match the surfaces with their parametrizations as well as with the description either in cylindrical coordinates (r, θ, z) or in spherical coordinates (ρ, ϕ, θ).



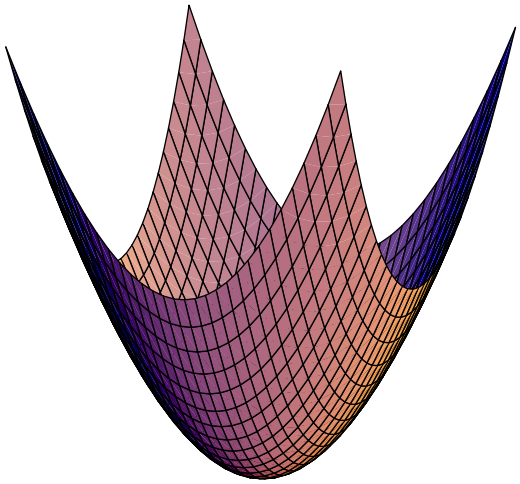
I



II



III



IV

Enter I,II,III,IV here	Parametrization of the surface
	$\langle 3\cos(\theta), 3\sin(\theta), 2z\rangle$.
	$\langle x, y, 3x^2 + 3y^2\rangle$
	$\langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$
	$\langle 3z\cos(\theta), 3z\sin(\theta), 2z\rangle$

Enter I,II,III,IV here	Description in cylindrical or spherical coordinates
	$z = 3r^2$
	$3z = 2r$
	$\rho = 3$
	$r = 3$

Problem 3) (10 points)

A tetrahedron has the vertices $A = (1, 1, 0), B = (3, 2, 0), C = (2, 1, 1), D = (3, 2, 1)$ with base triangle A, B, C.

a) (5 points) Find the height of the tetrahedron.

b) (5 points) The volume of a tetrahedron is the base area times height divided by 3. What is the volume of the tetrahedron with vertices A,B,C,D.

Problem 4) (10 points)

Find a parametrization of the line containing the two planes

$$2x + y + z = 4$$

and

$$x - y + 2z = 5 \ .$$

Problem 5) (10 points)

What is the distance between the two cylinders $x^2 + y^2 = 1$ and $(z - 2)^2 + (x - 5)^2 = 4$?

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle 2\sin(t), \frac{t^4}{4} + \frac{1}{2t^2}, 2\cos(t) \rangle$$

from $t = 1$ to $t = 2$.

Problem 7) (10 points)

At time $t = 0$ two trapeze artists have positions $\vec{r}(0) = \langle 0, 0, 25\rangle$ and $\vec{s}(0) = \langle 10, 0, 23\rangle$ and velocities $\vec{r}\,'(0) = \langle 2, 0, 1\rangle$ and $\vec{s}\,'(0) = \langle -3, 0, 2\rangle$. They both experience a constant gravitational acceleration $\langle 0, 0, -10\rangle$. Find the paths $\vec{r}(t), \vec{s}(t)$ and determine at which point the artists meet.



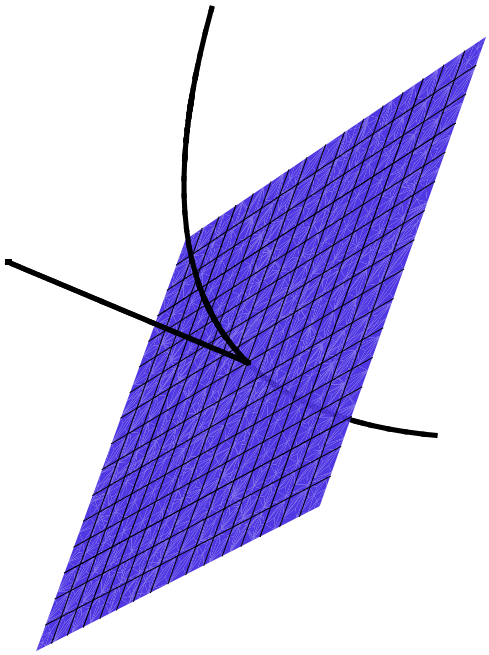
Problem 8) (10 points)

The angle between a curve and a plane is defined as $\pi/2 - \alpha$, where α is the angle between the normal vector to the plane and the velocity vector of the curve at the point of intersection.

a) (3 points) Find a normal vector to the plane $x + y - z/2 = 1$.

b) (3 points) What is the velocity vector to the curve $C : \vec{r}(t) = \langle 1, 0, 0\rangle + t\langle 1, 1, 3\rangle + t^2\langle 1, 1, 1\rangle$ at time $t = 0$?

c) (4 points) Find the angle (in radians) between the plane $x + y - z/2 = 1$ and the curve C at the point of intersection $\vec{r}(0)$.



Problem 9) (10 points)

The intersection of the paraboloid

$$x^2 + y^2 - z = 5$$

with the plane

$$x + y = 5$$

is a curve. Find the parametrization of this curve.

Problem 10) (10 points)

a) (3 points) Parametrize the plane containing the three points $A = (1, 1, 1), B = (1, 3, 2)$ and $C = (3, 4, 5)$.
b) (4 points) Parametrize the sphere which is centered at $(1, 1, 1)$ and has radius 3.
c) (3 points) Parametrize the surface which is given in spherical coordinates as $\rho = 3 + \sin(\phi)\sin(\theta)$.

11/1/2017 Second Hourly **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

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6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) T F The identity $f_{yxyx} = f_{xyxy}$ holds for all smooth functions $f(x, y)$.

2) T F Using linearization we can estimate $(1.003)^2(1.0001)^4 \approx 2 \cdot 0.003 + 4 \cdot 0.0001$.

3) T F We have $d/dt(x^2(t)y(t)) = \langle 2x(t)y(t), x^2(t)\rangle \cdot \langle x'(t), y'(t)\rangle$.

4) T F The function $f(x, y) = 3y^2 - 2x^3$ takes no maximal value on the "squircle" $x^4 + y^4 = 8$.

5) T F If $f(x, t)$ solves the heat equation then $f(x, -t)$ solves the heat equation.

6) T F If $f(x, t)$ solves the wave equation, then $f(x, -t)$ solves the wave equation.

7) T F There exists a smooth function on the region $x^2 + y^2 < 1$ so that it has exactly two local minima and no other critical points.

8) T F For a function $f(x, y)$, the vector $\langle f_x(0, 0), f_y(0, 0), -1\rangle$ is perpendicular to the graph $f(x, y) = z$ at $(0, 0, f(0, 0))$.

9) T F If a function $f(x, y)$ is equal to its linearization $L(x, y)$ at some point, then $f_{xx}(x, y) = f_{yy}(x, y)$ at every point.

10) T F The equation $f(x, y) = 9x - 5x^2 - y^2 = -9$ implicitly defines $y(x)$ near $(0, 3)$ and $y'(0) = f_x(0, 3)/f_y(0, 3)$.

11) T F If a tangent plane to a surface S intersects S at infinitely many points, then S must be a plane.

12) T F If $\vec{u} = \langle 1, 0, 0\rangle$ and $\vec{v} = \langle 0, 1, 0\rangle$ then $(D_{\vec{u}}D_{\vec{v}} - D_{\vec{v}}D_{\vec{u}})f = D_{\vec{u}\times\vec{v}}f$.

13) T F The surface area of the parametrized surface $\vec{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ is $\int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta| r\,dr\,d\theta$.

14) T F Let D be the unit disk $x^2 + y^2 \le 1$. Any function $f(x, y)$ which satisfies $|\iint_D f(x, y)\,dA| = \iint_D |f(x, y)|\,dA$ must have $f(x, y) \ge 0$ on D.

15) T F The iterated integral $\int_{-1}^{1}\int_{10}^{20} e^{x^2}y^{11}\,dx\,dy$ is zero.

16) T F The tangent plane to the graph of $f(x, y) = xy$ at $(2, 3, 6)$ is given by $6 + 3(x - 2) + 3(y - 3) = 0$.

17) T F If the gradient of $f(x, y)$ at $(1, 2)$ is zero, then $f(1, 2)$ must be either a local minimum or maximum value of $f(x, y)$ at $(1, 2)$.

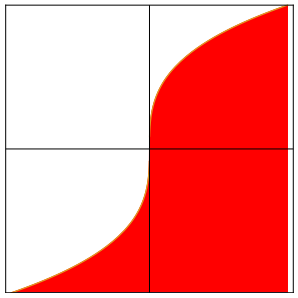
18) T F If $\vec{r}(t)$ is a parametrization of the level curve $f(x, y) = 5$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

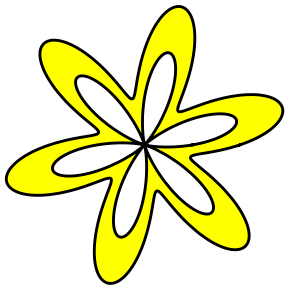
19) T F The function $f(x, y) = (x^3 + y^3)/(x^2 + y^2)^2$ has a limiting value at $(0, 0)$ so that it is continuous everywhere.

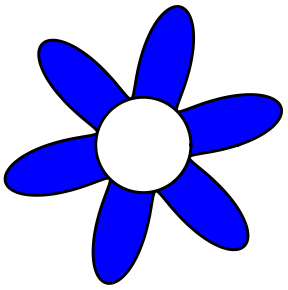
20) T F If the contour curves $f(x, y) = 1$ and $g(x, y) = 1$ have a common tangent line at $(1, 2)$ and $|\nabla f(1, 2)| = 1 = |\nabla g(1, 2)| = 1$, then $(1, 2)$ is a solution to the Lagrange equations for extremizing f under the constraint $g = 1$.

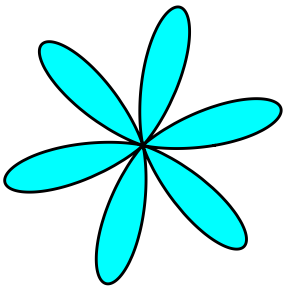
Problem 2) (10 points) No justifications needed

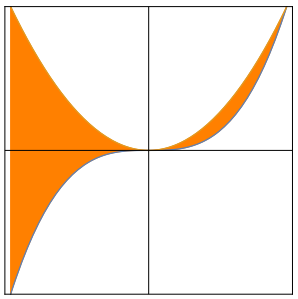
a) (6 points) Please match each picture below with the double integral that computes the area of the region:

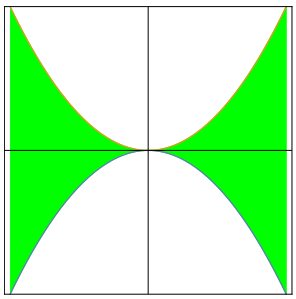
A 

B 

C 

D 

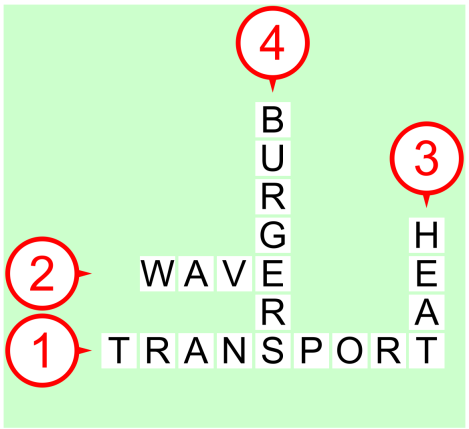
E 

F 

Enter A-F	Integral
	$\int_0^{2\pi} \int_{1+\sin(6\theta)}^{2+\sin(6\theta)} r \; dr d\theta$
	$\int_{-1}^{1} \int_{-x^2}^{x^2} 1 \; dy dx$
	$\int_0^{2\pi} \int_0^{1+\sin(6\theta)} r \; dr d\theta$
	$\int_{-1}^{1} \int_{y^3}^{1} 1 \; dx dy$
	$\int_{-1}^{1} \int_{x^3}^{x^2} 1 \; dy dx$
	$\int_0^{2\pi} \int_1^{2+\sin(6\theta)} r \; dr d\theta$

b) (4 points) We design a crossword puzzle. Match the PDEs:

Enter 1-4	
	$u_t = u_x$
	$u_t = u_{xx}$
	$u_{tt} = u_{xx}$
	$u_t + uu_x = u_{xx}$



Problem 3) (10 points)

3a) (7 points) Fill in the points A-G. There is an exact match. You see the level curves of a function $f(x, y)$ inspired from one of your homework submissions. The circular curve is $g(x, y) = x^2 + y^2 = 1$.

a) At the point [ ], the function f is a global maximum on $g = 1$.

b) At the point [ ], the function f is a global minimum on $g = 1$.

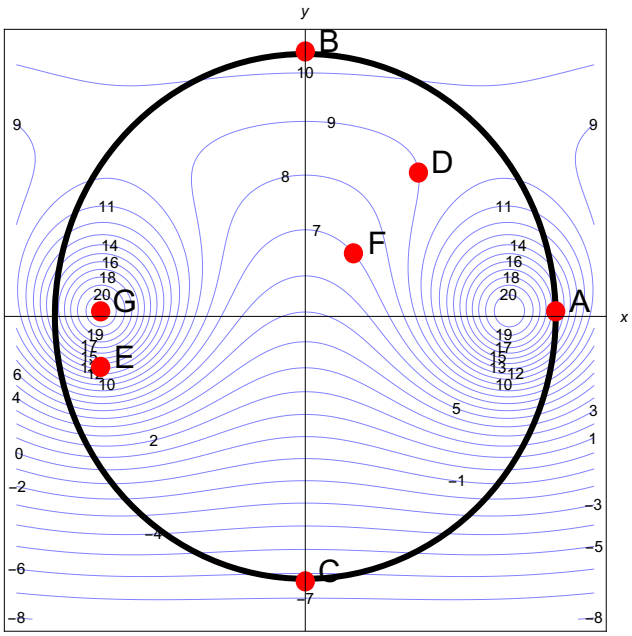
c) At the point [ ], $|f_y|$ is maximal among all points A-G.

d) At the point [ ], $f_x > 0$ and $f_y = 0$.

e) At the point [ ], $f_x > 0$ and $f_y > 0$.

f) At the point [ ], $\nabla f = \lambda \nabla g$ and $g = 1$ for some $\lambda > 0$.

g) At the point [ ], $|\nabla f|$ is minimal among all points A-G.



3b) (3 points) Fill in the numbers $1, -1$, or 0. In all cases, the vector $\vec{v}$ is a general unit vector.

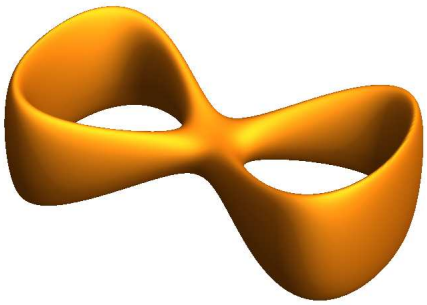
a) At a maximum point of $f(x, y)$, we have $D_{\vec{v}} f =$ [ ]

b) At any point (x, y), we have $|D_{\vec{v}} f| / |\nabla f| \leq$ [ ]

c) If $D_{\vec{v}} f = 1$, then $D_{-\vec{v}} f =$ [ ]

Problem 4) (10 points)

The surface $f(x,y,z) = 1/10$ for $f(x,y,z) = 10z^2 - x^2 - y^2 + 100x^4 - 200x^6 + 100x^8 - 200x^2y^2 + 200x^4y^2 + 100y^4$ is a blueprint for a new sour-sweet gelatin candy brand.



a) (4 points) Find the equation $ax + by + cz = d$ for the tangent plane of f at $(0, 0, 1/10)$.

b) (3 points) Find the linearization $L(x,y,z)$ of f at $(0, 0, 1/10)$.

c) (3 points) Estimate $f(0.01, 0.001, 0.10001)$.

Problem 5) (10 points)

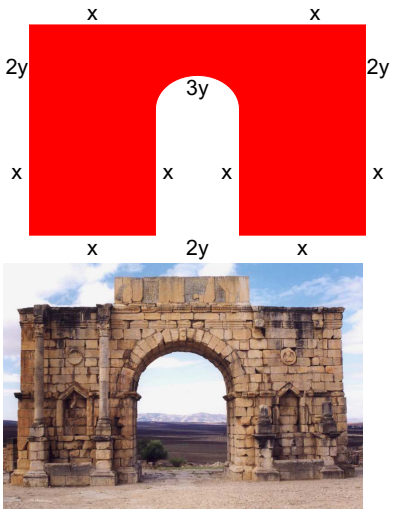
The **marble arch of Caracalla** is a Roman monument, built in the year 211. We look at a region modelling the arch. Using the Lagrange optimization method, find the parameters (x, y) for which the area

$$f(x,y) = 2x^2 + 4xy + 3y^2$$

is minimal, while the perimeter

$$g(x,y) = 8x + 9y = 33$$

is fixed.



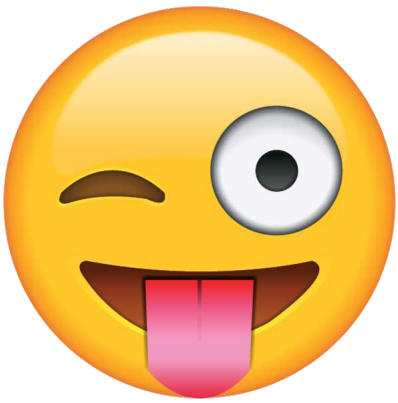
Problem 6) (10 points)

a) (8 points) Find and classify the critical points of the function

$$f(x,y) = x^2 - y^2 - xy^3 \ .$$

b) (2 points) Decide whether f has a global maximum or minimum on the entire 2D plane.

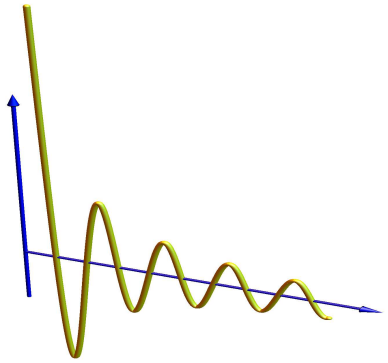
We don't know of any application for f. But if you read out the function aloud, it rolls beautifully off your tongue!



Problem 7) (10 points)

We look at the integral

$$\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} \sin(x)/x^2 \, dxdy \, .$$



Just for illustration, we have drawn the graph of the function $f(x) = \sin(x)/x^2$.

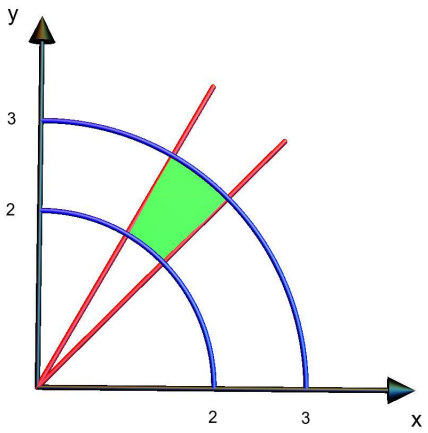
a) (5 points) Draw the region over which the double integral is taken.

b) (5 points) Find the value of the integral.

Problem 8) (10 points)

"Heat-assisted magnetic recording" (HAMR) promises high density hard drives like 20 TB drives in 2019. The information is stored on sectors, now typically 4KB per sector. Let's assume that the magnetisation density on the drive is given by a function $f(x, y) = \sin(x^2 + y^2)$. We are interested in the total magnetization on the sector R in the first quadrant bounded by the lines $x = y$, $y = \sqrt{3}x$ and the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. In other words, find the integral

$$\int \int_R \sin(x^2 + y^2) \, dxdy \, .$$

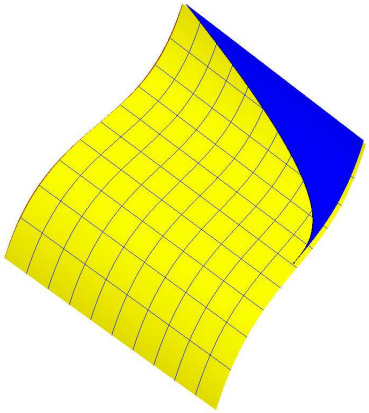


Problem 9) (10 points)

a) (4 points) Write down the double integral for the surface area of

$$\vec{r}(x, y) = \langle 2x, y, x^3/3 \ + y \rangle$$

with $0 \leq x \leq 2$ and $0 \leq y \leq x^3$.



b) (6 points) Find the surface area.

11/1/2017 Second Hourly Practice 1 **Math 21a, Fall 2017**

Name:

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MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The function $f(x,y) = x^3y/(x^6+y^5)$ can be filled in at the origin with a value $f(0,0) = a$ so that f is continuous everywhere.

2) [T] [F] The chain rule assures that $\int_0^1 (\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(1)) - f(\vec{r}(0))$.

3) [T] [F] The formula $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 \int_0^1 f(y,x)\, dy\, dx$ holds.

4) [T] [F] If $u(x,t)$ solves the partial differential equation $u_t = u_x$, then so does the function u_x.

5) [T] [F] There is a surface S containing the curve $\vec{r}(t) = \langle t, t^2, t^3\rangle$ for which the tangent plane to S at $(0,0,0)$ is $x + 2y + 3z = 0$.

6) [T] [F] For any two unit vectors $\vec{u}$ and $\vec{v}$, and any f, we have $D_{\vec{u}}D_{\vec{v}}f = D_{\vec{v}}D_{\vec{u}}f$.

7) [T] [F] If the tangent plane to $z = f(x,y)$ at $(0,0,f(0,0))$ is $4 + 3x + 2y + z = 0$, then $L(x,y) = 4 + 3x + 2y$ is the linearization of $f(x,y)$ at $(0,0)$.

8) [T] [F] For $f(x,y) = x^3 e^{y^2 \cos y} - x^4 \cos y$ the function $f_{xyxyxyxyxy}$ is zero everywhere.

9) [T] [F] The point $(0,0)$ is a critical point of $f(x,y) = x^3y^2$.

10) [T] [F] The gradient of $f(x,y) = x^2 + y^2$ is a vector perpendicular to the surface $z = f(x,y)$.

11) [T] [F] If the function $f(x,y)$ attains an absolute maximum on the region $x^2+y^2 \le 4$ at the point $(2,0)$, then we must have $f_{xx}(2,0) \le 0$.

12) [T] [F] If $f(x,y) \le 5$ for all values of (x,y), then $\int_0^{2\pi} \int_0^7 f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \le 5\pi(7^2)$.

13) [T] [F] For any constant a, we have $\int_{-a}^{a} \int_0^a \left(e^{x^2} \sin y\right) dx\, dy = 0$.

14) [T] [F] The linearization of the function $f(x,y) = e^{x^2+y}$ at the point $(0,0)$ is the function $L(x,y) = 1 + 2x^2e^{x^2+y} + ye^{x^2+y}$.

15) [T] [F] Let $\vec{u}$ be the unit vector in the direction $\langle 1,1\rangle/\sqrt{2}$. Then $D_{\vec{u}}f = f_{xy}$.

16) [T] [F] The integral of $f(x,y) = \sqrt{x^2+y^2}$ over the unit disk is $\int_0^{2\pi} \int_0^1 r\, dr d\theta$.

17) [T] [F] There is a function $f(x,y)$ for which $D_{\vec{v}}f(0,0) = 1$ for all directions $\vec{v}$.

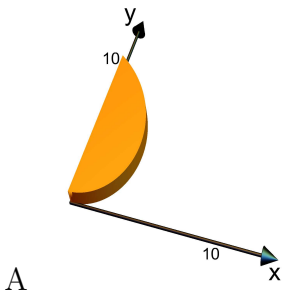
18) [T] [F] Given $f(x,y(x)) = 0$, then $f_x + f_y \frac{dy}{dx} = 0$.

19) [T] [F] Any function on a closed and bounded region must have a critical point.

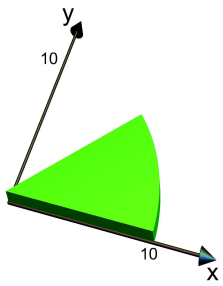
20) [T] [F] The integral $\iint_{x^2+y^2\le 1} |f(x,y)|\, dxdy$ computes the surface area of the surface $z = f(x,y), x^2 + y^2 \le 1$.

Problem 2) (10 points) No justifications needed

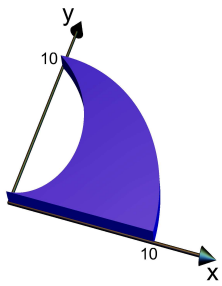
a) (6 points) Double integrals like $\iint_R 1\ dxdy$ or $\iint_R r\ drd\theta$ can be interpreted both as the **area** of the region R as well as the **volume** of the solid under the graph of the constant function $f(x,y) = 1$ or $f(\theta, r) = 1$. Match the regions with the integrals:



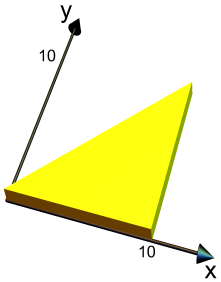
A



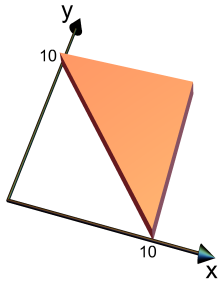
B



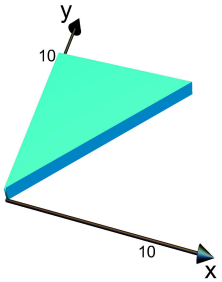
C



D



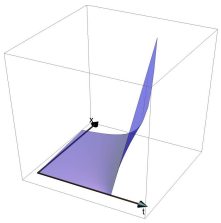
E



F

Enter A-F	Integral
	$\int_0^{10}\int_0^{x} 1\ dydx$
	$\int_0^{10}\int_0^{\pi/4} r\ d\theta dr$
	$\int_0^{\pi/2}\int_0^{20\theta/\pi} r\ drd\theta$
	$\int_0^{\pi/2}\int_{20\theta/\pi}^{10} r\ drd\theta$
	$\int_0^{10}\int_0^{y} 1\ dxdy$
	$\int_0^{10}\int_{10-x}^{10} 1\ dydx$

b) (4 points)



You know the **Transport**, **Wave**, **Heat**, or **Burgers** equation. Given in a possibly different order, these differential equations are $u_t = u_{xx}, u_t = u_x, u_{tt} = u_{xx}, u_t + uu_x = u_{xx}$. Check all the boxes where the given function solves the given PDE.

$f(x,t) = x/(1+t)$ solves	Name
	Burgers
	Transport

$f(x,t) = xt$ solves	Name
	Heat
	Wave

Problem 3) (10 points)

3a) (7 points) All the parts of this problem refer to the labeled points and the differentiable function $f(x, y)$ whose level curves are shown in the following plot:

a) At the point ☐, the gradient ∇f has maximal length

b) At the point ☐, $f_x > 0$ and $f_y = 0$

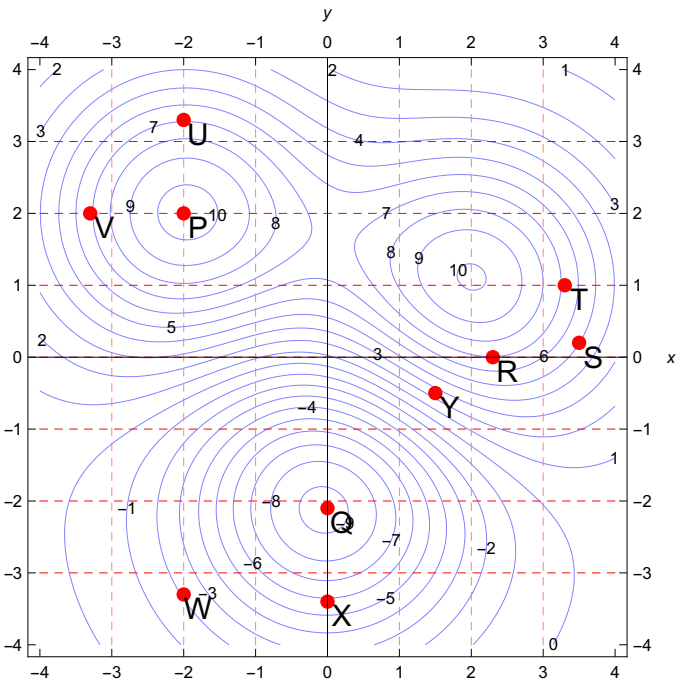
c) At the point ☐, $f_x < 0$ and $f_y < 0$

d) At the point ☐, $D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f = 0$ and $f_x \neq 0$

e) At the point ☐, f achieves a global min on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$

f) At the point ☐, $\nabla f = \vec{0}$ and $f_{xx} < 0$

g) At the point ☐, ∇f points straight toward the top of the page.



3b) (3 points) Check the cases where the maximum, minimum or saddle point of the function can be established **conclusively** using the second derivative test. Don't check the box if the test does not apply, (even if it might be a sort of minimum, maximum or saddle).

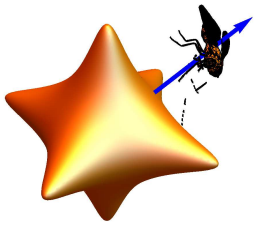
Critical Point	$x^4 + y^2$	xy	$x^2 - y^4$
Maximum			
Minimum			
Saddle point			

Problem 4) (10 points)

a) (4 points) A **math candy** of the form

$$f(x,y,z) = 3x^2y^2 + 3y^2z^2 + 3x^2z^2 + x^2 + y^2 + z^2 = 12$$

is leaning at $(1,1,1)$ at the plane tangent to it. Find that plane.

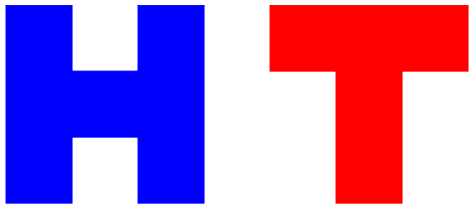


b) (3 points) Estimate $f(1.1, 1.01, 0.98)$ using linearization.

c) (3 points) A fruit fly just dipped some sugar from the candy at $(1,1,1)$ and moves along a path $\vec{r}(t)$ with constant speed 1 perpendicularly away from the candy. What is $\frac{d}{dt} f(\vec{r}(t))$ at the moment of take-off?

Problem 5) (10 points)

In order to figure out the **Egos** x and y of the US presidential candidates, we want to minimize the sum of the perimeter of the letters H and T written in units x and y if the total area is fixed. The letter H has area $7x^2$ and perimeter $16x$, the letter T has area $5y^2$ and perimeter $12y$. Minimize



$$f(x,y) = 16x + 12y \ .$$

under the constraint

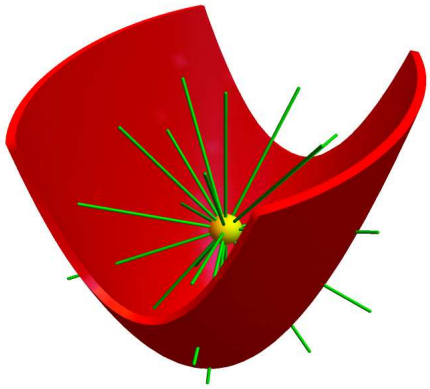
$$g(x,y) = 7x^2 + 5y^2 = 2016 \ .$$

We don't actually need to know x and y. **As political pundits, we are only interested in the ratio y/x at the minimum.** Find this ratio!

Problem 6) (10 points)

With $F(x,y,z) = 2x^2 + y^2 + z^2$ and the surface S parametrized by $\vec{r}(x,y) = \langle 2x, y, 2x^2+y^2-1\rangle$, the function $f(x,y) = F(\vec{r}(x,y))$ giving the value F on S is

$$f(x,y) = 4x^4 + 4x^2y^2 + 4x^2 + y^4 - y^2 + 1 \; .$$



a) (8 points) Find all the critical points of f and classify them with the second derivative test. Please organize your work carefully so that we can see your method and your conclusions easily.

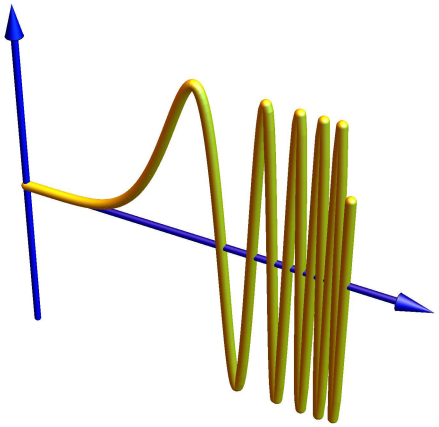
b) (2 points) The minimum could be obtained by minimizing $F(x,y,z)$ on the surface $G(x,y,z) = x^2/2 + y^2 - 1 - z = 0$. We would then use a method found by some mathematician. Which one? Just check the name. No additional work is needed in b).

Fubini	Burgers	Laplace	Lagrange	Bolzano	Clairaut

Problem 7) (10 points)

Integrate

$$\int_0^1 \int_{(1-y)^{1/4}}^1 \sin(x^5)\, dxdy \; .$$



The figure just shows a fancy plot of the function $\sin(x^5)$.

Problem 8) (10 points)

Integrate the double integral

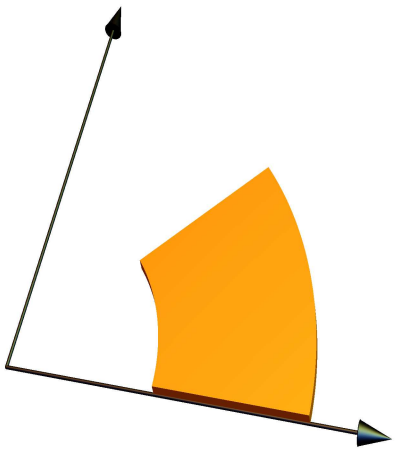
$$\iint_R x^2 \, dxdy \, ,$$

where R is the region

$$1 \leq x^2 + y^2 \leq 4$$

and

$$x \geq 0, y \geq 0, y \leq x \, .$$

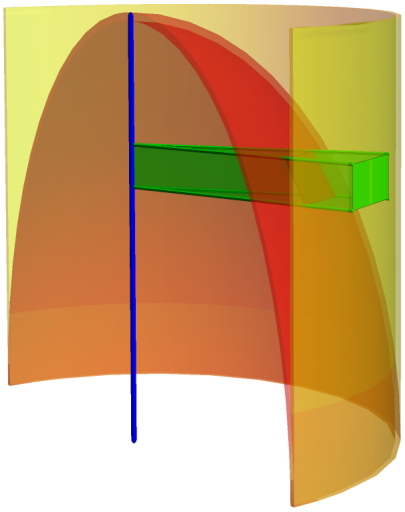


Problem 9) (10 points)

a) (7 points) Compute $A = |\vec{r}_\theta \times \vec{r}_\phi|$ for the half cylinder parametrized by

$$\vec{r}(\theta, \phi) = \langle \cos(\theta), \sin(\theta), \cos(\phi) \rangle \, .$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ and use this to find the surface area of the half cylinder



b) (3 points) Compute $B = |\vec{r}_\theta \times \vec{r}_\phi|$ for the quarter sphere parametrized by

$$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ to show that (remarkably!) it is the same factor than in part a).

Remark: The fact that the surface area elements A and B are the same has been realized by Archimedes already. It allowed him to compute the surface area of the sphere in terms of the surface area of the cylinder.

11/1/2017 Second Hourly Practice 2 **Math 21a, Fall 2017**

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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F The point $(-5, 3)$ is a critical point of $f(x, y) = 3x^2 + 5y^2$.

2) T F If a function $f(x, y, z)$ has gradient satisfying $|\nabla f| = 1$ everywhere, then the level surface $f(x, y, z) = 1$ is a sphere.

3) T F The chain rule assures that the vector $\nabla f(\vec{r}(t))$ and the velocity vector $\vec{r}\,'(t)$ for any curve $\vec{r}(t)$ on the level surface are perpendicular.

4) T F The function $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ satisfies the formula $D_{\vec{d}}D = \nabla D \cdot \vec{d}$ for any unit vector $\vec{d}$.

5) T F The function $f(x, y) = x^4 - 1$ has infinitely many critical points.

6) T F The points $(0, 1)$ and $(0, -1)$ are maxima of $f(x, y) = y^2$ under the constraint $g(x, y) = x^2 + y^2 = 1$.

7) T F The function $f(x, y) = y^2$ satisfies the partial differential equation $u_{yx}(x, y) = u_x(x, y)$.

8) T F Let $f(x, y) = x^3y$. At every point (x, y) there is a direction $\vec{v}$ for which $D_{\vec{v}}f(x, y) = 0$.

9) T F If $f_{xy} = f_{yx}$ then $f(x, y) = xy$.

10) T F $g(x, y) = \int_0^x \int_0^y f(s, t)\ dtds$ satisfies the partial differential equation $g_{xy}(x, y) = f(x, y)$.

11) T F If $f(x, y) = g(x, y) = x^2 + y^4$, then the Lagrange problem for maximizing f under the constraint $g(x, y) = 1$ has infinitely many solutions.

12) T F The number $|\nabla f(0, 0)|$ is the maximal directional derivative $|D_{\vec{v}}f(0, 0)|$ among all unit vectors $\vec{v}$.

13) T F Any continuous function $f(x, y)$ takes a global maximum as well as a global minimum on the region $0 \leq x^2 + y^2 \leq 1$.

14) T F For any continuous function, $\int_0^1 \int_0^1 f(r, \theta)r\ drd\theta = \int_0^1 \int_0^1 f(x, y)\ dxdy$.

15) T F If the Lagrange multiplier λ at a solution to a Lagrange problem is positive then this point is a minimum.

16) T F The equation $f_{xy}f_{xx}f_{yy} = 1$ is an example of a partial differential equation.

17) T F If the discriminant D appearing in the second derivative test of $f(x, y)$ is positive at $(0, 0)$ then $|\nabla f(0, 0)| > 0$.

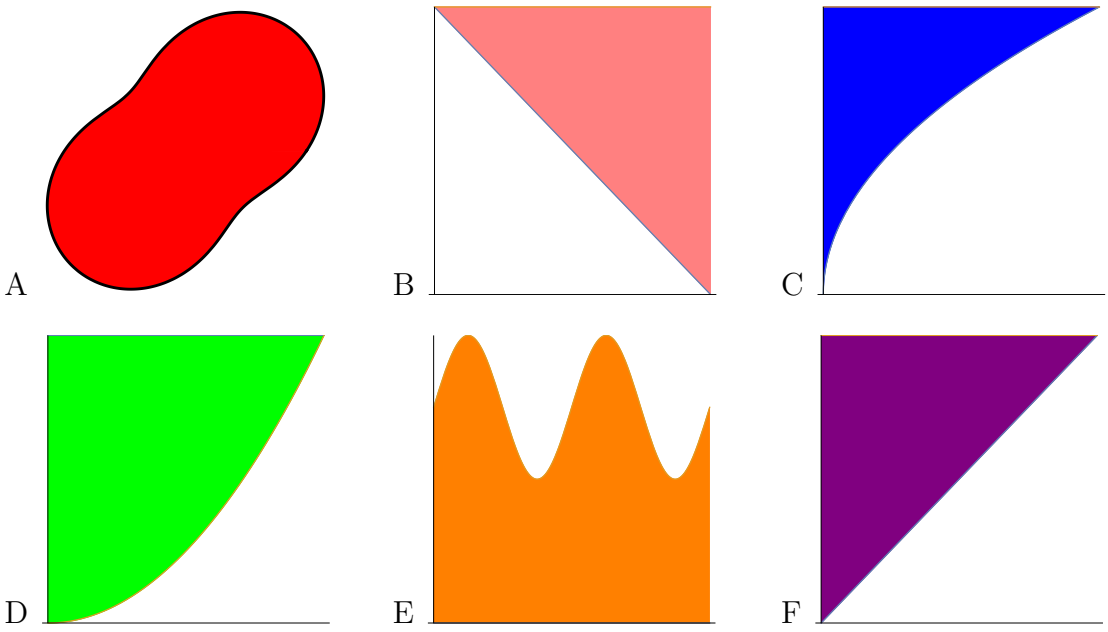
18) T F If $f(x, y)$ is a continuous function then $\int_7^9 \int_5^7 f(x, y)\ dxdy = \int_5^7 \int_7^9 f(x, y)\ dxdy$.

19) T F If f has the critical point $(0, 0)$, then $f_y + f_x$ has the critical point $(0, 0)$.

20) T F If $f(x, y)$ takes arbitrary large values, then $g(x, y) = |\nabla f(x, y)|$ takes arbitrary large values.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A - F$.



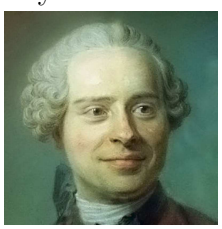
Enter A-F	Integral
	$\int_0^{2\pi} \int_0^{y} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_{2\pi-x}^{2\pi} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{y^2} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{3+\sin(2x)} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{3+\sin(2t)} f(r,t)\ rdrdt$

b) (4 points) We define the **complexity** of a partial differential equation for $u(t,x)$ or $u(x,y)$ as the number of derivatives appearing in total. For example, the partial differential equation $u_{xxx} = u_{tx}$ has complexity 5 because 5 derivatives have been taken in total. As an expert in PDEs, you know a few of them. Write down the complexities of the partial differential equations. These are integers ≥ 2 in each case.

Complexity	Name
	Laplace for $u(x,y)$
	Wave for $u(t,x)$
	Transport for $u(t,x)$
	Heat for $u(t,x)$


Taylor


Fourier

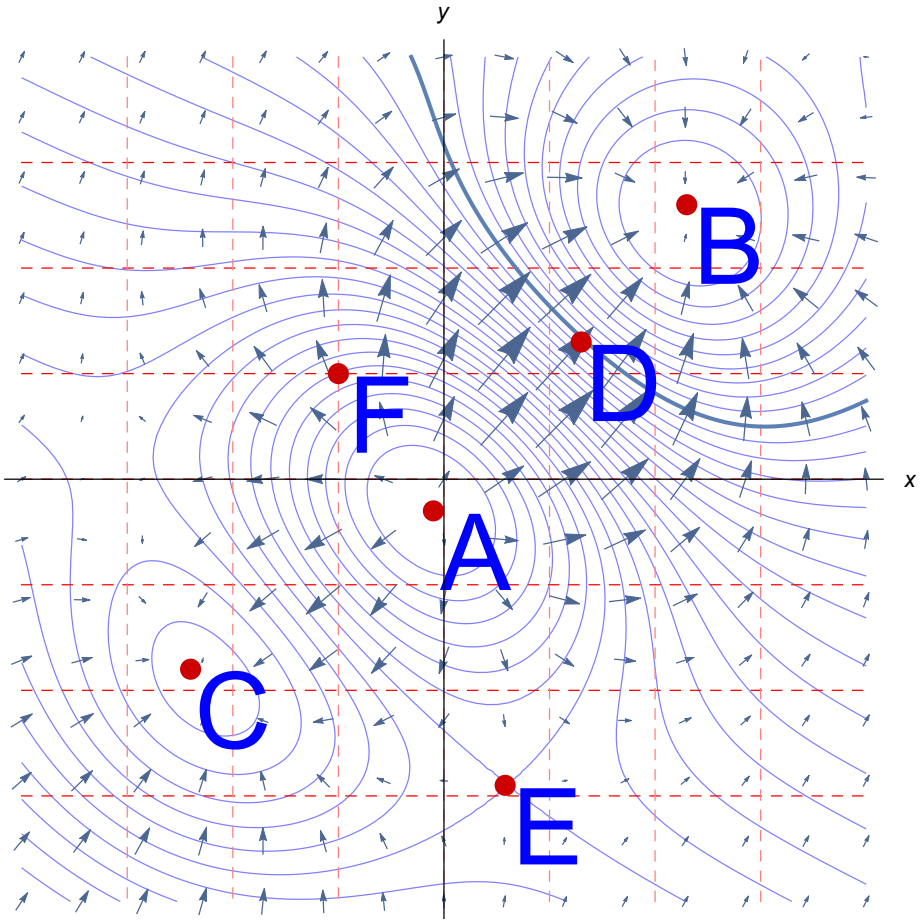

d'Alembert


Laplace

Problem 3) (10 points)

3a) (5 points) In the following contour plot of a height function $f(x, y)$, neighboring contours $f(x, y) = c$ have height distance 1. The arrows indicate the gradient of f. Every point A-F occurs at most once.

Which of the points is the global maximum on the visible region?	
Which of the points is a global minimum on the visible region?	
Which of the points is a global maximum for the function $\|\nabla f(x, y)\|^2$	
Which of the points is a saddle point?	
Which of the points has the property that $f_x f_y < 0$ at this point?	



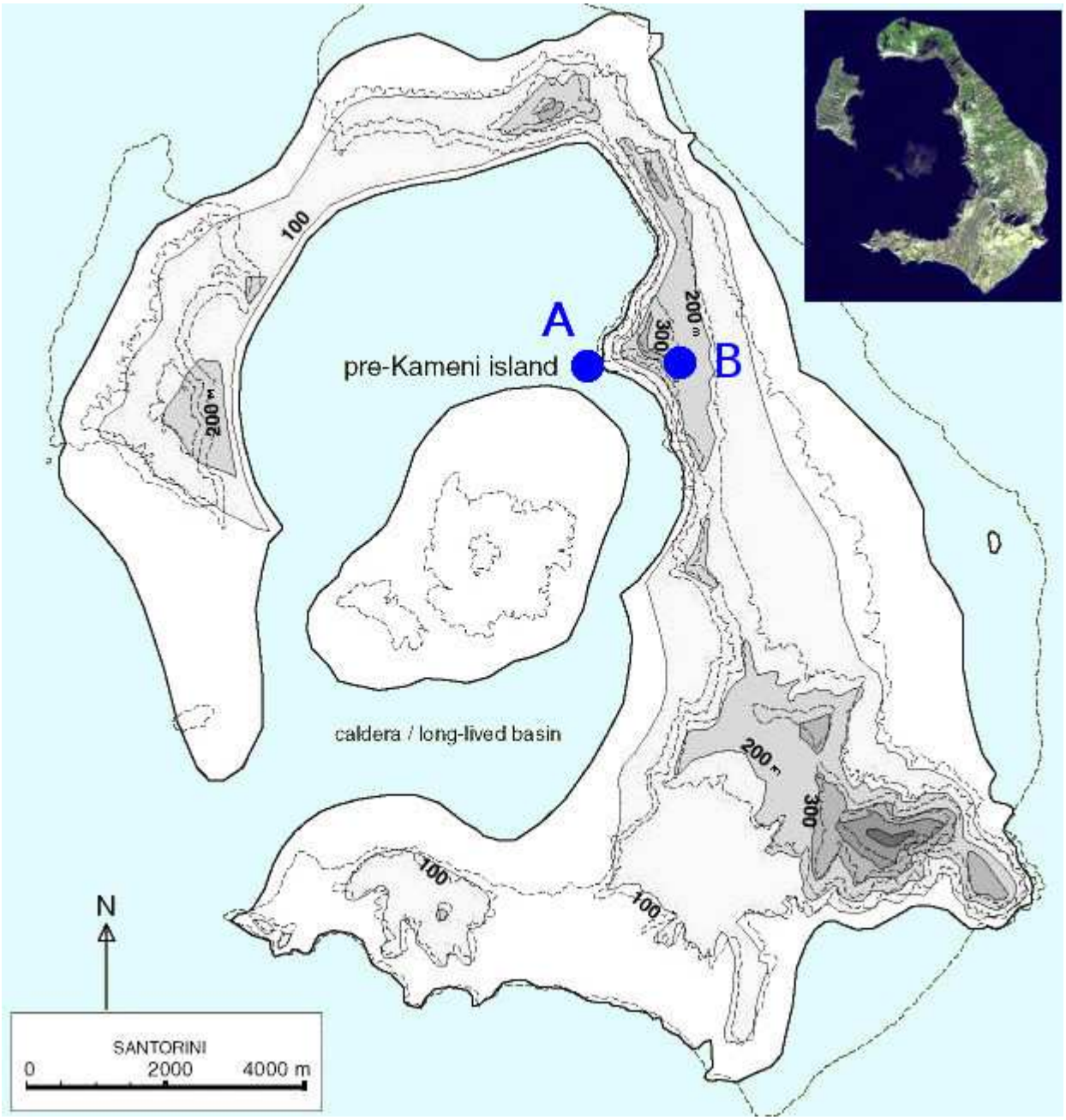
Part b) and c) of the problem are unrelated and on the new page.



Santorini panorama from Imerovigli with view onto Skaros rock, Caldera basin and volcanic island Nea Kameni. Photo: Oliver Knill, June 2015

3b) (2 points) You see a contour map of the Greek island of **Santorini**. Point A is on the water (0 elevation) Point B is **Skaros rock**, which used to be a fortification protecting merchants from pirates. Estimate the average directional derivative between A and B in the direction from A to B. Given elevation markers 100,200,300 are in meters.

Derivative	Check one
2	
0.2	
0.02	



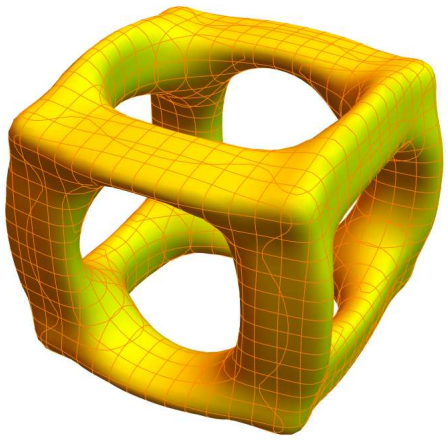
Source: http://www.decadevolcano.net, the picture shows a reconstruction of pre-Minoan Thera done by Druitt and Francaviglia from 1991. The island of today is shown in dottet curves. A satelite picture of the Santorini Caldera with the Nea Kameni volcano in the center is seen in the upper right corner.

3c) (3 points) Which statements about a critical point with discriminant $D \neq 0$ always hold for a smooth function $f(x, y)$?

Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum				
Minimum				
Saddle point				

Problem 4) (10 points)

a) (6 points) Let $g(x,y) = (6y^2-5)^2(x^2+y^2-1)^2$. Find the gradient of g at the points $(1,-1)$, $(-1,1)$ and $(1,1)$.



b) (4 points) A student from the **Harvard graduate school of design** contemplates the surface

$$f(x,y,z) = g(x,y) + g(y,z) + g(z,x) = 3$$

shown in the picture. She first discovers the formula

$$\begin{aligned} \nabla f(1,-1,1) &= \langle g_x(1,-1) + g_y(1,1)\ , \\ & \quad g_x(-1,1) + g_y(1,-1)\ , \\ & \quad g_x(1,1) + g_y(-1,1)\rangle\ . \end{aligned}$$

Without verifying this, find the tangent plane at $(1,-1,1)$.

Problem 5) (10 points)

Octagons are used in architecture designs, in symbolism, for rugs or in traffic signs. Use the Lagrange method to find the octagon with maximal area

$$f(x,y) = (x+2y)^2 - 2y^2$$

if the circumference

$$g(x,y) = 4x + 4y\sqrt{2} = 8\ .$$

is fixed.



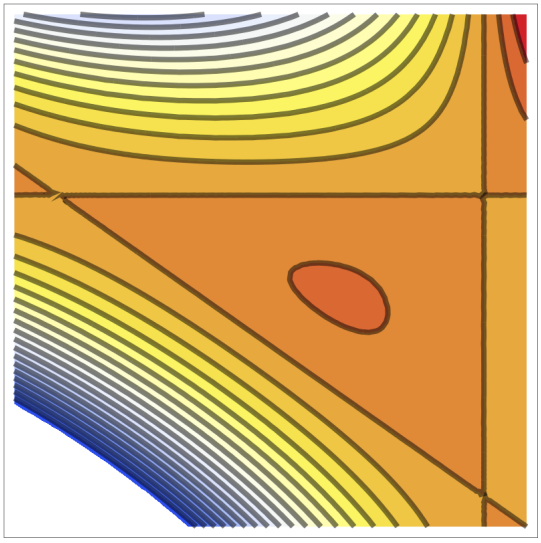
Problem 6) (10 points)

a) (8 points) Find and classify the four critical points of the **"triangle function"**

$$f(x, y) = x^2y + y^2x - y^2 - y$$

using the second derivative test. There is no need to find the values of f.

b) (2 points) State whether any of the four points is a global maximum or minimum on the entire plane.



Problem 7) (10 points)

The region R defined by

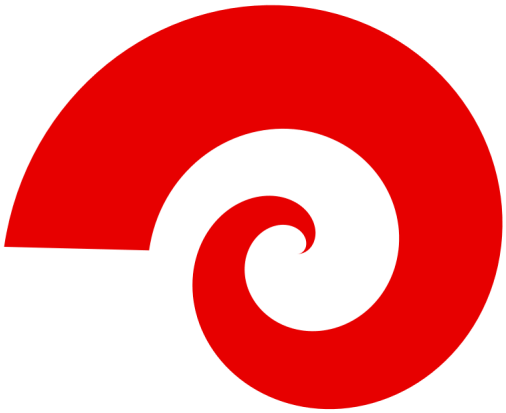
$$\theta \leq r(\theta) \leq 2\theta$$

with

$$0 \leq \theta \leq 3\pi$$

is shown in the picture. Compute its **moment of inertia**

$$\iint_R x^2 + y^2 \, dA \, .$$

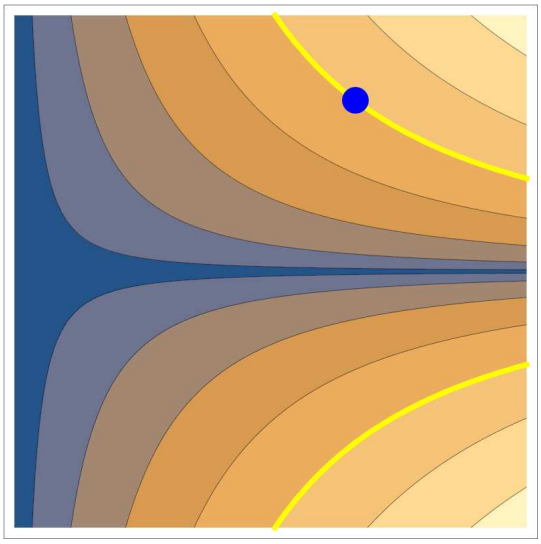


Problem 8) (10 points)

a) (5 points) Find a vector perpendicular to the **tangent line** of the curve

$$f(x,y) = 5(x^3y^2)^{1/5} = 100$$

at $(20, 20)$. The picture shows a contour map of f.



b) (5 points) Use the same function in a) to estimate

$$f(21, 19) = 5(21^3 \cdot 19^2)^{1/5}$$

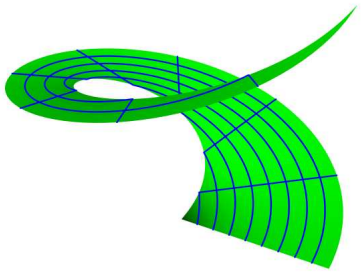
by **linearizing** f near $(20, 20)$.

Problem 9) (10 points)

We compute the **surface area** of the surface

$$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), u\rangle$$

over the region $R : 0 \leq u \leq 2\pi, u \leq v \leq 2\pi$.



a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2}\, dudv\ .$$

b) (5 points) Now compute the surface area integral.

Problem 9) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$ is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

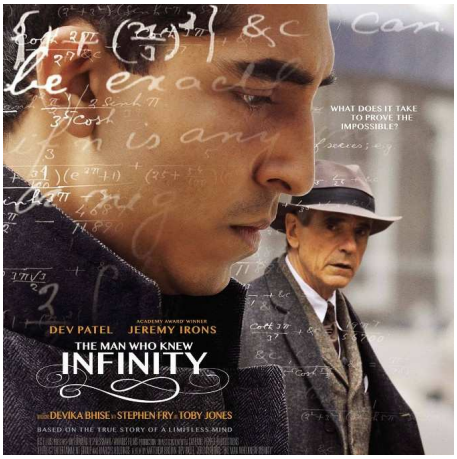
Let

$$f(x, y) = x\sqrt{y}\ .$$

Estimate

$$f(3.141, 163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



Ramanujan is featured in the movie: "The Man who knew infinity", 2015

11/1/2017 Second Hourly Practice 3 **Math 21a, Fall 2017**

Name:

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- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
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- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $|D_{\vec{v}} f(0,0)|$ among all unit vectors $\vec{v}$.

2) T F The relation $f_{xxyyxx} = f_{xxyxyx}$ holds everywhere for $f(x,y) = \cos(\exp(x^{10}) + \sin(x-y))$.

3) T F $\int_0^4 \int_0^{4x} f(x,y)\, dydx = \int_0^{16} \int_{y/4}^{16} f(x,y)\, dxdy$.

4) T F $g(x,y) = \int_y^0 \int_0^x f(s,t)\, dsdt$ satisfies $g_{xy} = -f(x,y)$.

5) T F If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$.

6) T F If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(a,b) = 3$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(a,b) = 5$, then $D_{\vec{v}} f(a,b) \geq 0$ for all unit directions $\vec{v}$.

7) T F Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x,y)$ we have $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.

8) T F If $u(t,x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant c.

9) T F If the Lagrange multiplier λ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.

10) T F The equation $f_x^2 + f_y^2 + f_z^2 = 1$ is an example of a partial differential equation.

11) T F If the discriminant D of $f(x,y)$ is zero at $(0,0)$ then $\nabla f(0,0) = \langle 0,0\rangle$.

12) T F If $f(x,y,z) = 0$ describes the unit sphere, then the gradient ∇f points outwards.

13) T F If $f(x,y)$ is a continuous function then $\int_0^2 \int_0^1 f(x,y)\ dxdy = \int_0^2 \int_0^1 f(y,x)\, dxdy$.

14) T F The point $(5,5,5)$ is a critical point of $f(x,y,z) = x + y + z$.

15) T F Assume $\nabla f(0,0) = \langle 0,0\rangle$ with discriminant $D > 0$, then $-f(x,y)$ has the same critical point $(0,0)$ with discriminant $D < 0$.

16) T F $\int\int_R |\nabla f|^2\ dxdy$ is the surface area of the cubic paraboloid $z = f(x,y) = x^3 + y^3$ defined over the region R.

17) T F If $D(x,y)$ is the discriminant of f at (x,y) then the following poetic formula of the **d**irectional **d**erivative of the **d**iscriminant holds: $D_{\langle 1,0\rangle} D = \partial_x D$.

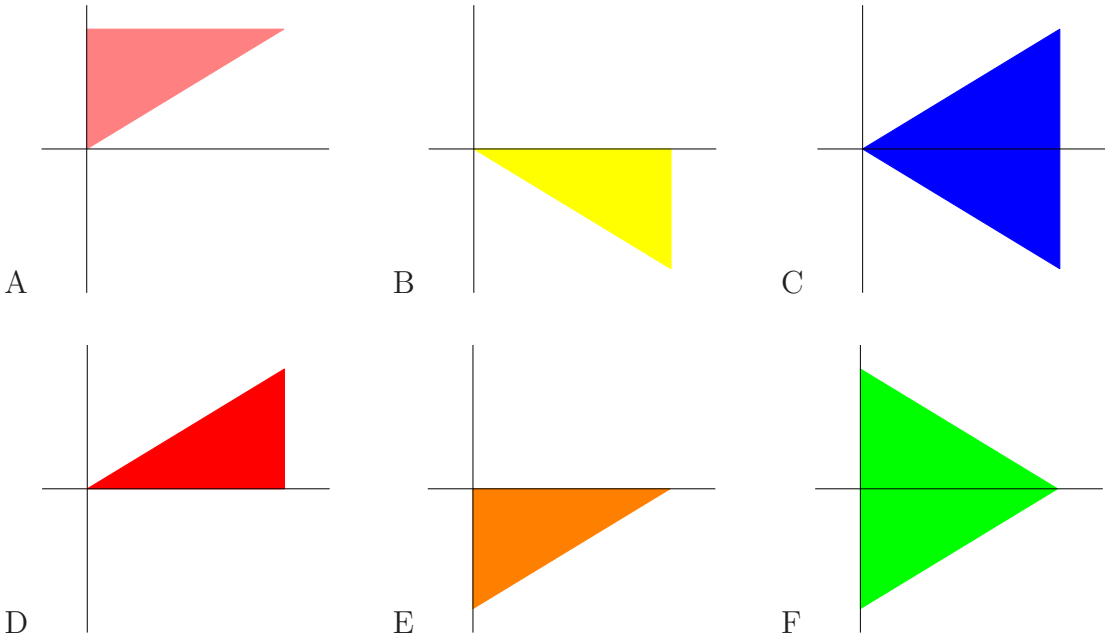
18) T F Assume $f(x,y) = -x^2 + y^4$ and a curve $\vec{r}(t)$ satisfying $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$ for all t.

19) T F The Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y) = c$ have the same solutions as the Lagrange equations for extremizing $F = f + g$ under the constraint $g = c$.

20) T F If f is a maximum under the constraint $g = 1$ at $(0,0)$, and $(0,0)$ is not a critical point for both f and g, then the level curves of f and g have the same tangent line at $(0,0)$.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.



Enter A-F	Integral
	$\int_{-1}^{1}\int_{\lvert y\rvert}^{1} f(x,y)\,dxdy$
	$\int_{0}^{1}\int_{0}^{x} f(x,y)\,dydx$
	$\int_{-1}^{0}\int_{-y}^{1} f(x,y)\,dxdy$
	$\int_{0}^{1}\int_{x-1}^{0} f(x,y)\,dydx$
	$\int_{0}^{1}\int_{x-1}^{1-x} f(x,y)\,dydx$
	$\int_{0}^{1}\int_{x}^{1} f(x,y)\,dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

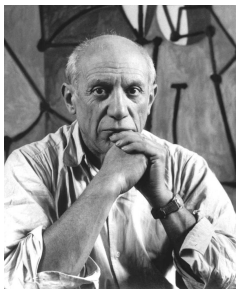
Equation number	Formula for PDE
1	$\frac{\partial}{\partial_t}u - \frac{\partial}{\partial_y}u = 0$
2	$\frac{\partial}{\partial_t}u - \frac{\partial^2}{\partial_x^2}u = 0$
3	$\frac{\partial^2}{\partial_t^2}u - \frac{\partial^2}{\partial_x^2}u = 0$
4	$\frac{\partial^2}{\partial_x^2}u + \frac{\partial^2}{\partial_y^2}u = 0$

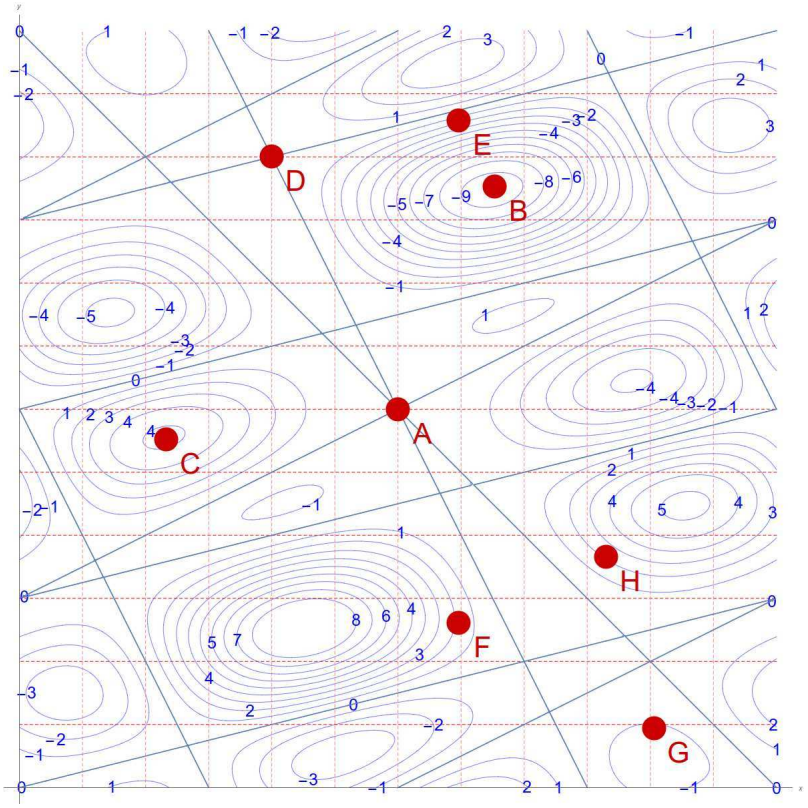
Problem 3) (10 points)

a) (7 points) The following contour map is inspired by a cubistic style of Picasso. Each of the points A-H fit exactly once

A point where $f_x \neq 0, f_y = 0$	
A saddle point of f	
A local maximum of f	
A critical point with $D = 0$	
A local minimum	
A point with $f_y \neq 0, f_x = 0$	
$\lvert\nabla f\rvert$ maximal among A-H	
Point where $D_{\langle -1,1\rangle/\sqrt{2}} f = 0$	





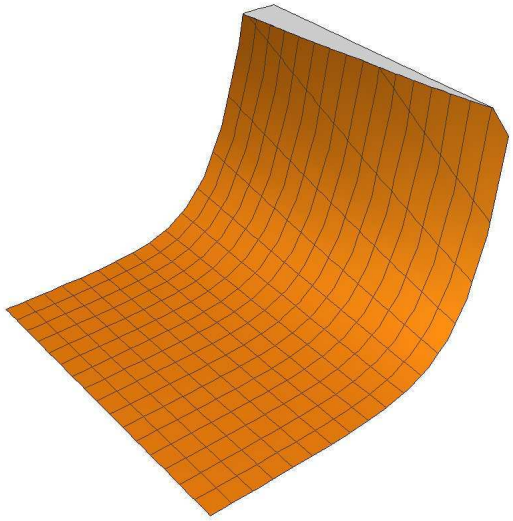


The painting "Pigeon with Green Peas" by Pablo Picasso was stolen in 2010. The thief got scared and disposed it to trash shortly after the theft. The garbage was emptied and taken away, the painting lost for ever. Or the thief had been clever ...

b) (3 points) Given a function $f(x, y)$ and a curve $\vec{r}(t)$. Let L be the linearization of f at $\vec{r}(0)$. Each of the following 3 vectors $\vec{a}, \vec{b}, \vec{c}$ is placed exactly twice in the puzzle to the right below.

$\vec{a} = \nabla f(\vec{r}(0))$	$\frac{d}{dt} f(\vec{r}(t))_{t=0} =$ ☐ $\cdot$ ☐
$\vec{b} = \vec{r}\,'(0)$	$L(\vec{r}(t)) =$ $f(\vec{r}(0)) +$ ☐ $\cdot$ ☐
$\vec{c} = \vec{r}(t) - \vec{r}(0)$	☐ $= \lim_{t\to 0} \frac{1}{t}$ ☐

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the **skate board ramp**

$$z - f(x,y) = z - \sqrt{x^{30}y^3 + x} = 0$$

at the point $(1, 2, 3)$.

b) (5 points) Estimate

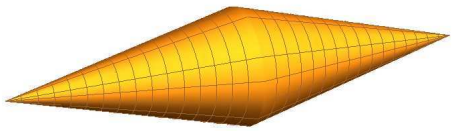
$$f(1.006, 1.98) = \sqrt{1.006^{30} \cdot 1.98^3 + 1.006}$$

by linearizing the function $f(x, y)$ at $(1, 2)$.

Problem 5) (10 points)

A **croissant** of length $2h$ and radius r in the shape of two cones has fixed volume

$$V(r,h) = \frac{2\pi r^2 h}{3} = 18 \, .$$

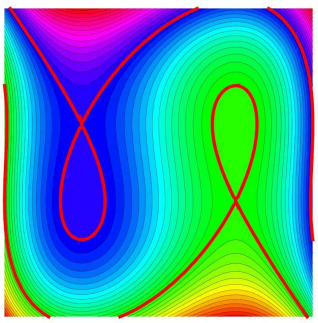


Use Lagrange to find the values r and h for which the surface area

$$A(r,h) = 2\pi r\sqrt{r^2 + h^2}$$

is minimal. **Hint:** as you have seen in homework, it is much more convenient to minimize $f(r,h) = A(r,h)^2$ instead.

Problem 6) (10 points)



Find the local maxima, minima and saddle points of the **tadpole function**

$$f(x,y) = 3y^2 + 4x^3 + 2y^3 - 12x \, .$$

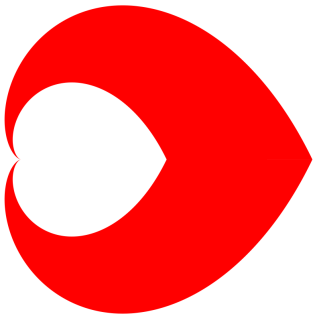
First tadpole: "What is your favorite book?" Second tadpole: "Metamorphosis by Kafka. What is your favorite year?" First tadpole: "Leap year!" Both croak with laughter. Then, sick of frog jokes, they turn green.

Problem 7) (10 points)

Find the area of the heart shaped polar region

$$(\theta - \pi)^2 \leq r \leq 2(\theta - \pi)^2$$

with $0 \leq \theta \leq 2\pi$.



Warning: Valentine cards displaying **"You are my** $r < (\theta - \pi)^2$**!"** *do not always work.*

Problem 8) (10 points)

Mathematica 10 does not give an elementary expression for the integral

$$\int_0^1 \int_{\exp(y)}^{e} \frac{1}{\log(x)}\, dx\, dy\,,$$

where log is the natural log. You can! "Humans are awesome 2014"! https://www.youtube.com/watch?v=ZBCOMG2F2Zk



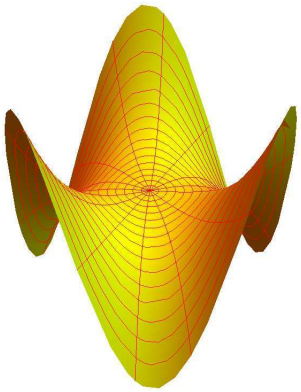
The logarithmic integral $\mathrm{Li}(x) = \int_0^x dt/\log(t)$ is important in number theory. It was Gauss who proposed first that the number $\pi(x)$ of primes smaller than x is about $\mathrm{Li}(x)$. It is now known that $0.89\ \mathrm{Li}(x) \leq \pi(x) \leq 1.11\ \mathrm{Li}(x)$ for all large enough x. P.S. Mathematica can solve the double integral of course, but only if told to "FullSimplify".

Problem 9) (10 points)

Compute the weighted surface area

$$\int\int_R (u^2 + v^2)|\vec{r}_u \times \vec{r}_v|\, dudv$$

of the monkey saddle parametrized by $\vec{r}(u, v) = \langle u, v, u^3 - 3uv^2 \rangle$ over the domain $R : u^2 + v^2 \leq 1$. This quantity is also known as the moment of inertia of the surface. Spin that monkey!



Problem 10) (10 points)



The following two integrals are called "Mad Max" integrals because they were written while watching that movie:

a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)}\, dxdy\ .$$

b) (5 points) Integrate the double integral

$$\int\int_R \sin(x^2+y^2)\, dxdy$$

where R is the disk of radius $\sqrt{\pi/2}$.

11/1/2017 Second Hourly Practice 4 **Math 21a, Fall 2017**

Name:

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- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F For any continuous function $f(x,y)$, we have $\int_0^1 \int_1^2 f(x,y)\,dx\,dy = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$.

2) T F If $\vec{u}$ is a unit vector tangent to $f(x,y) = 1$ at $(0,0)$ and $f(0,0) = 1$, then $D_{\vec{u}} f(0,0)$ is zero.

3) T F Assume f is zero on $x = y$ and $x = -y$, then $(0,0)$ is a critical point of f.

4) T F If $(0,0)$ is the only local minimum of a function f and f has no local maxima, then $(0,0)$ is a global minimum.

5) T F If $(0,0)$ is a critical point for f, and $f_{yy}(0,0) < 0$ then $(0,0)$ is not a local minimum.

6) T F If $f(x,y)$ and $g(x,y)$ have the same non-constant linearization $L(x,y)$ at $(0,0)$ and $f(0,0) = g(0,0) = 0$, then the level sets $f = 0$ and $g = 0$ have the same tangent line at $(0,0)$.

7) T F There are saddle points with positive discriminant $D > 0$.

8) T F If R is the unit disc, then $\int\int_R x^2 - y^2\,dxdy$ is zero.

9) T F There is a nonzero function $f(x,y)$ for which the linearization $L(x,y)$ is equal to $2f(x,y)$.

10) T F The directional derivative at a local minimum $(0,0)$ is positive in every direction.

11) T F If $\vec{r}(t)$ is a curve on the surface $g(x,y,z) = 1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

12) T F If $|\nabla f(0,0)| = 2$, there is a direction in which the directional derivative at $(0,0)$ is 2.

13) T F If $D > 0$ at $(0,0)$ and $\nabla f(0,0) = 0$ and $f_{xx}(0,0) < 0$ then $f_{yy}(0,0) < 0$.

14) T F $\int_0^1 \int_0^x f(x,y)\,dydx = \int_0^1 \int_y^1 f(x,y)\,dxdy$.

15) T F The surface area of the sphere of radius L is $\int_0^\pi L^2 \sin(\phi)\,d\phi$.

16) T F If $f(x,y) = g(x)$ is a function of x only, then $D = 0$ at every critical point.

17) T F The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x,y)$.

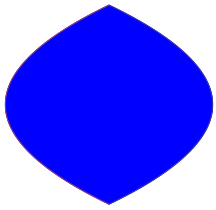
18) T F If $|\nabla f(0,0)| = 2$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}} f(0,0) = 1$.

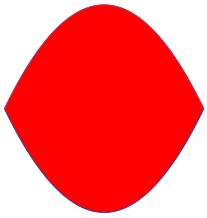
19) T F The gradient of the function $f(x,y) = \int_x^y \sin(t)\,dt$ is $\langle -\sin(x), \sin(y)\rangle$.

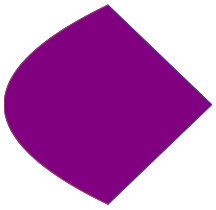
20) T F Assume $f(x,y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$.

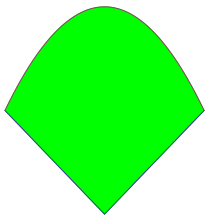
Problem 2) (10 points) No justifications needed

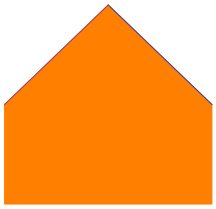
a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.

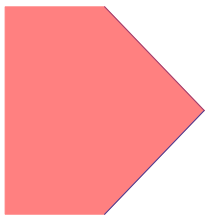
A 

B 

C 

D 

E 

F 

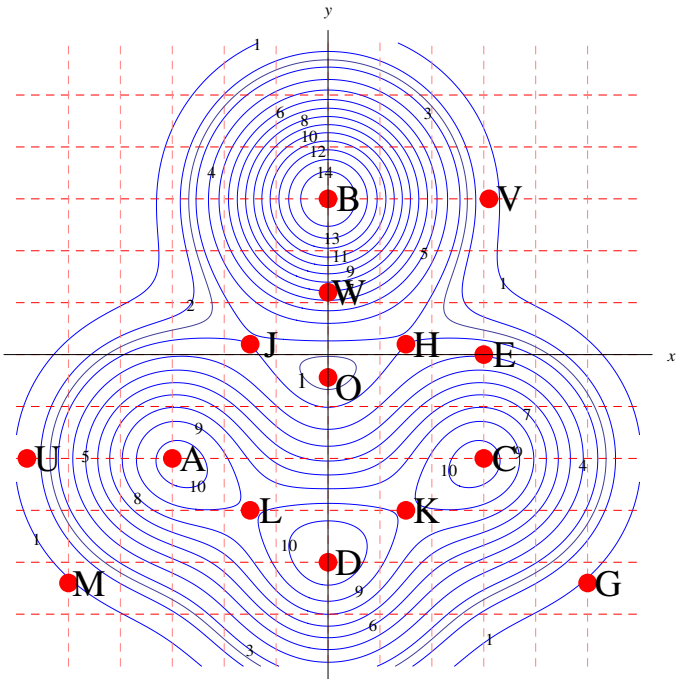
Enter A-F	Integral
	$\int_{-1}^{1}\int_{-1}^{2-\lvert y\rvert} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{y^2}^{2-\lvert y\rvert} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{x^2}^{2-x^2} f(x,y)\,dydx$
	$\int_{-1}^{1}\int_{\lvert x\rvert}^{2-x^2} f(x,y)\,dydx$
	$\int_{-1}^{1}\int_{y^2}^{2-y^2} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{-1}^{2-\lvert x\rvert} f(x,y)\,dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation Number	PDE
1	$g_x - g_y = 0$
2	$g_{xx} - g_{yy} = 0$
3	$g_x - g_{yy} = 0$
4	$g_{xx} + g_{yy} = 0$

Problem 3) (10 points)



a) (6 points) Enter one label into each of the boxes.

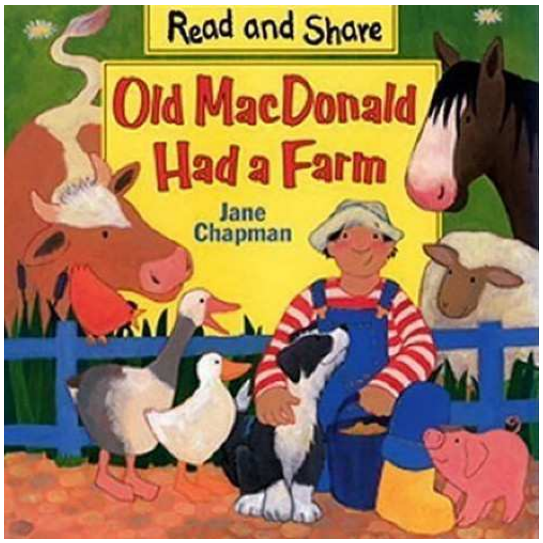
	At which point is the length of the gradient maximal?
	At which point is the global maximum?
	At which point is $f_x > 0, f_y = 0$?
	At which point is $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$, $D_{\langle 1,-1\rangle/\sqrt{2}} f < 0$?
	At which point is f maximal under the constraint $g(x,y) = y = 0$?
	At which point does f have a local minimum?

b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

	parallel	perp	
The gradient ∇f is always			to the surface $f = c$.
For a Lagrange minimum, ∇g is			to ∇f.
If $(0,0)$ is a min. of f then $\nabla f(0,0)$ is			to $\langle 1, 0\rangle$.
If $(0,0)$ is max. of f and $g = z - f(x,y)$ then ∇g is			to $\langle 0, 0, 1\rangle$.

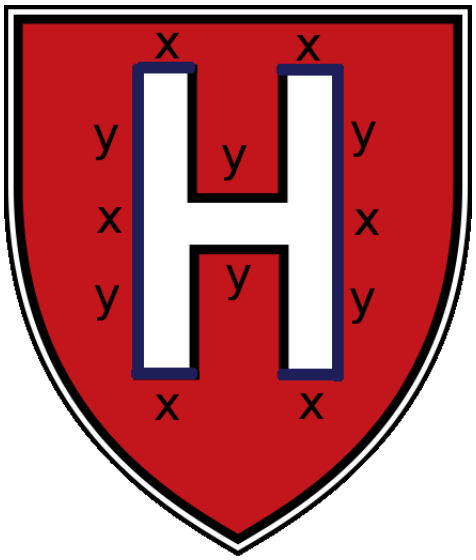
Problem 4) (10 points)

A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y. Estimate $f(12, 19)$.



"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"

Problem 5) (10 points)



Find the Harvard H which has maximal area

$$f(x, y) = 5xy + 2x^2$$

with fixed exposed perimeter

$$6x + 4y = 88 \, .$$

Find the maximum using Lagrange.

Problem 6) (10 points)

a) (7 points) A minigolf on the cape has a hole at a local minimum of the function

$$f(x,y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 \ .$$

Find all the critical points and classify them.

b) (3 points) A golfer hits tangent to the level curve $f(x,y) = 2$ through $(1,1)$. Find this line.



About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.

Problem 7) (10 points)



A circular track near Salem is a circle of radius 500 which is centered at the origin $(0,0)$. A go-kart goes counterclockwise around the track $\vec{r}(t)$. The cheering intensity is given by a function $f(x,y)$. The go-kart passes the point $(300,400)$ at time $t=0$ with velocity $\langle -4, 3\rangle$. We know that $f_x(300,400) = 2$ and $f_y(300,400) = 10$. Find the rate of change

$$\frac{d}{dt} f(\vec{r}(t))$$

at $t = 0$.

Problem 8) (10 points)

a) (6 points) Find the integral

$$\int_0^1 \int_y^{y^{1/5}} \frac{e^x + x^7}{x - x^5}\, dx\, dy \ .$$

b) (4 points) Integrate

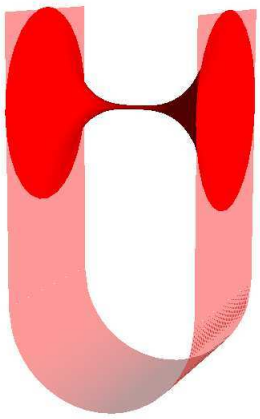
$$\int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dx dy \ .$$

Problem 9) (10 points)

Find the surface area of the "wormhole"

$$\vec{r}(u,v) = \langle 3v^3, v^9\cos(u), v^9\sin(u)\rangle \ ,$$

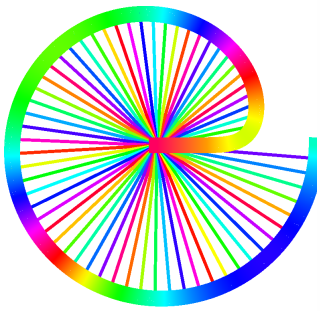
where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.



Einstein-Rosen bridges *are hypothetical topological constructions which would allow shortcuts through space-time. Tunnels connecting different parts of the universe appear frequently in science fiction.*

Problem 10) (10 points)

a) (5 points) We become typographer and design new mathematically defined **typeface** of the alphabet. The new letter "e" in this "21a" design is given by a polar region $r(t) \le t^{1/7}$, with $0 \le t \le 2\pi$. Find the area of this region.



b) (5 points) Integrate

$$\int_0^1 \int_0^{\arccos(y)} \frac{1}{\cos(x)}\, dxdy \ .$$

Remark: Computer scientist **Donald Knuth** once wrote an entire article about "The Letter S".



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The Letter S

Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 25 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analytic geometry may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematics underlying printed S's, and also to give an example of the METAFONT language I have recently been developing. (A complete description of METAFONT, which is a computer system and language intended to aid in the design of letter shapes, appears in [1, part 3].

Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central reason is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of metals or of movable type. The task of making a plate for a printed page is now essentially that of constructing a gigantic matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I wanted the second edition of one of my books to look like the first edition, although the first edition had been typeset with the old hot-lead technology; and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and it also discusses the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct S's

.t., con quello tondo qual ch'i lo son puncto de meco fora del quadro, longe da la inferiore linea del quadro puncto meco. Poi largo le circino puncti 2., procedo una puncti dove finisti la inferiore parte del .3. qual fu facta a dicta linea, cioè longe da la linea del quinto da parte dritta puncti 2., e altri

punct .4. da la (linea) inferiore del quadro. L'altro puncta longe da quella del quinto da parte sinistra puncti .4. descenderai in tondo verso mun dritta tanto che giongi sopra la media linea. Poi con dicta larghezza de circino ponenda l'una puncta dove al presente finisti, l'altra puncta longe da la linea del quinto da parte sinistra puncti .2., venendo dal dicto ultimo loco del .5. tanto che sia lontano da la inferiore linea del quadro puncti .2. Poi da questa ultima parte in tondo vengati a dritta linea a congiungere con lo inferiore tondo longe da la linea da parte sinistra del quadro puncti .1. e serre cerchi, e' sarà finito la littera .S., come apertamente si vede.

Fig. 1. Francesco Torniello's method of "squaring the S" in 1517. (This is page 65 of [4], reproduced by kind permission of Officina Bodoni in Verona, Italy.)

11/1/2017 Second Hourly Practice 5 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F If $\vec{r}(t)$ is a space curve satisfying $\vec{r}\,'(0) = 0$ and $f(x, y, z)$ is a function of three variables then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at $t = 0$.

2) T F The integral $\int\int_R 1 \, dxdy$ is the area of the region R in the xy-plane.

3) T F If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}} f(x, y)$ is independent of $\vec{u}$.

4) T F If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}} f(x, y)$ is independent of (x, y).

5) T F If $(0, 0)$ is a saddle point of $f(x, y)$ it is possible that $(0, 0)$ is a minimum of $f(x, y)$ under the constraint $x = y$.

6) T F The equation $f_{xy}(x, y) = 0$ is an example of a partial differential equation.

7) T F The linearization of $f(x, y) = 4 + x^3 + y^3$ at $(x_0, y_0) = (0, 0)$ is $L(x, y) = 4 + 3x^2 + 3y^2$.

8) T F Assume $(1, 1)$ is a saddle point of $f(x, y)$. Then $D_{\vec{v}} f(1, 1)$ takes both positive and negative values as $\vec{v}$ varies over all directions.

9) T F The integral $\int_{\pi/2}^{\pi} \int_0^2 r \, drd\theta$ is equal to π.

10) T F If $|\nabla f(0, 0)| = 1$, then there is a direction in which the slope of the graph of f at $(0, 0)$ is 1.

11) T F The vector $\nabla f(a, b)$ is a vector in space orthogonal to surface defined by $z = f(x, y)$ at the point (a, b).

12) T F If $f(x, y, z) = 1$ defines y as a function of x and z, then $\partial y(x, z)/\partial x = -f_x(x, y, z)/f_y(x, y, z)$.

13) T F In a constrained optimization problem it is possible that the Lagrange multiplier λ is 0.

14) T F The area $\int\int_R |\vec{r}_u \times \vec{r}_v| \, dudv$ of a surface is independent of the parametrization.

15) T F The function $f(x, y) = x^6 + y^6 - x^5$ has a global minimum in the plane.

16) T F The area of a graph $z = f(x, y)$ where (x, y) is in a region R is the integral $\int\int_R |f_x \times f_y| \, dxdy$.

17) T F The gradient of a function $f(x, y)$ of two variables can be written as $\langle D_{\vec{i}} f(x, y), D_{\vec{j}} f(x, y) \rangle$, where $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.

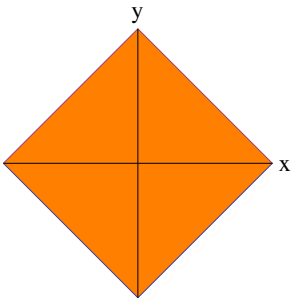
18) T F The length of the gradient of f at a critical point is positive if the discriminant $D(x, y) = f_{xx} f_{yy} - f_{xy}^2$ is strictly positive.

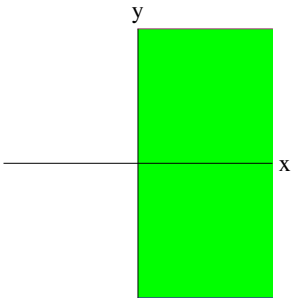
19) T F If $f(0, 0) = 0$ and $f(1, 0) = 2$ then there is a point on the line segment between $(0, 0)$ and $(1, 0)$, where the gradient has length at least 2.

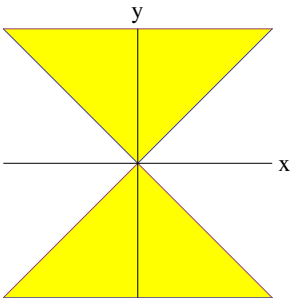
20) T F The tangent plane of the surface $-x^2 - y^2 + z^2 = 1$ at $(0, 0, 1)$ intersects the surface at exactly one point.

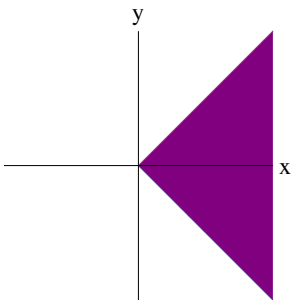
Problem 2) (10 points)

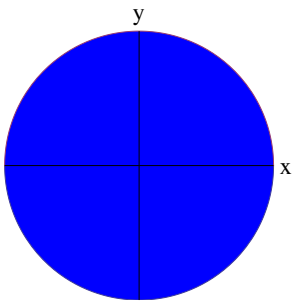
a) (6 points) Match the regions with the integrals. Each integral matches exactly one region $A - F$.

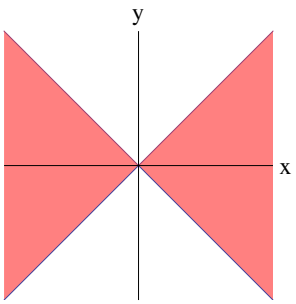
A 

B 

C 

D 

E 

F 

Enter A-F	Integral
	$\int_{-\pi}^{\pi} \int_{-\lvert y\rvert}^{\lvert y\rvert} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(r,\theta) r\, drd\theta$
	$\int_{-\pi}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{0}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{-\pi+\lvert x\rvert}^{\pi-\lvert x\rvert} f(x,y)\, dydx$

b) (4 points) Name the partial differential equations correctly. Each equation appears once to the left.

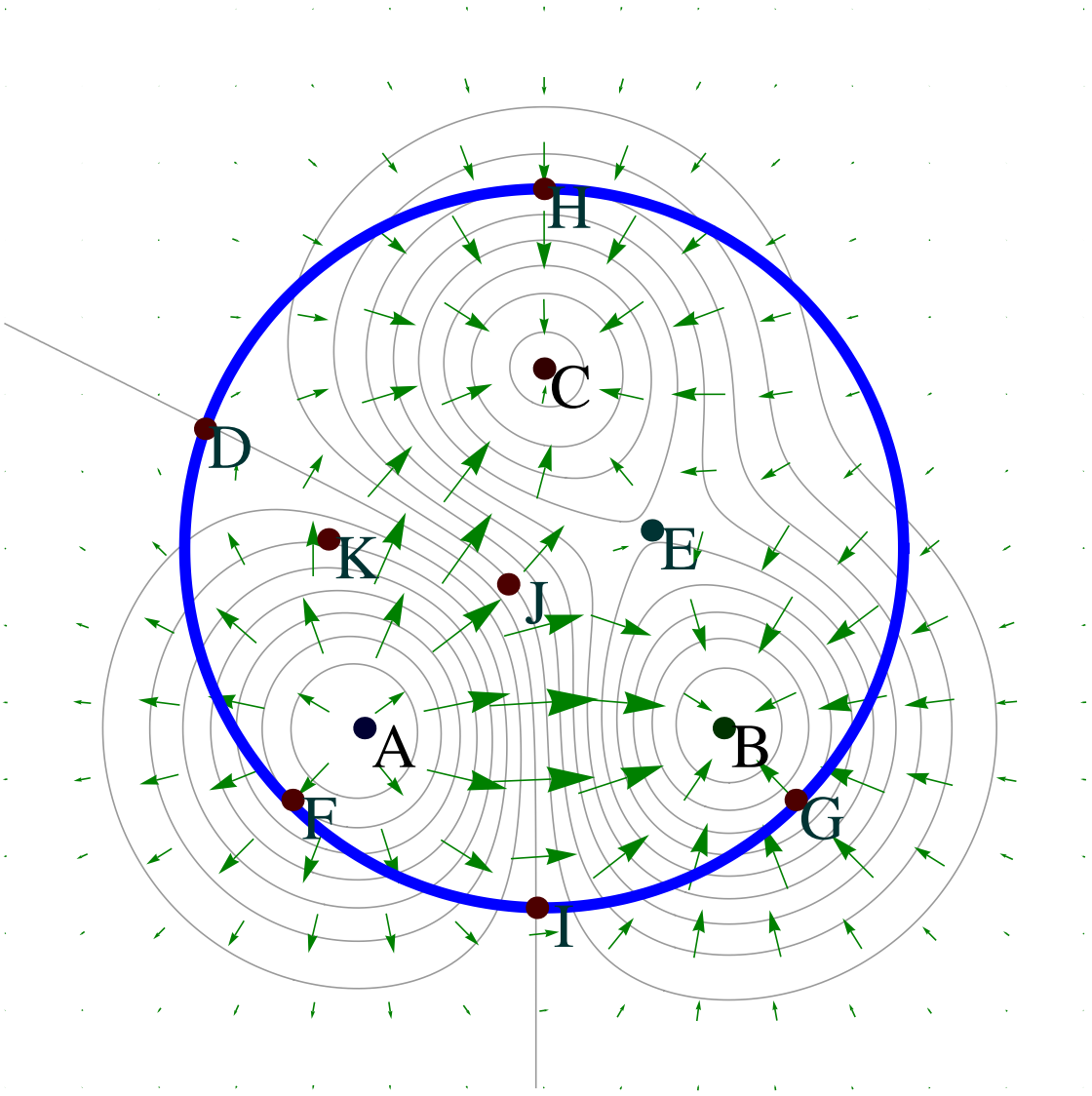
Fill in 1-4	Order
	Burgers
	Transport
	Heat
	Wave

Equation Number	PDE
1	$u_x - u_y = 0$
2	$u_{xx} - u_{yy} = 0$
3	$u_x - u_{yy} = 0$
4	$u_x + uu_x - u_{xx} = 0$

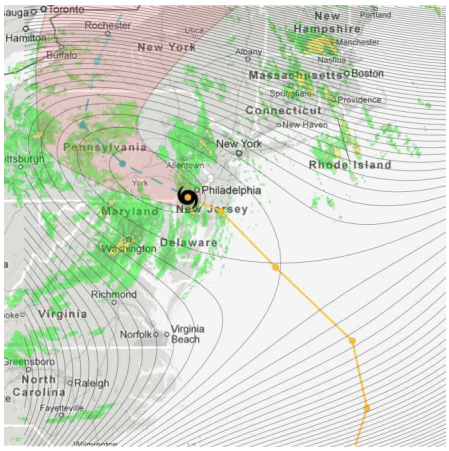
Problem 3) (10 points)

(10 points) Let's label some points in the following contour map of a function $f(x, y)$ indicating the height of a region. The arrows indicate the gradient $\nabla f(x, y)$ at the point. Each of the 11 selected points appears each exactly once.

Enter A-K	description
	a local minimum of $f(x, y)$ inside the circle
	a saddle point of $f(x, y)$ inside the circle
	a point, where $f_x \neq 0$ and $f_y = 0$
	a point, where $f_x = 0$ and $f_y > 0$
	a point, where $f_x = 0$ and $f_y < 0$
	a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$.
	the lowest point on the circle
	the highest point on the circle
	the local but not global maximum inside or on the circle
	the global maximum inside or on the circle
	the steepest point inside the circle



Problem 4) (10 points)

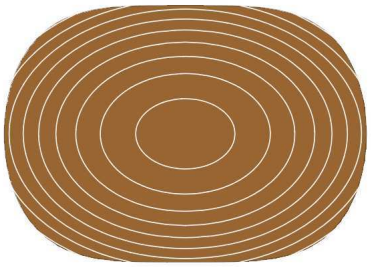


On October 30, 2012, the wind speed of Hurricane Sandy was given by the function

$$f(x, y) = 60 - x^3 + 3xy + y^3 \ .$$

Classify the critical points (maxima, minima and saddle points) of this function. Compute also the values of f at these points.

Problem 5) (10 points)



Use the second derivative test and the method of Lagrange multipliers to find the global maximum and minimum of the sugar concentration $f(x, y) = 10 + x^2 + 2y^2$ on a cake given by

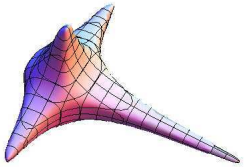
$$g(x, y) = x^4 + 4y^2 \leq 4 \ .$$

Note that this means you have to look both inside the cake and on the boundary.

Problem 6) (10 points)

a) (5 points) A seed of "Tribulus terrestris" has the shape

$$x^2 + y^2 + z^2 + x^4y^4 + x^4z^4 + y^4z^4 - 9z = 21$$



Find the tangent plane at $(1, 1, 2)$.

b) (5 points) The seed intersects with the xy-plane in a curve

$$x^2 + y^2 + x^4y^4 = 21 \ .$$

Find the tangent line to this curve at $(1, 2)$.

Problem 7) (10 points)

Let $f(x, y)$ model the time that it takes a rat to complete a maze of length x given that the rat has already run the maze y times. We know $f_y(10, 20) = -5$ and $f_x(10, 20) = 1$ as well as $f(10, 20) = 45$. Use this to estimate $f(11, 18)$.



Picture by Ellen van Deelen,

South west News service (UK)

Problem 8) (10 points)

a) (5 points) Find the double integral

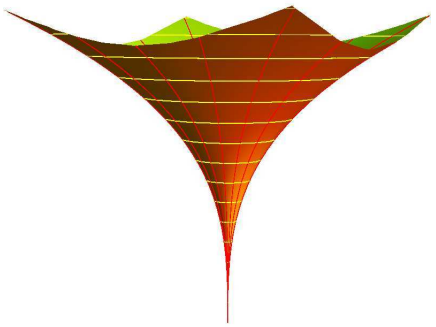
$$\int\int_R x \, dydx \ ,$$

where R is the region obtained by intersecting $x \leq |y|$ with $x^2 + y^2 \leq 1$.

b) (5 points) The square $\sin^2(x)/x^2$ of the sync function $\sin(x)/x$ does not have a known antiderivative. Compute nevertheless the integral

$$\int_0^{\pi^2/4} \int_{\sqrt{y}}^{\pi/2} \frac{\sin^2(x)}{x^2} \, dx \, dy \ .$$

Problem 9) (10 points)



Find the surface area of the surface of revolution $x^2 + y^2 = z^6$ where $0 \leq z \leq 1$. The surface is parametrized by

$$\vec{r}(t, z) = \langle z^3 \cos(t), z^3 \sin(t), z \rangle$$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$.

Problem 10) (10 points)



It turns out that there is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{x^7}{\sqrt{x} - x^2} \, dx \, dy \, .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{10} \, dx dy \, .$$

11/1/2017 Second Hourly Practice 6 **Math 21a, Fall 2017**

Name:

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MWF 9 Dennis Tseng
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MWF 10 Koji Shimizu
MWF 11 Oliver Knill
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MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
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TTH 11:30 Aukosh Jagannath
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F There is a function $f(x,y)$ for which the linearization at $(0,0)$ is $L(x,y) = x^2+y^2$.

2) T F For any two functions f, g and unit vector $\vec{u}$ we have $D_{\vec{u}}(f+g) = D_{\vec{u}}f + D_{\vec{u}}g$.

3) T F $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dydx = \int_0^2 \int_0^{\pi/2} r^2\, d\theta dr$.

4) T F If we solve $\sin(y) - xy^2 = 0$ for y, then $y' = -y^2/(\cos(y) - 2xy)$.

5) T F If $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y, then $g(x,y) = \int_0^x \int_0^y f(s,t)\, dtds$ solves $g_{xy}(x,y) = f(x,y)$.

6) T F If $|\nabla f| = 1$ at $(0,0)$, then there exists a direction in which the slope of the graph of f at $(0,0)$ is 1.

7) T F The function $f(x,y) = x^2+y^2$ satisfies the partial differential equation $f_{xx}f_{yy} - f_{xy}^2 = 4$.

8) T F The height of Mount Wachusett is $f(x,y) = 4 - 2x^2 - y^2$. On the trail $x^2+y^2 = 1$, the point $(1,0)$ is a maximum.

9) T F Mount Wachusett has height $f(x,y) = 4-2x^2-y^2$. Except at the maximum $(0,0)$, the gradient vector is perpendicular to the graph of the function.

10) T F If $f_x(a,b) > 0$ and $f_y(a,b) > 0$ then for any unit vector $\vec{u}$ we must have $D_{\vec{u}}f(a,b) > 0$.

11) T F If $f(x,y)$ has two local minima, then f must have at least one local maximum.

12) T F If $\vec{r}(t)$ is a curve on the surface $g(x,y,z) = x^2+y^2-z^2 = 6$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

13) T F If f and g have the same trace $\{x = 5\}$ then $f_x(5,y) = g_x(5,y)$ for all y.

14) T F If f and g have the same trace $\{x = 5\}$ then $f_y(5,y) = g_y(5,y)$ for all y.

15) T F The surface area of $\vec{r}_1(u,v) = \langle u\cos(v), u\sin(v), u^2\rangle$ and $\vec{r}_2(u,v) = \langle \sqrt{u}\cos(v), \sqrt{u}\sin(v), u\rangle$ defined on $\{0 \le u, v \le 1\}$ are the same.

16) T F If $\vec{r}(t)$ is a curve on a graph $z = f(x,y)$ of a function $f(x,y)$, then the velocity vector of $\vec{r}$ is perpendicular to the vector $\langle f_x, f_y, -1\rangle$.

17) T F A continuous function $f(x,y)$ on the closed disc $R = \{x^2+y^2 \le 51^2\}$ (of course, R is called **"area** 51π**"**) has a global maximum on R.

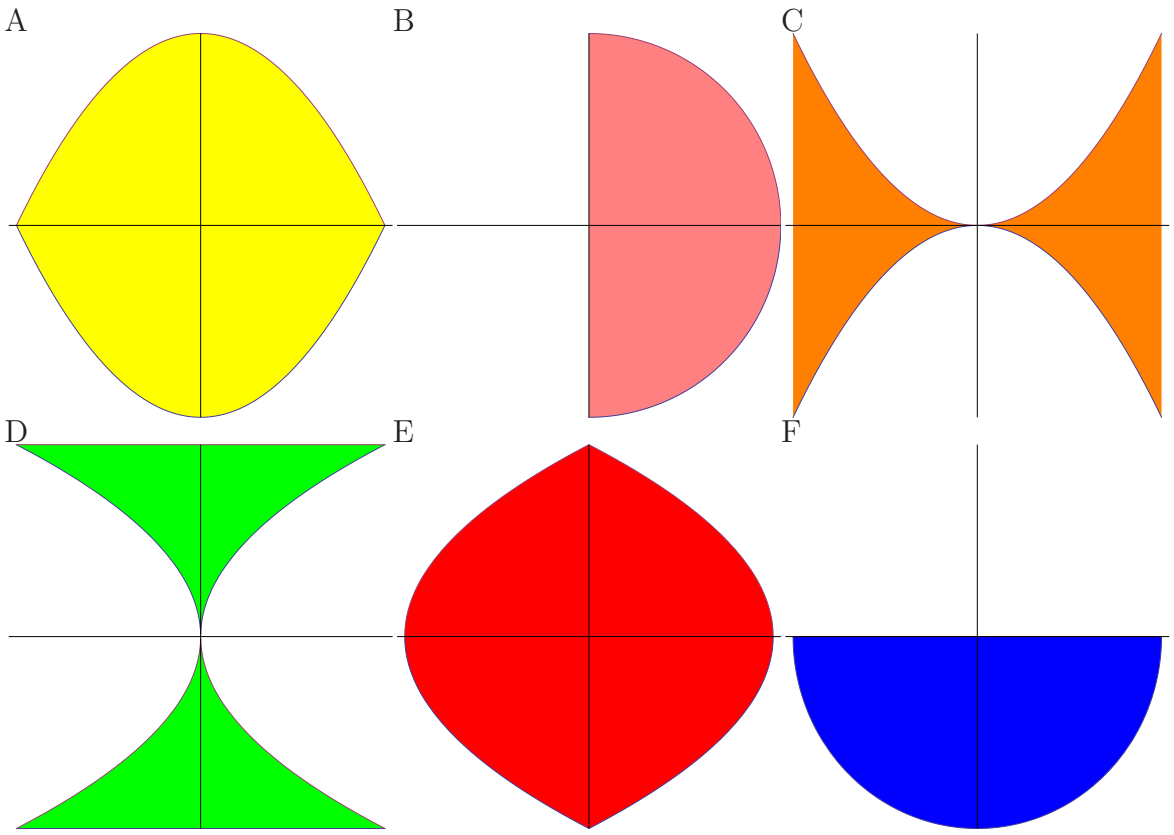
18) T F Any continuous function $f(x,y)$ has a global minimum and maximum on the curve $y = x^2$.

19) T F Fubini's theorem assures that $\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \int_c^d f(x,y)\, dxdy$.

20) T F $\int\int_R \sin(x+y)\, dxdy = 0$ for $R = \{-\pi \le x \le \pi, -\pi \le y \le \pi\}$.

Problem 2) (10 points)

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region $A - F$.



Enter A-F	Integral
	$\int_{-1}^{1}\int_{-x^2}^{x^2} f(x,y)\,dydx.$
	$\int_{-1}^{1}\int_{-y^2}^{y^2} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{y^2-1}^{1-y^2} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} f(x,y)\,dydx.$
	$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dydx.$

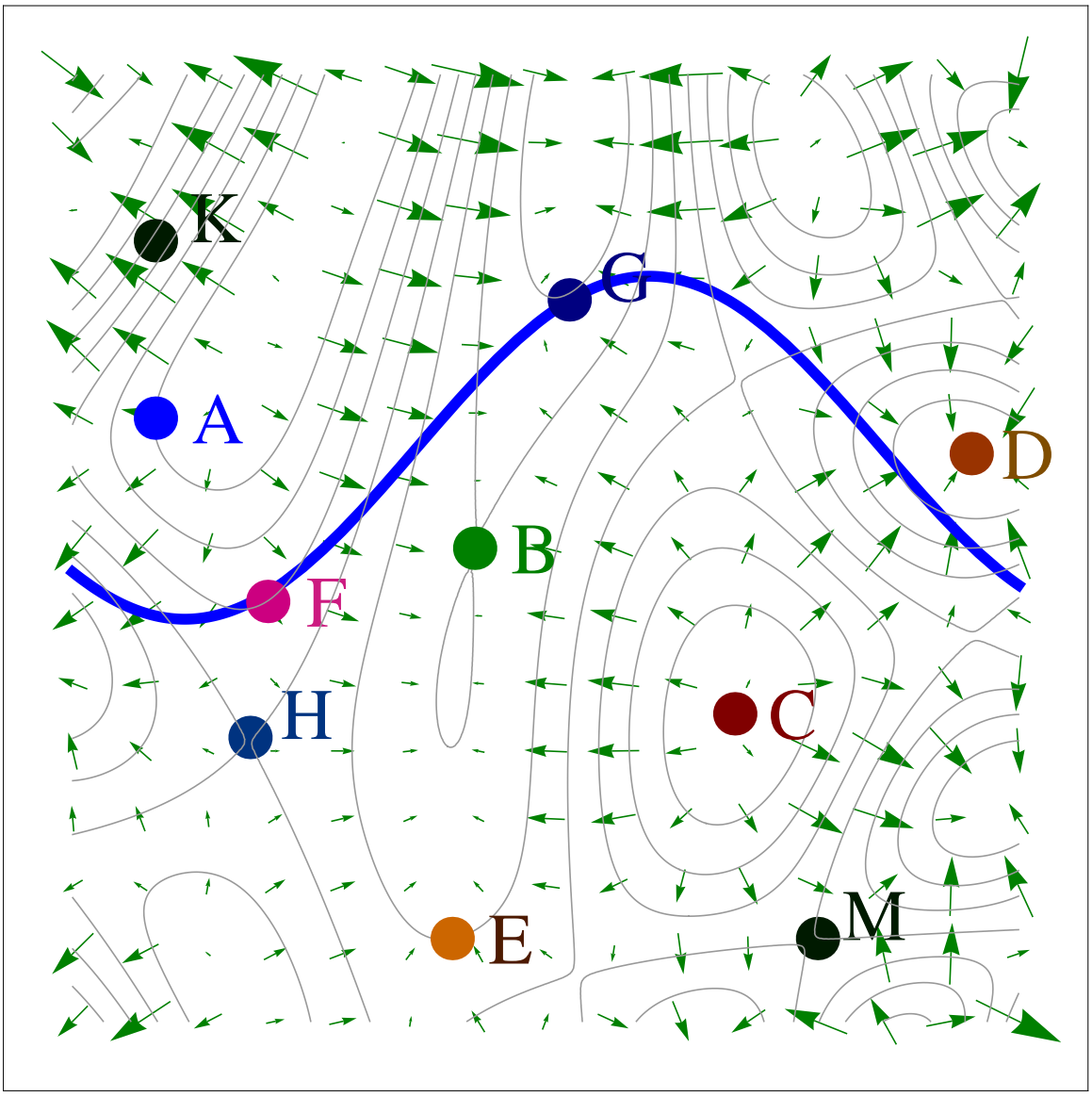
b) (4 points) Fill in one word names (like "Heat","Wave" etc) for the partial differential equations:

Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = g_{yy}$
	$g_{xx} = -g_{yy}$
	$g_x = g_{yy}$

Problem 3) (10 points)

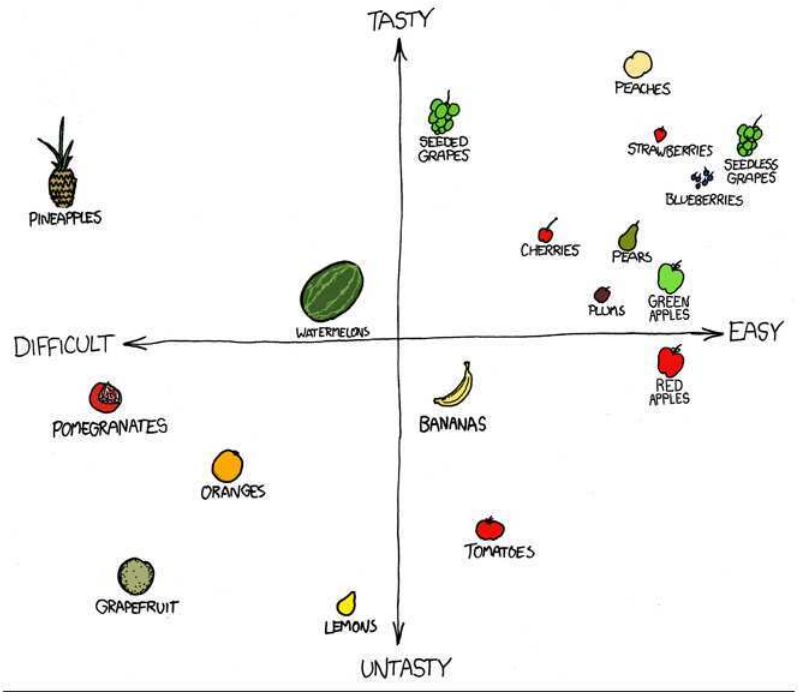
(10 points) A function $f(x, y)$ of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve $g(x, y) = 0$ which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters $A, B, C, D, E, F, G, H, K, M$ appears exactly once.

Enter A-P	Description
	a local maximum of $f(x, y)$.
	a local minimum of $f(x, y)$.
	a saddle point of $f(x, y)$ where $f_{xx} < 0$.
	a saddle point of $f(x, y)$ where $f_{xx} > 0$.
	a saddle point of $f(x, y)$ where f_{xx} is close to zero
	a point, where $f_x = 0$ and $f_y \neq 0$
	a point, where $f_y = 0$ and $f_x \neq 0$
	the point, where $\|\nabla f\|$ is largest
	a local maximum of $f(x, y)$ under the constraint $g(x, y) = 0$.
	a local minimum of $f(x, y)$ under the constraint $g(x, y) = 0$.



Problem 4) (10 points)

Find and classify all the extrema of the function $f(x,y) = x^5 + y^3 - 5x - 3y$. This function measures "eat temptation" in the x=Easy-y=Tasty plane. Is there a global minimum or global maximum?

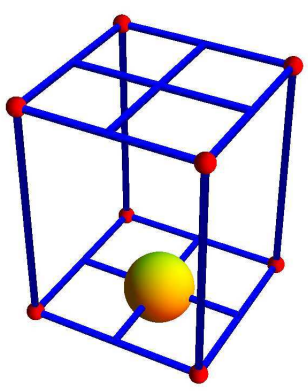


The "Easy-Tasty plane" was introduced in the XKCD cartoon titled "F&#% Grapefruits".

Problem 5) (10 points)

After having watched the latest Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x,y,z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x,y,z) = xyz$.





Problem 6) (10 points)

a) (8 points) Find the tangent plane to the surface $f(x, y, z) = x^2 - y^2 + z = 6$ at the point $(2, 1, 3)$.

b) (2 points) A curve $\vec{r}(t)$ on that tangent plane of the function $f(x, y, z)$ in a) has constant speed $|\vec{r}\,'| = 1$ and passes through the point $(2, 1, 3)$ at $t = 0$. What is $\frac{d}{dt} f(\vec{r}(t))$ at $t = 0$?

Problem 7) (10 points)

a) (5 points) Estimate $\sqrt{\sin(0.0004) + 1.001^2}$ using linear approximation.

b) (5 points) We know $f(0, 0) = 1$, $D_{\langle \frac{3}{5}, \frac{4}{5} \rangle} f(0, 0) = 2$ and $D_{\langle -\frac{4}{5}, \frac{3}{5} \rangle} f(0, 0) = -1$. If $L(x, y)$ is the linear approximation to $f(x, y)$ at the point $(0, 0)$, find $L(0.06, 0.08)$.

Problem 8) (10 points)

a) (5 points) Find the following double integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\pi \sin(\pi y)}{y^2 - \sqrt{y}} \, dy dx \, .$$

b) (5 points) Evaluate the following double integral

$$\int \int_R \frac{\sin(\pi \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \, dx dy$$

over the region

$$R = \{ x^2 + y^2 \leq 1, x > 0 \} \, .$$

Problem 9) (10 points)

a) (8 points) Find the surface area of the surface parametrized as

$$\vec{r}(u, v) = \langle u - v, u + v, (u^2 - v^2)/2 \rangle \, ,$$

where (u, v) is in the unit disc $R = \{ u^2 + v^2 \leq 1 \}$.

b) (2 points) Give a nonzero vector $\vec{n}$ normal to the surface at $\vec{r}(4, 2) = \langle 2, 6, 6 \rangle$.

Problem 10) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y}\, dydx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2 + y^2)\, dydx\ ,$$

where R is the ring $1 \leq x^2 + y^2 \leq 9$.

11/1/2017 Second Hourly Practice 7 **Math 21a, Fall 2017**

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5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F If $f(x,y)=1$ is a curve, and near $(2,3)$ one can write y as a function of x, then $y' = -f_y(2,3)/f_x(2,3)$.

2) T F If $\iint_R f(x,y)\ dA = 0$, then the function $f(x,y)$ is everywhere zero on $R = \{x^2+y^2 \leq 1\ \}$.

3) T F The directional derivative in the direction of the gradient is $|\nabla f|$.

4) T F The linearization of $f(x,y) = x^3+y^3$ at $(1,1)$ is the quadratic function $L(x,y) = 3x^2+3y^2$.

5) T F The function $f(x,y) = x^2+y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.

6) T F The function x^2y^2 has no local minimum at $(0,0)$ because the discriminant function D is zero there.

7) T F The double integral $\int_0^{\pi/4}\int_0^2 r^3\ drd\theta$ is the volume of the part of a solid cylinder $x^2+y^2 \leq 4$ which is below the paraboloid $z = x^2+y^2$ and above the xy plane.

8) T F The gradient of $f(x,y,z)$ at (x_0,y_0,z_0) is perpendicular to the level surface of f through (x_0,y_0,z_0).

9) T F If $f(x,y,z) = 3x-4z$, then the minimal possible directional derivative $D_{\vec{u}}f$ at any point in space is -5.

10) T F If (x,y) is not a critical point, then the directional derivative $D_{\vec{v}}f$ can take both positive and negative values for different choices of $\vec{v}$.

11) T F Using linearization of $f(x,y) = x/y$ we can estimate $1.01/1.001 = f(1.01,1.001) \sim 1+0.01-0.001 = 1.009$.

12) T F If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.

13) T F For a rectangular region R, Fubini tells that $\int_0^2\int_0^3 f(x,y)\ dxdy = \int_0^2\int_0^3 f(x,y)\ dydx$ for any continuous function $f(x,y)$.

14) T F If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2+y^2 \leq 1\ \}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y)\ dxdy > 0$.

15) T F If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dtf(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

16) T F $\int_0^1\int_0^1 f_{xy}(x,y)\ dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.

17) T F If $f_{yy}(x,y) > 0$ everywhere, then f can not have any local maximum.

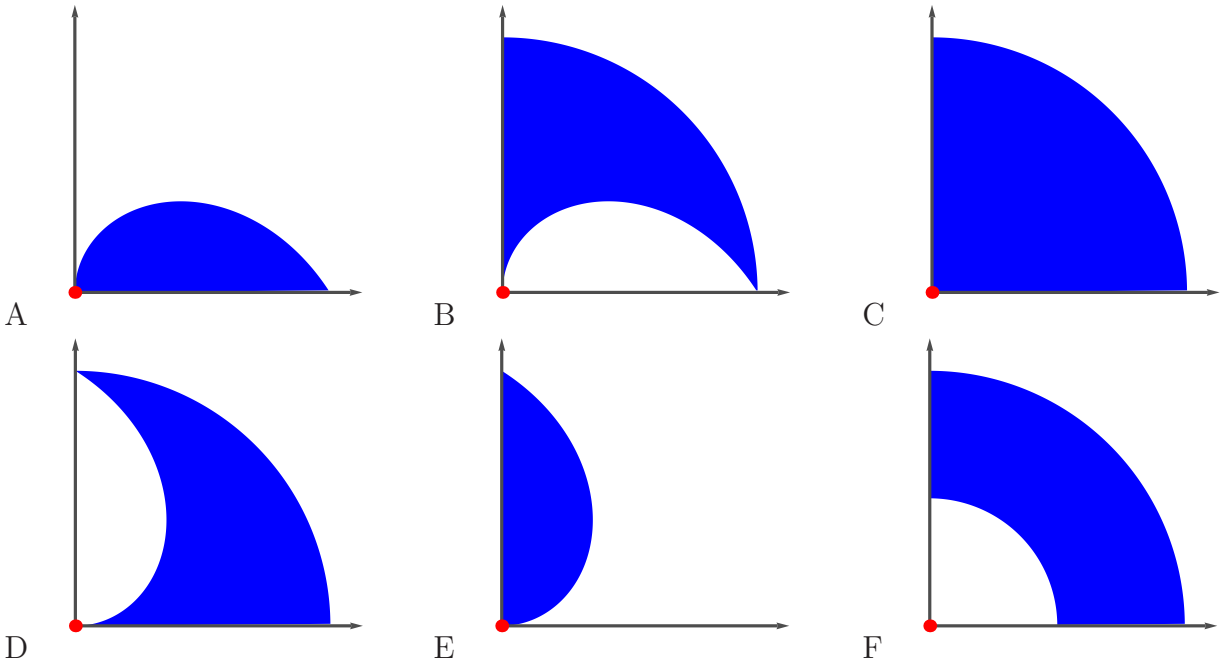
18) T F The double integral $\int_0^1\int_0^1 x^2-y^2\ dxdy$ is the volume of the solid below the graph of $f(x,y) = x^2-y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy-plane.

19) T F For any unit vector $\vec{v}$ and any differentiable function f, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

20) T F The surfaces $x+y+z = 0$ and $x^2+y^2+z^2+x+y+z = 0$ have the same tangent plane at $(0,0,0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta)r\ drd\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_\theta^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta)r\ drd\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_t + uu_s - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$

Enter A,B,C,D here	PDE
	$u_{tt} - u_{ss} = 0$
	$u_t - u_{ss} = 0$

A) Wave equation	B) Heat equation	C) Burgers equation	D) Laplace equation

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x,y)$?

Problem 4) (10 points)

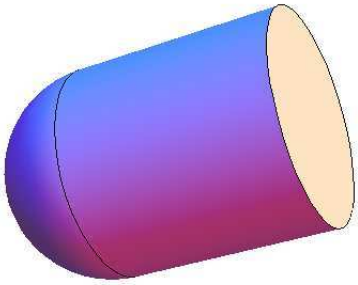
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

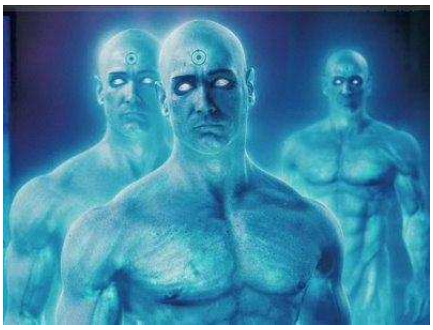
$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \ .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.



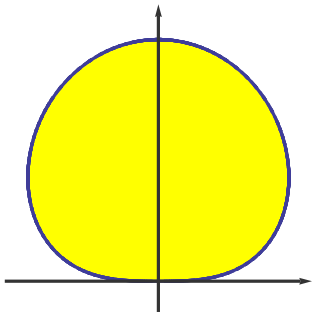


Problem 5) (10 points)

A region R in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int\int_R 1 \, dxdy$$

of this nihilistic object.



Problem 6) (10 points)

a) (4 points) If

$$f(x, y) = y \cos(x - y) \ ,$$

find equation of plane tangent to $z = f(x, y)$ at the point $(2, 2, 2)$.

b) (3 points) Find the equation of the tangent line to $f(x, y) = 2$ at $(2, 2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.



Image source: Harvard Press release on robobees.seas.harvard.edu

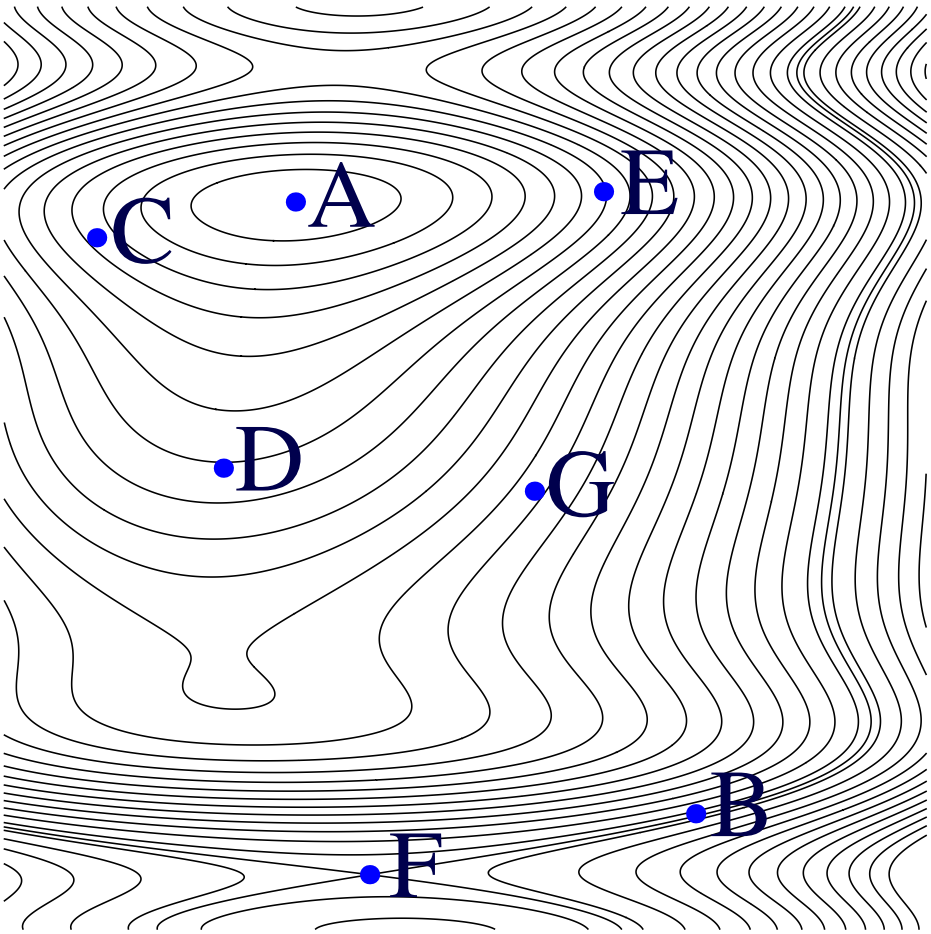
a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b?

b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x,y) = 0$ and $f_y(x,y) \neq 0$.
	$f_y(x,y) = 0$ and $f_x(x,y) \neq 0$.
	$f(x,y)$ has either a max or a min.
	$f(x,y)$ has a saddle point.
	$f(x,y)$ has no max nor min but is extremal under a constraint $y = c$ for some c.
	$f(x,y)$ has no max nor min but is extremal under a constraint $x = c$ for some c.
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2},1/\sqrt{2}\rangle} f(x,y) = 0$ and $D_{\langle 1/\sqrt{2},-1/\sqrt{2}\rangle} f(x,y) \neq 0$.
	$D_{\langle 1/\sqrt{2},-1/\sqrt{2}\rangle} f(x,y) = 0$ and $D_{\langle 1/\sqrt{2},1/\sqrt{2}\rangle} f(x,y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d.



Problem 9) (10 points)

Evaluate the following double integral

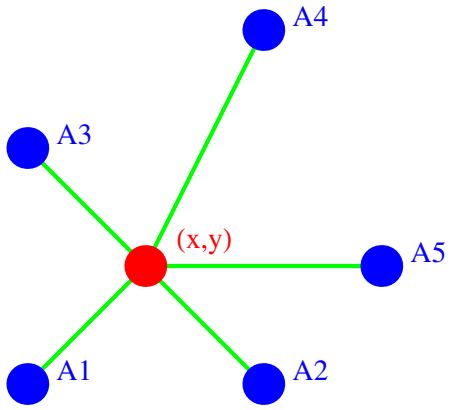
$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4}\, dydx\ .$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.



11/1/2017 Second Hourly Practice 8 **Math 21a, Fall 2017**

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10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F Every function $f(x,y)$ of two variables has either a global minimum or a global maximum.

2) T F The linearization of the function $f(x,y) = e^{x+3y}$ at $(0,0)$ is $L(x,y) = 1 + x + 3y$.

3) T F The function $f(x,y,z) = x^2\cos(z) + x^3y^2z + (y-2)^3y^5$ satisfies the partial differential equation $f_{xyxzxy} = 12$.

4) T F If $xe^z = y^2z$, then $\partial z/\partial x = e^z/(y^2 - xe^z)$.

5) T F The function $\cos(x^2)\cos(y^2)$ has a local maximum at $(0,0)$.

6) T F The value of the double integral $\int_0^{\pi/4}\int_0^2 x^3\cos(y)\ dxdy$ is the same as $(\int_0^2 x^3\ dx)(\int_0^{\pi/4}\cos(y)\ dy)$.

7) T F The gradient of $f(x,y)$ is always tangent to the level curves of f.

8) T F If $f(x,y,z) = x - 2y + z$, then the largest possible directional derivative $D_{\vec{u}}f$ at any point in space is $\sqrt{6}$.

9) T F $\int_0^1\int_0^1 (x^2+y^2)\ dxdy = \int_0^1\int_0^1 r^3\ drd\theta$.

10) T F It is possible that the directional derivative $D_{\vec{v}}f$ is positive for all unit vectors $\vec{v}$.

11) T F Using linearization of $f(x,y) = xy$ we can estimate $f(0.999, 1.01) \sim 1 - 0.001 + 0.01 = 1.009$.

12) T F Given a curve $\vec{r}(t)$ on a surface $g(x,y,z) = -1$, then $\frac{d}{dt}g(\vec{r}(t)) < 0$.

13) T F If $f(x,y)$ has a local minimum at $(0,0)$ then it is possible that $f_{xy}(0,0) > 0$.

14) T F The function $f(x,y) = -x^8 - 2x^6 - y^8$ has a local minimum at $(0,0)$.

15) T F If $\vec{r}(t)$ is a curve in space and f is a function of three variables, then $\frac{d}{dt}f(\vec{r}(t)) = 0$ for $t = 0$ implies that $\vec{r}(0)$ is a critical point of $f(x,y,z)$.

16) T F Let a, b, c be the number of saddle points, maxima and minima of a function $f(x,y)$. Then $a \leq b + c$.

17) T F If $f(x,y)$ is a nonzero function of two variables and R is a region, then $\int\int_R f(x,y)\ dxdy$ is the volume under the graph of f and therefore a positive value.

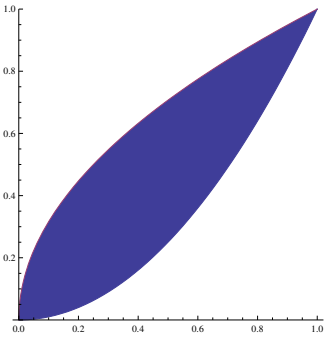
18) T F We extremize $f(x,y)$ under the constraint $g(x,y) = c$ and obtain a solution (x_0, y_0). If the Lagrange multiplier λ is positive, then the solution is a minimum.

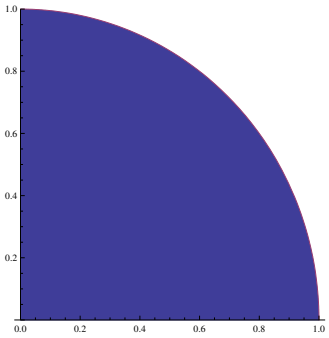
19) T F The tangent plane to a surface $f(x,y,z) = 1$ intersects the surface in exactly one point.

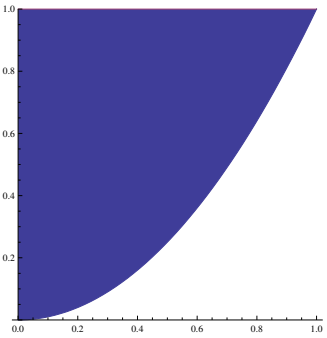
20) T F Let $\vec{v}$ be a vector of length 1 in space. Given a function $f(x,y,z)$ of three variables. If (x_0, y_0, z_0) is a critical point of f, then it is a critical point of $g(x,y,z) = D_v f(x,y,z)$.

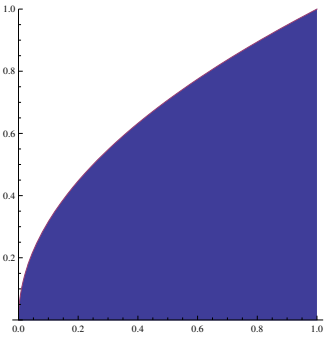
Problem 2) (10 points)

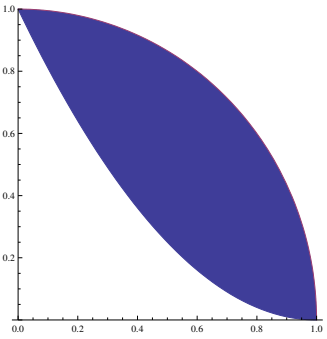
a) (6 points) Match the regions with the corresponding double integrals

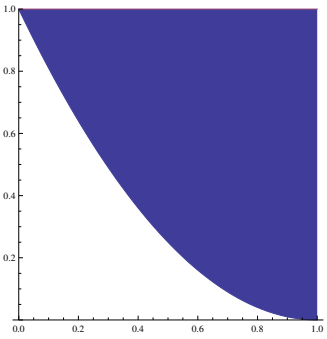
a 

b 

c 

d 

e 

f 

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\, dydx$
	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\, dxdy$
	$\int_0^1 \int_{y^2}^1 f(x,y)\, dxdy$

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dydx$
	$\int_0^1 \int_{(1-x)^2}^1 f(x,y)\, dydx$
	$\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x,y)\, dydx$

b) (4 points) Match the PDE's with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$f_{xx} = -f_{yy}$
	$f_x = f_y$

Enter A,B,C,D here	PDE
	$f_{xx} = f_{yy}$
	$f_x = f_{yy}$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Problem 3) (10 points)

a) (3 points) Find and classify all the critical points of $f(x, y) = xy - x$ on the plane.

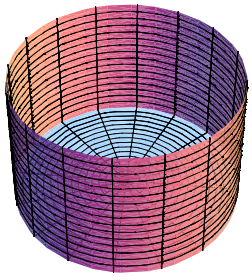
b) (2 points) Decide whether an absolute maximum or an absolute minimum of f exists on the plane R^2.

c) (3 points) Use the method of Lagrange multipliers to find the maximum and minimum of f on the boundary $x^2 + 4y^2 = 12$ of the elliptical region $G : x^2 + 4y^2 \leq 12$.

d) (2 points) Find the absolute maximum and absolute minimum of f on the region G given in c).

Problem 4) (10 points)

Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

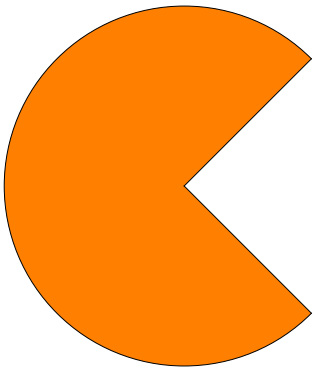


Problem 5) (10 points)

The Pac-Man region R is bounded by the lines $y = x, y = -x$ and the unit circle. The number

$$a = \frac{\int\int_R x \, dxdy}{\int\int_R 1 \, dxdy}$$

defines the point $C = (a, 0)$ called center of mass of the region. Find it.



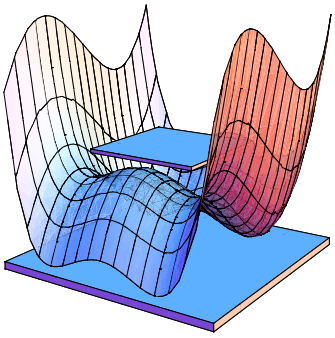
Problem 6) (10 points)

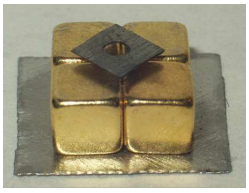
a) (5 points) Find the tangent plane to the surface $\sqrt{xyz} = 60$ at $(x, y, z) = (100, 36, 1)$.

b) (5 points) Estimate $\sqrt{100.1 * 36.1 * 0.999}$ using linear approximation. Here, for clarity reasons, we use $*$ for the usual multiplication for numbers.

Problem 7) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form $f(x, y) = x^4+y^3-2x^2-3y$. Find all critical points of this function and classify them. Is there a global minimum?





Right picture credit: Wikipedia.

Problem 8) (10 points)

Let $f(x, y) = xy$.

a) (2 points) Find the direction of maximal increase at the point $(1, 1)$.

b) (3 points) Find the directional derivative at $(1, 1)$ in the direction $\langle 3/5, 4/5\rangle$.

c) (2 points) The curve $\vec{r}(t) = \langle\sqrt{2}\sin(t), \sqrt{2}\cos(t)\rangle$ passes through the point $(1, 1)$ at some time t_0. Find $\frac{d}{dt}f(\vec{r}(t))$ at time t_0 directly.

d) (3 points) Find $\frac{d}{dt}f(\vec{r}(t))$ at time t_0 using the multivariable chain rule.

Problem 9) (10 points)

Integrate the function

$$f(x, y) = \frac{y^5 - 1}{y^{1/3} - y^{1/4}}$$

on the finite region bounded by the curves $y = x^3$ and $y = x^4$.

Problem 10) (10 points)

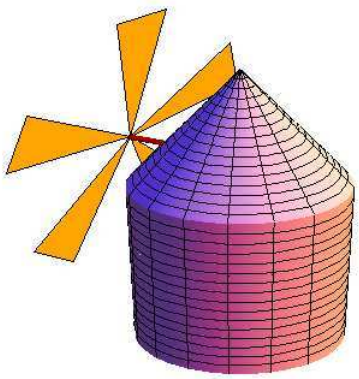
The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius r, the height of the side wall is h and the height of the roof is h, then the volume is

$$V(h, r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and assume the cost of the building is

$$A(h, r) = \pi r^2 + 2\pi rh + \pi 2r^2 = \pi(3r^2 + 2rh)$$

which is the area of the ground plus the area of the wall plus $2\pi rh$, the cost for the roof. For fixed volume $V(h, r) = 4\pi/3$, minimize the cost $A(h, r)$ using the Lagrange multiplier method.



11/1/2017 Second Hourly Practice 9 **Math 21a, Fall 2017**

Name:

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MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The directional derivative $D_{\vec{v}}f$ is a vector perpendicular to $\vec{v}$.

2) [T] [F] Using linearization of $f(x,y) = xy$ we can estimate $f(0.9, 1.2) \sim 1 - 0.1 + 0.2 = 1.1$.

3) [T] [F] Given a curve $\vec{r}(t)$ on a surface $g(x,y,z) = 1$, then $\frac{d}{dt}g(\vec{r}(t)) = 0$.

4) [T] [F] Given a function $f(x,y)$ such that $\nabla f(0,0) = \langle 2, -1\rangle$. Then $D_{\langle 0,-1\rangle}f(0,0) = 0$.

5) [T] [F] $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ is a surface of revolution.

6) [T] [F] If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2, y^2)$.

7) [T] [F] If $f(x,y)$ has a local maximum at $(0,0)$ then it is possible that $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) < 0$.

8) [T] [F] The integral $\int_0^x \int_0^y 1\, dxdy$ computes the area of a region in the plane.

9) [T] [F] The function $f(x,y) = x^2 + y^4$ has a local minimum at $(0,0)$.

10) [T] [F] The integral $\int_0^1 \int_0^1 x^2 + y^2\, dxdy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.

11) [T] [F] There exists a region in the plane, which is neither a type I integral, nor a type II integral.

12) [T] [F] Fubini's theorem assures that $\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_0^y f(x,y)\, dxdy$.

13) [T] [F] The function $f(x,y) = \sin(x)\cos(y)$ satisfies the partial differential equation $f_{xx} + f_{yy} = 0$.

14) [T] [F] Let $L(x,y)$ be the linearization of $f(x,y) = \sin(x(y+1))$ at $(0,0)$. Then, the level curves of $L(x,y)$ consist of lines.

15) [T] [F] For any smooth function $f(x,y)$, the inequality $|\nabla f| \geq |f_x + f_y|$ is true.

16) [T] [F] Any differentiable function $f(x,y)$ which satisfies the partial differential equation $||\nabla f||^2 = 0$ is constant.

17) [T] [F] If $x + \sin(xy) = 1$, $dy/dx = \frac{-(1+y\cos(yx))}{(x\cos(xy))}$.

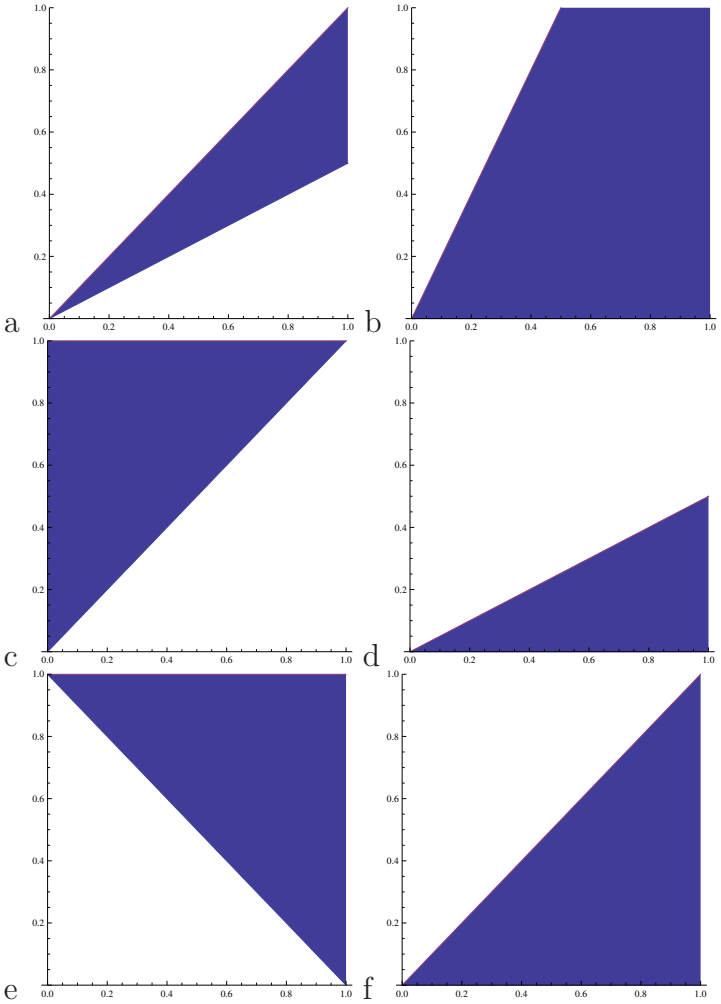
18) [T] [F] The directional derivative $D_v f(1,1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1,1)$.

19) [T] [F] If (a,b) is a maximum of $f(x,y)$ under the constraint $g(x,y) = 0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a,b).

20) [T] [F] If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle}f(1,2) = 0$ and $D_{\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle}f(1,2) = 0$, then $(1,2)$ is a critical point.

Problem 2) (10 points)

Match the regions with the corresponding double integrals



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x/2}^{x} f(x,y)\ dydx$
	$\int_0^1 \int_0^{y} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{x/2} f(x,y)\ dydx$
	$\int_0^1 \int_{y/2}^{1} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{x} f(x,y)\ dydx$
	$\int_0^1 \int_{1-x}^{1} f(x,y)\ dydx$

Problem 3) (10 points)

Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.
a) (5 points) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.
b) (5 points) The surface in a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.

Problem 4) (10 points)

a) (5 points) Use the technique of linear approximation to estimate $f(\pi/2 + 0.1, 2.9)$ for

$$f(x, y) = (10 \sin(x) - 5y^2 + 8)^{1/3} .$$

b) (5 points) Find the unit vector at $(\pi/2, 3)$, in the direction where the function increases fastest.

Problem 5) (10 points)

The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$.
a) (2 points) State the chain rule which applies in this situation.
b) (4 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.
c) (4 points) At which time t does the observer go in the direction, in which the pressure decreases most?

Problem 6) (10 points)

The coffee chain **Astrbucks** [1] has branches at $(0, 0), (0, 3)$ and $(3, 3)$ (JFK street, Church street, and Broadway) near Harvard square. A caffeine addicted [politically correct: loving] mathematician wants to rent an apartment at a location, where the sum of the squared distances $f(x, y)$ to all those shops is a local minimum. The function is

$$f(x, y) = (x-0)^2+(y-0)^2+(x-0)^2+(y-3)^2+(x-3)^2+(y-3)^2 = 27-6x+3x^2-12y+3y^2 .$$

a) (5 points) Where does the mathematician have to live to locally minimize $f(x, y)$?
b) (3 points) For every local minimum answer: Is this local minimum a **global** minimum?
c) (2 points) Is there a global maximum to this problem? If yes, give it. If no, why not?

[1] This problem was sponsored by *Astrbucks*©.



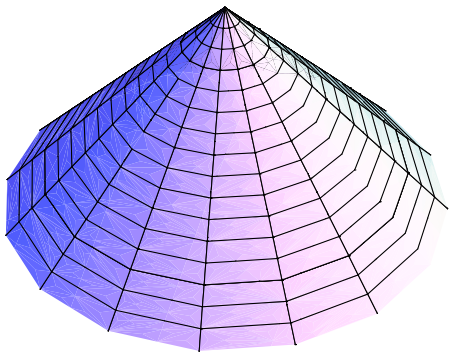


Problem 7) (10 points)

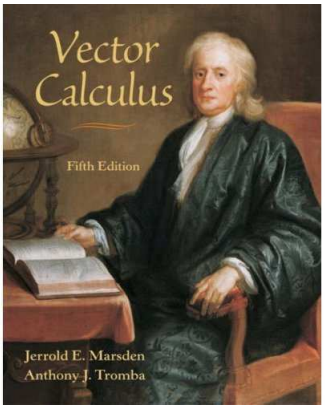
Find all the critical points of $f(x, y) = 3xy + x^2y + xy^2$ and classify them as saddle points, local maxima or local minima.

Problem 8) (10 points)

A solid cone of height h and with base radius r has the volume $f(h, r) = h\pi r^2/3$ and the surface area $g(h, r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h, r) = \pi$ use the Lagrange method to find the cone with maximal volume.



Problem 9) (10 points)



Marsden and Tromba pose in their textbook the following riddle: Suppose $w = f(x, y)$ and $y = x^2$. By the chain rule

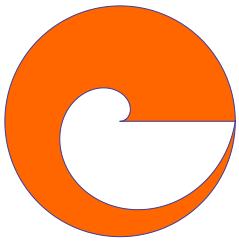
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + 2x\frac{\partial w}{\partial y}$$

so that $0 = 2x\frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$.

a) Find an explicit example of a function $f(x, y)$, where you see the argument is false.

b) What is flawed in the above application of the chain rule?

Problem 10) (10 points)



Evaluate the double integral

$$\int\int_R \sqrt{x^2 + y^2}\, dxdy$$

where R is the region bounded by the positive x-axes, the spiral curve $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle, 0 \le t \le 2\pi$ and the circle with radius 2π.

11/1/2017 Second Hourly Practice 10 **Math 21a, Fall 2017**

Name:

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MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
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- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] $f(x$

2) [T] [F] If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

3) [T] [F] Given 2 arbitrary points in the plane, there is a function $f(x,y)$ which has these points as critical points and no other critical points.

4) [T] [F] If (x_0, y_0) is the maximum of $f(x,y)$ on the disc $x^2+y^2 \leq 1$ then $x_0^2+y_0^2 < 1$.

5) [T] [F] There are no functions $f(x,y)$ for which every point on the unit circle is a critical point.

6) [T] [F] An absolute maximum (x_0, y_0) of $f(x,y)$ is also an absolute maximum of $f(x,y)$ constrained to a curve $g(x,y) = c$ that goes through the point (x_0, y_0).

7) [T] [F] If $f(x,y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

8) [T] [F] There exists a function $f(x,y)$ of two variables which has no critical points at all.

9) [T] [F] If $f_x(x,y) = f_y(x,y) = 0$ for all (x,y) then $f(x,y) = 0$ for all (x,y) .

10) [T] [F] $(0,0)$ is a local maximum of the function $f(x,y) = x^2 - y^2 + x^4 + y^4$.

11) [T] [F] If $f(x,y)$ has a local maximum at the point $(0,0)$ with discriminant $D > 0$ then $g(x,y) = f(x,y) - x^4 + y^3$ has a local maximum at the point $(0,0)$ too.

12) [T] [F] Every critical point (x,y) of a function $f(x,y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

13) [T] [F] If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

14) [T] [F] In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

15) [T] [F] The function $f(x,y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0,0)$.

16) [T] [F] It is possible to find a function of two variables which has no maximum and no minimum.

17) [T] [F] The value of the function $f(x,y) = \sqrt{1+3x+5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

18) [T] [F] The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

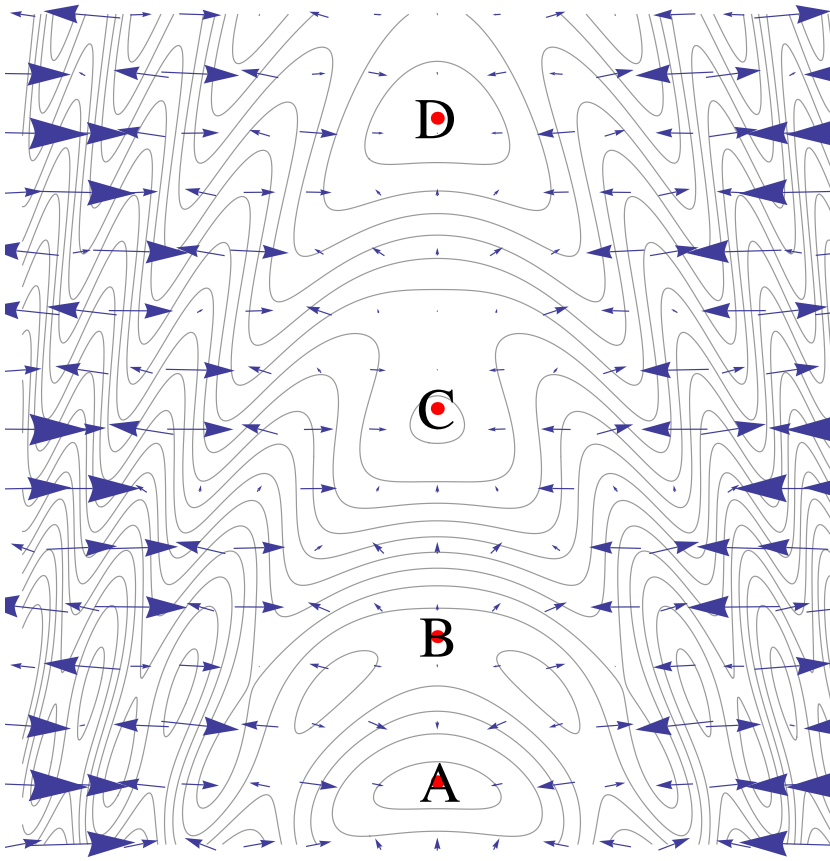
19) [T] [F] If $\vec{r}(t)$ is a curve with unit speed in the plane with $\vec{r}(0) = (0,0)$ and $D_{\vec{r}\,'(0)} f(0,0) = 0$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at the time $t = 0$.

20) [T] [F] If a function $f(x,y)$ satisfies the partial differential equation $f_x^2 - f_y^2 = 0$, then f is the constant function.

Problem 2) (10 points)

a) (5 points) The picture below shows the contour map of a function $f(x, y)$ which has many critical points. Four of them are outlined for you on the y axes and are labeled A, B, C, D and ordered in increasing y value. The picture shows also the gradient vectors. Determine from each of the 4 points whether it is a local maximum, a local minimum or a saddle point. No justification is necessary in this problem.

Point	Max	Min	Saddle
D			
C			
B			
A			



b) (5 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x,y)\, dxdy$
	$\int_0^1 \int_y^1 f(x,y)\, dxdy$
	$\int_0^1 \int_0^{1-y} f(x,y)\, dxdy$
	$\int_0^1 \int_0^y f(x,y)\, dxdy$

I) $\int_0^1 \int_0^x f(x,y)\, dydx$
II) $\int_0^1 \int_0^{1-x} f(x,y)\, dydx$

III) $\int_0^1 \int_x^1 f(x,y)\, dydx$
IV) $\int_0^1 \int_0^{x-1} f(x,y)\, dydx$
V) $\int_0^1 \int_{1-x}^1 f(x,y)\, dydx$

Problem 3) (10 points)

When Ramanujan, the amazing India born mathematician was sick in the hospital in england and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered: "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.



a) (5 points) Find the linearization $L(x,y,z)$ of the function $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$.

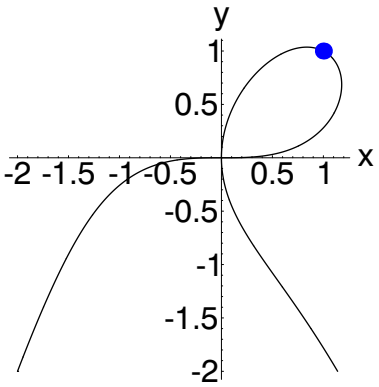
b) (5 points) Use the technique of linear approximation to estimate $9.001^3 + 10.02^3 - 12.001^3$. Since we are not all Ramanujans, you can leave the end result as a product and sum of numbers. For example, $234 \cdot 0.001 - 100 \cdot 0.002$ would be an acceptable end result.

Problem 4) (10 points)

Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x = 1, y = 1$, the function can be written as a graph $y = y(x)$. Find the slope of that graph at the point $(1,1)$.



Problem 5) (10 points)

a) Find a point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ which is locally closest to the origin.
b) Is this a global minimum? Hint: look at points $(x,y,z) = (1, -1/n, 8/n)$ where n is an integer.

Problem 6) (10 points)

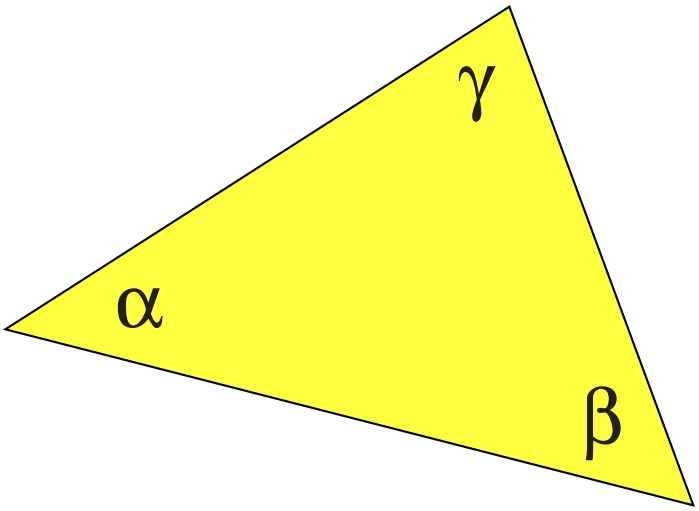
Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum among them?

Problem 7) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha,\beta,\gamma) = \log\left(\sin(\alpha)\sin(\beta)\sin(\gamma)\right)$$

is maximal?



Problem 8) (10 points)

Let $g(x,y)$ be the distance from a point (x,y) to the curve $x^2 + 2y^2 + y^4/10 = 1$. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve.

Hint: no computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f?

Remark: This problem only needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Problem 9) (10 points)

a) (6 points) Find all critical points of $f(x, y) = 3xe^y - e^{3y} - x^3$ and classify them.

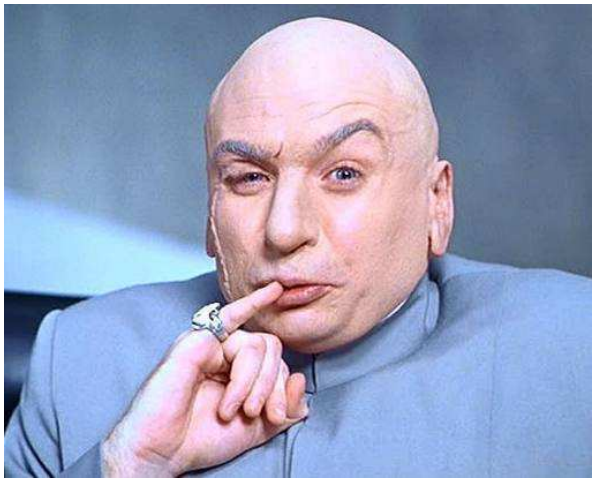
b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Problem 10) (10 points)

a) (5 points) Integrate $f(x, y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \leq 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \; .$$



11/1/2017 Second Hourly Practice 11 **Math 21a, Fall 2017**

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MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

2) [T] [F] At a critical point of a function f, the gradient vector has length 1.

3) [T] [F] At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

4) [T] [F] For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x, y)$ is nonzero.

5) [T] [F] If $f_{xx}(0, 0) = 0, D = f_{xx} f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0, 0) = \langle 0, 0 \rangle$, then $(0, 0)$ is a saddle point.

6) [T] [F] A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

7) [T] [F] The function $f(x, y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0, 0)$.

8) [T] [F] If (x, y) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 5$ then it is also a maximum of $f(x, y) + g(x, y)$ under the constraint $g(x, y) = 5$.

9) [T] [F] The functions $f(x, y)$ and $g(x, y) = (f(x, y))^6$ always have the same critical points.

10) [T] [F] For $f(x, y, z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1, 1, 1)$ is perpendicular to the surface $f(x, y, z) = 4$ at the point $(1, 1, 1)$.

11) [T] [F] $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

12) [T] [F] If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

13) [T] [F] If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c, h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1, 1, 1)$ is a critical point of f.

14) [T] [F] Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

15) [T] [F] At a saddle point, all directional derivatives are zero.

16) [T] [F] The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

17) [T] [F] At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

18) [T] [F] It is possible that $f(x, y)$ attains a maximum under the constraint $g(x, y) = 0$ at a point, where $\nabla f \neq \lambda \nabla g$.

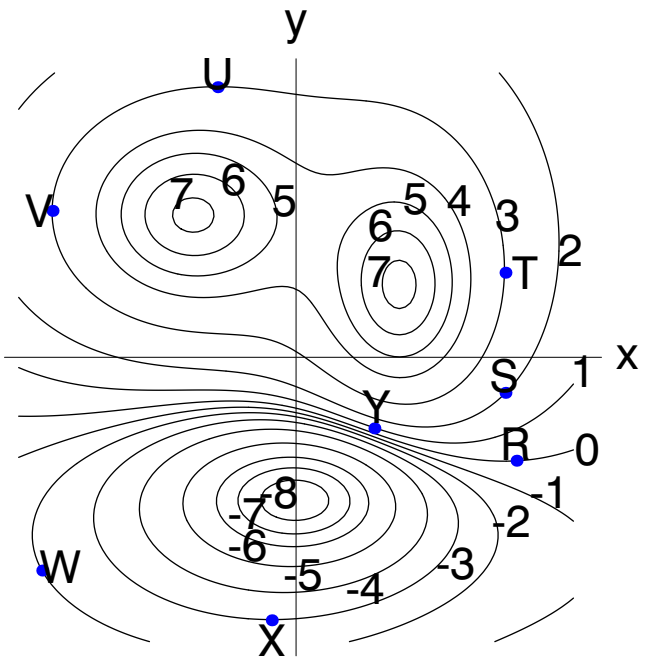
19) [T] [F] Any Lagrange problem which asks for an extremum of $f(x, y)$ under a constraint $g(x, y) = 0$ has either a maximum or a minimum.

20) [T] [F] The function $u(x, y) = \sin(x + y)$ satisfies the PDE $u_{xx} + u_{yy} - 2u_{xy} = 0$.

Problem 2) (10 points) No justifications needed.

a) (4 points) Fill in the boxes. You do not need to give additional explanations.

Chain rule:	$\frac{d}{dt} f(\vec{r}(t)) =$ ☐ $\cdot\, \vec{r}\,'(t)$
Directional derivative D_v	$D_{\langle 2,3\rangle/\sqrt{13}} f(1,1) = \nabla f(1,1) \cdot$ ☐
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) =$ ☐ $+ \nabla f(1,1) \cdot (x-1, y-1)$
Equation of tangent line at $(1,1)$	$\nabla f(1,1) \cdot \langle x-1, y-1\rangle =$ ☐
Critical point $(1,1)$ of f	$\nabla f(1,1) =$ ☐
Lagrange equations	$\nabla f(x,y) =$ ☐ $\nabla g(x,y),\ g(x,y) = c.$
Type I integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y)$ ☐ .
Type II integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y)$ ☐ .
Integration in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)}$ ☐ $f(r\cos(\theta), r\sin(\theta))\ dr d\theta.$
Area	$\int\int_R$ ☐ $dxdy$



b) (2 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

c) (2 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

d) (2 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Problem 3) (10 points)

a) Locate and classify all the critical points of

$$f(x,y) = 3y - y^3 - 3x^2y \ .$$

b) Where on the parameterized surface

$$\vec{r}(x,y) = \langle u,v,w\rangle = \langle xy^3, x^2/2, 3y^2/2\rangle$$

is the function $g(u,v,w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x,y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Problem 4) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4-\sqrt{x}}\, dx\, dy \ .$$

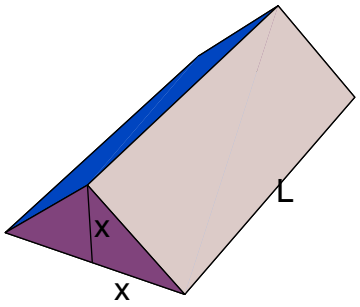
Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x,y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have an absolute maximum or absolute minimum? Make sure to justify also this answer.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x,y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.



Problem 7) (10 points)

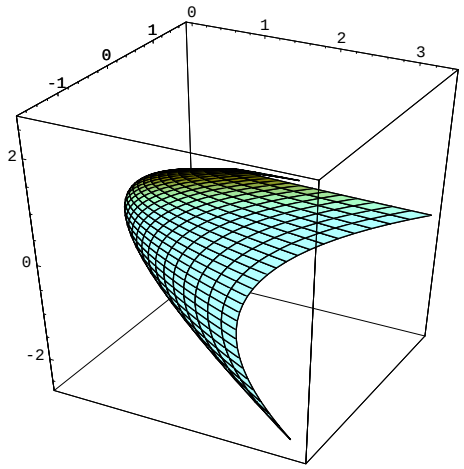
Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Problem 8) (10 points)

The temperature distribution in a room is $T(x,y,z) = x + y + z$. On which point of the parametrized surface

$$\vec{r}(s,t) = \langle x, y, z \rangle = \langle s^2 + t^2, st, 2s - t \rangle$$

is the temperature extremal? Is it a maximum or a minimum?



Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta^2$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi - (x^2+y^2)^{1/4})}\, dx\, dy\ .$$



Problem 10) (10 points)

Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$.

a) Find the linear approximation of $f(1.01, 0.98)$.
b) What is the gradient ∇f at $(1, 1)$?
c) What is the equation $ax + by = d$ of the tangent line at $(1, 1)$?

11/1/2017 Second Hourly Practice 12 **Math 21a, Fall 2017**

Name:

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MWF 11 Oliver Knill
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- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.

2) [T] [F] Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$.

3) [T] [F] The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface.

4) [T] [F] Even with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x, y)$ at (a, b) is non-zero.

5) [T] [F] There exists a pair of different points on a sphere, for which the tangent planes are parallel.

6) [T] [F] If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_u f(x, y, z) = 0$.

7) [T] [F] Assume we have a smooth function $f(x, y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x, y) = 0$. Then $(0, 0)$ is a critical point with $D < 0$.

8) [T] [F] The gradient of $f(x, y)$ is perpendicular to the graph of f.

9) [T] [F] The level curves of a linearization $L(x, y)$ of a function $f(x, y) = \sin(x + y)$ at $(0, 0)$ consist of lines.

10) [T] [F] If $x^4 y + \sin(y) = 0$ then $y' = 4x^3 y/(x^4 + \cos(y))$.

11) [T] [F] The linearization $L(x, y)$ at a critical point (x_0, y_0) of a function $f(x, y)$ is a constant function.

12) [T] [F] The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s, t), y(s, t), z(s, t)\rangle = \langle s, t, f(s, t)\rangle$ for some function $f(s, t)$ for which the parametrization covers the entire surface.

13) [T] [F] The tangent plane to the graph of $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x, y, z)$ of $z - f(x, y)$.

14) [T] [F] The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

15) [T] [F] The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle.

16) [T] [F] If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

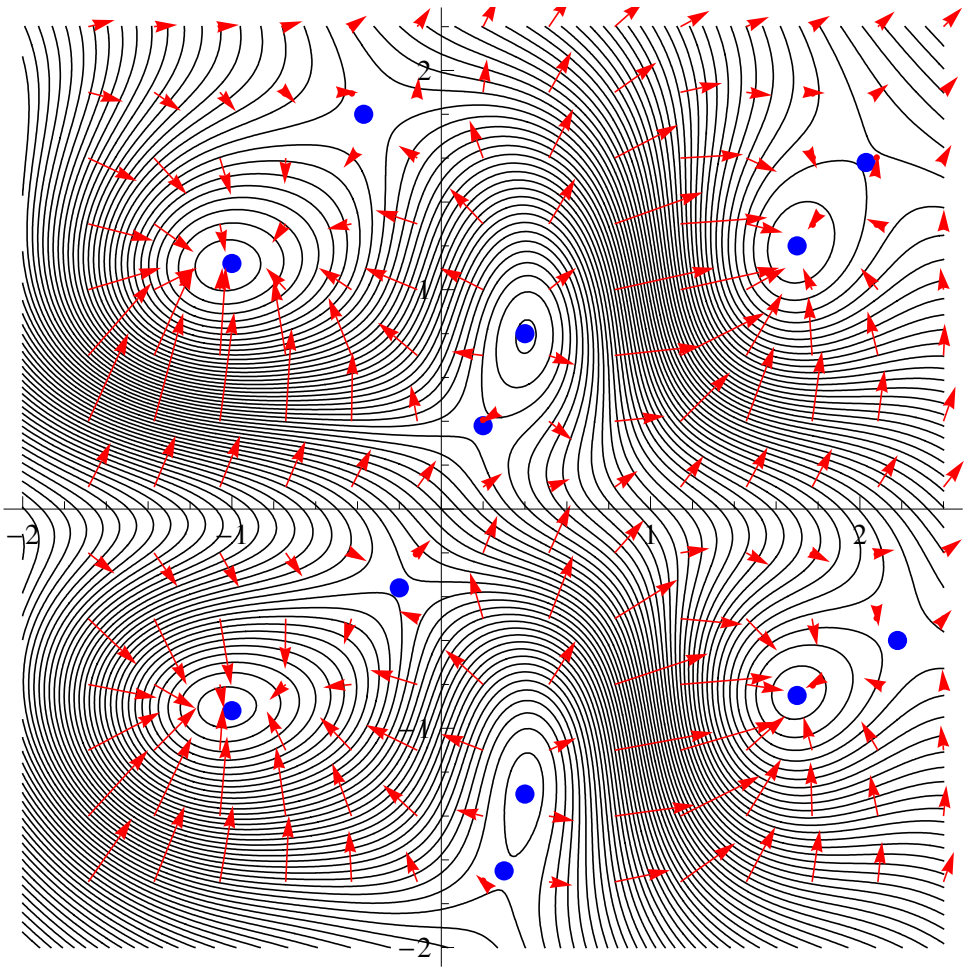
17) [T] [F] If $f(x, y, z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$.

18) [T] [F] A function $f(x, y)$ in the plane always has a local minimum or a local maximum.

19) [T] [F] For any smooth function $f(x, y)$, the inequality $\|\nabla f\| \geq |f_x + f_y|$ is true.

20) [T] [F] If a function $f(x, y)$ satisfies $|\nabla f(x, y)| = 1$ everywhere in the plane, then f is constant.

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x, y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x, y)$ has		local maxima
The function $f(x, y)$ has		local minima
The function $f(x, y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Problem 3) (10 points)

Find the cos of the angle between the sphere

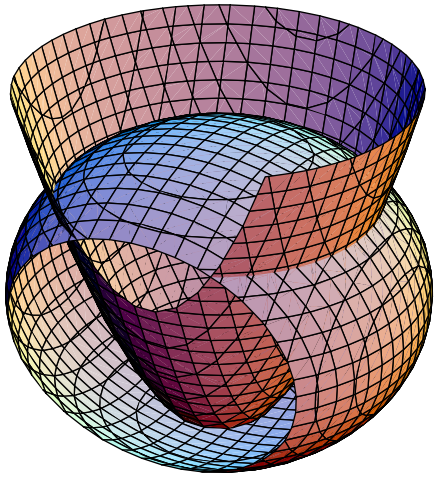
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P.



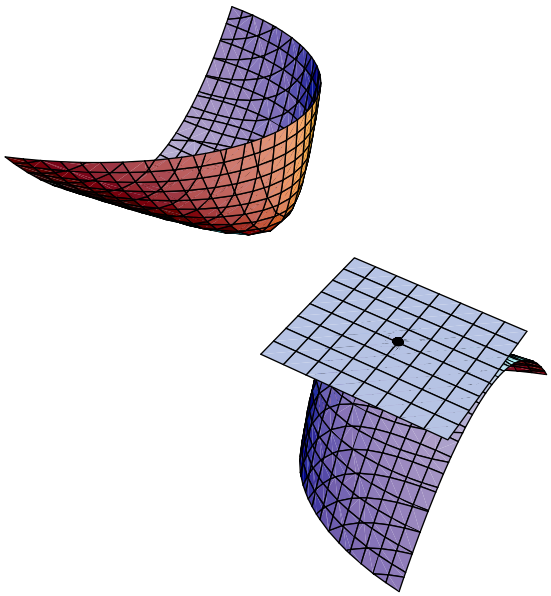
Problem 4) (10 points)

a) You know that

$$-2x + 5y + 10z = 2$$

is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$. Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.



Problem 5) (10 points)

a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 \ .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 \ .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Problem 6) (10 points)

a) Assume $f(x, y) = e^{2x-y-2} + y + \sin(x - 1)$ and $x(t) = \cos(5t), y(t) = \sin(5t)$. What is

$$\frac{d}{dt} f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x, y, z) = (1, 1, 1)$. Find the gradient

$$\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \rangle$$

of $z(x, y)$ at the point $(1, 1)$.

Problem 7) (10 points)

The temperature in a room is given by $T(x, y, z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1, 0, 0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3, 2, 1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.



Problem 8) (10 points)

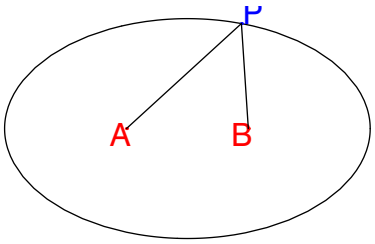
Let $g(x, y)$ denote the distance of a point $P = (x, y)$ to a point A and $h(x, y)$ the distance from P to a point B. The set of points (x, y) for which $f(x, y) = g(x, y) + h(x, y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot r' = -\nabla h \cdot r'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P. (Hint: check that $|\nabla f| = |\nabla g| = 1$).

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.

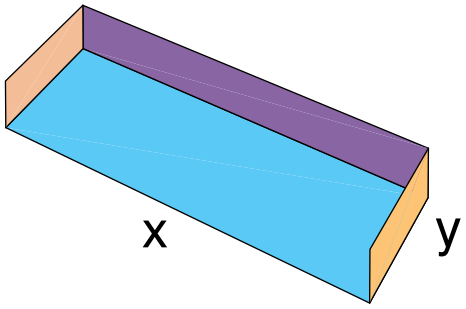


Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x, y) = xy + 2x + 2y$$

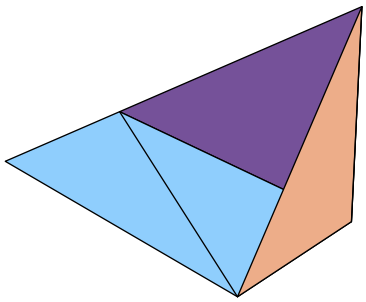
of length x, width y and height 1 under the constraint that the volume $g(x, y) = xy$ is constant and equal to 4.



Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one

triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?



12/15/2017 FINAL EXAM **Math 21a, Fall 2017**

Name:

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1		20
2		10
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6		10
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8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The length of the vector $\vec{v} = \langle 3, 4, 5\rangle$ is equal to the distance from the point $(1, 1, 1)$ to the point $(-2, -3, -4)$.

2) T F There is a non-constant vector field $\vec{F}(x, y, z)$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.

3) T F For every $\vec{v}$ and $\vec{w}$, the projection of $\vec{v}$ onto $\vec{w}$ always has the same length as the projection of $\vec{w}$ onto $\vec{v}$.

4) T F If $\vec{F}(x, y, z) = \langle \sin(z), \cos(z), 0\rangle$, then $\text{curl}(\text{curl}(\text{curl}(\vec{F}))) = \vec{F}$.

5) T F If S is the graph $z = x^6 + y^6$ above $x^2 + y^2 \leq 1$ oriented upwards and $\vec{F}(x, y, z) = \langle 0, 0, z\rangle$, then the flux of $\vec{F}$ through S is positive.

6) T F If S is the unit sphere oriented outwards and $\vec{F}(x, y, z) = \langle 0, 0, z^2\rangle$, then the flux of $\vec{F}$ through the upper hemisphere of S is the same as the flux through the lower hemisphere.

7) T F If the divergence of $\vec{F}$ is zero, then $\vec{F}$ is a gradient vector field.

8) T F If E is the unit ball $x^2 + y^2 + z^2 \leq 1$ and $\vec{F}$ is the curl of some other vector field, then $\int\int\int_E \text{div}(\vec{F})dV = 4\pi/3$.

9) T F The curve $\vec{r}(t) = \langle t, t\rangle$ is a flow line of $\vec{F}(x, y) = \langle x, 2y\rangle$.

10) T F The integral $\iint_R \sqrt{1 + f(u, v)^2}\ dudv$ is the surface area of the surface parametrized by $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$ for $(u, v) \in R$.

11) T F The volume of a parallelepiped with corners $A, B = A + \vec{v}, C = A + \vec{w}, D = A + \vec{v} + \vec{w}$ and $A + \vec{u}, B + \vec{u}, C + \vec{u}, D + \vec{u}$ is $|\vec{u} \cdot (\vec{v} \times \vec{w})|$.

12) T F The curvature of $y = x^2$ at $(0, 0)$ is larger than the curvature of $y = 3x^2$ at $(0, 0)$.

13) T F If $\vec{F}$ and $\vec{G}$ are two vector fields for which the divergence is the same, then $\vec{F} - \vec{G}$ is a constant vector field.

14) T F If $\vec{F}, \vec{G}$ are two vector fields which have the same curl, then $\vec{F} - \vec{G}$ is irrotational.

15) T F The parametrization $\vec{r}(u, v) = \langle 1 + u, v, u + v\rangle$ describes a plane.

16) T F Any function $u(x, y)$ that obeys the partial differential equation $u_x + u_y - u_{xx} = 1$ has no local minima.

17) T F If $\vec{F} = \langle P, Q, R\rangle$ is a vector field so that $\langle P_x, Q_y, R_z\rangle = \langle 0, 0, 0\rangle$, then it is incompressible meaning that the divergence is zero everywhere.

18) T F If $f(x, g(x)) = 0$, then $g'(x) = -f_x/f_y$ provided f_y is not zero.

19) T F The equation $x^2 - (y - 1)^2 + z^2 + 2z = -1$ represents a two-sheeted hyperboloid.

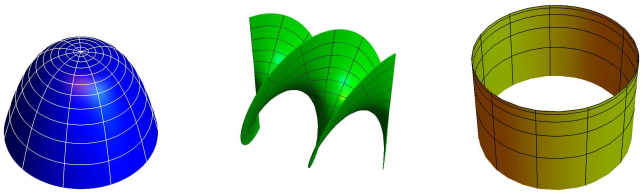
20) T F If $\vec{F}(\vec{r}(u, v)) = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$, then the absolute value of the flux of $\vec{F}$ through a closed bounded surface S parametrized by $\vec{r}(u, v)$ is the surface area of S.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(u,v)$	A-C
$\langle u\sin(v), u\cos(v), -u^3\rangle$	
$\langle u, v\sin(u), v\cos(u)\rangle$	
$\langle \sin(u), \cos(u), -v^3\rangle$	

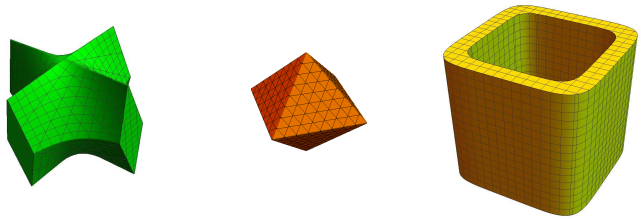
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$1 < x^6 + y^6 < 2$	
$\lvert x\rvert\lvert y\rvert + \lvert z\rvert < 1$	
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert < 3$	

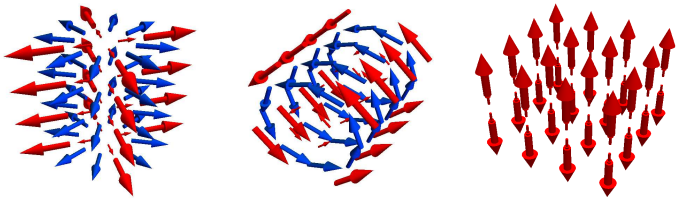
A B C



c) (2 points) The figures display vector fields $\vec{F}$. Match them.

Field	A-C
$\vec{F}(x,y,z) = \langle 0, 0, z\rangle$	
$\vec{F}(x,y,z) = \langle -z, 0, x\rangle$	
$\vec{F}(x,y,z) = \langle x, y, 0\rangle$	

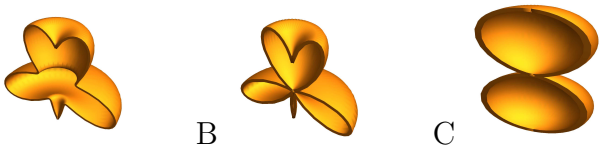
A B C



d) (2 points) Match the spherical plots.

Surface	A-C
$\rho(\theta,\phi) = 1 + \sin(4\phi)$	
$\rho(\theta,\phi) = 2 + \sin(4\phi)$	
$\rho(\theta,\phi) = 1 + \cos(2\phi)$	

A B C

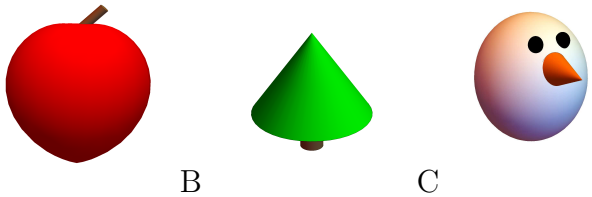


e) (1 point) Name a partial differential equation (PDE) for a function $u(t,x)$ discussed in this course which involves a term uu_x.

f) (1 point) Match each surface S to a graphic that contains S.

Surface S	A-C
$r^2 - (1-z)^2 = 0$	
$x^2 + y^2 + z^2 = 1$	
$\rho(\theta,\phi) = \sin^2(\phi/2)\phi$	

A B C



Problem 3) (10 points)

a) (3 points) Ed Sheeran's **"Shape of You"**, released in January this year, has been a critical success: it peaked at number-one on the singles charts of 44 countries and is currently the most streamed song on **Spotify.** But what is the shape of you? Which of the letters are not simply connected (SC)?

	Check if not SC
S	
H	
A	
P	
E	

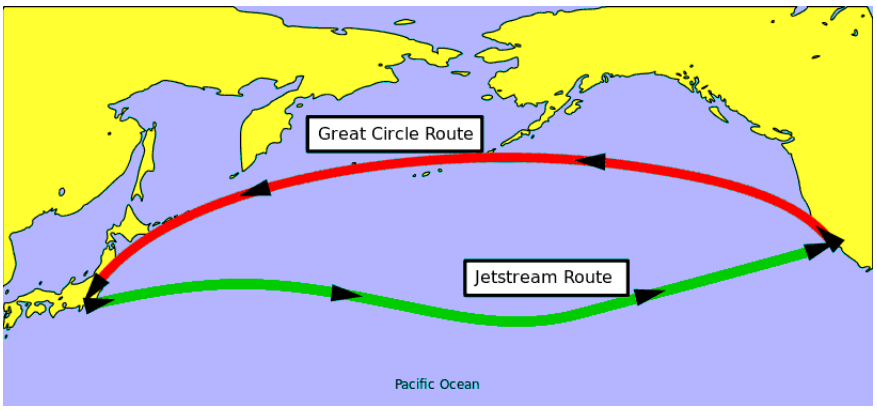
	Check if not SC
O	
F	
Y	
O	
U	



b) (4 points)

A plane flies from **Los Angeles** to **Tokyo** along the great circle route A and comes back via the jet stream route B. There is a force field $\vec{F}$ acting on the plane so that the work along A is $\int_A \vec{F} \cdot \vec{dr}$ and the work along B is $\int_B \vec{F} \cdot \vec{dr}$. You know that there is a potential function f such that $\vec{F} = \nabla f$. Check the statements that must be true.

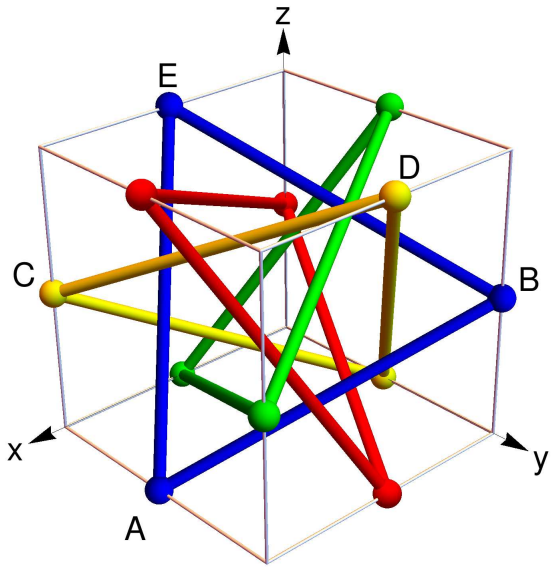
$\int_A \vec{F} \cdot \vec{dr} = 0$	
$\int_B \vec{F} \cdot \vec{dr} = 0$	
$\int_A \vec{F} \cdot \vec{dr} + \int_B \vec{F} \cdot \vec{dr} = 0$	
$\int_A \vec{F} \cdot \vec{dr} - \int_B \vec{F} \cdot \vec{dr} = 0$	



c) (3 points) Let E be the solid given by $x^8 + y^8 + z^8 \leq 4, y \geq 0$. Let S be the boundary of E with outward orientation. Consider the vector fields $\vec{F} = \langle x, y, z \rangle$, $\vec{G} = \langle x, y, -z \rangle$ and $\vec{H} = \langle x + y, y^2 + z^2, yz \rangle$. Check the correct box in each line:

Flux integral	$< \text{Vol}(E)$	$= \text{Vol}(E)$	$> \text{Vol}(E)$
$\iint_S \vec{F} \cdot d\vec{S}$			
$\iint_S \vec{G} \cdot d\vec{S}$			
$\iint_S \vec{H} \cdot d\vec{S}$			

Problem 4) (10 points)



In topology one knows the **Danzer cube**. It is an example of what one calls a "non-shellable triangulation" of the cube. The picture shows four of the triangles.

a) (5 points) Find the distance between the line joining

$$A = (2, 1, 0) \text{ and } B = (0, 2, 1)$$

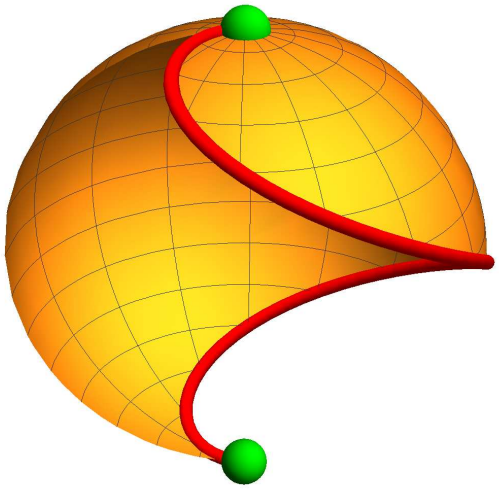
and the line joining

$$C = (2, 0, 1) \text{ and } D = (1, 2, 2) \ .$$

b) (5 points) Find the area of the triangle ABE, where

$$E = (1, 0, 2) \ .$$

Problem 5) (10 points)



a) (6 points) Find the **surface area** of

$$\vec{r}(t, s) = \langle \cos(t)\sin(s), \sin(t)\sin(s), \cos(s) \rangle$$

$0 \leq t \leq 2\pi, 0 \leq s \leq t/2.$

b) (4 points) The part of the boundary curve when $s = t/2$ is defined as

$$\vec{r}(t) = \langle \cos(t)\sin(t/2), \sin(t)\sin(t/2), \cos(t/2) \rangle \ .$$

Compute the number $|\int_0^{2\pi} \vec{r}\,'(t)\, dt|$.

Problem 6) (10 points)

The **Longy School of Music** at 27 Garden Street shows an abstract art work featuring a sphere, a cone and a cylinder. You build a model. Your parametrization should use the variables provided. No further justifications are needed in this problem.

a) (2 points) Parametrize the sphere $x^2 + y^2 + (z-5)^2 = 9$.

$$\vec{r}(\theta, \phi) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

b) (3 points) Parametrize the cylinder $x^2 + y^2 = 4$.

$$\vec{r}(\theta, z) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

c) (3 points) Parametrize the cone $4y^2 + 4(z-5)^2 = x^2$.

$$\vec{r}(\theta, x) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

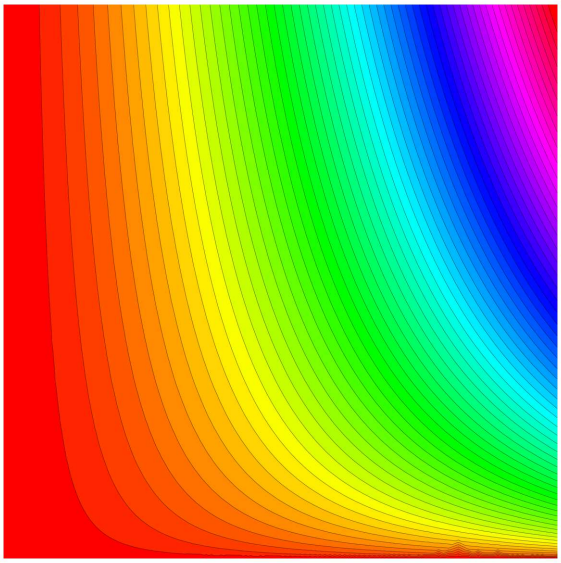
d) (2 points) Parametrize the grass floor $z = \sin(99x + 99y)$.

$$\vec{r}(x, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$



Photo: O. Knill, December 2017

Problem 7) (10 points)



a) (6 points) Find the **linearization** $L(x, y)$ of

$$f(x, y) = \sqrt{x^3 y}$$

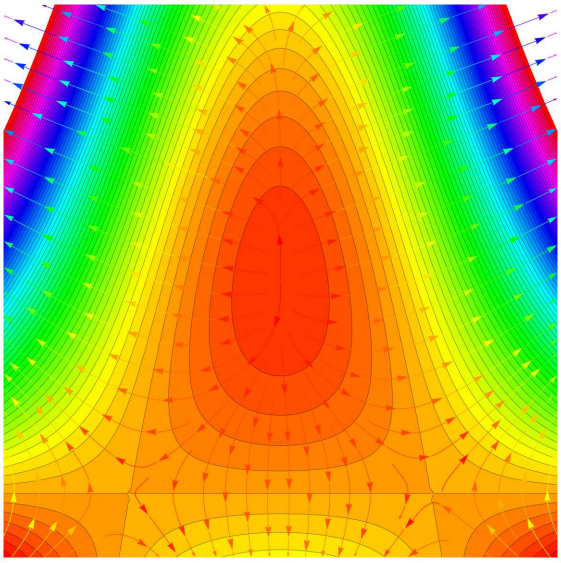
at $(x_0, y_0) = (10, 1000)$.

b) (4 points) Estimate the value

$$\sqrt{11^3 \cdot 999}$$

using the linearization in a).

Problem 8) (10 points)



The vector field $\vec{F}$ is a **gradient field** with potential function

$$f(x, y) = y^2 + 4yx^2 + 4x^2 \ .$$

a) (8 points) Find and classify all the critical points of f.

b) (2 points) Does f have a global maximum or global minimum on the whole xy-plane? Only a brief explanation is needed. Some context: the critical points of f are the equilibrium points of $\vec{F}$. A critical point is called a **sink** if all vectors nearby point towards it. **Sinks** correspond to maxima of f. The minima of f are also called **source** as all vectors points nearby point away of it. An equilibrium is called **hyperbolic** if there are vectors pointing both away and towards it. These are the saddle points of f.

Problem 9) (10 points)

The top tower of the Harvard **Memorial Hall** is a **square frustum** of height $h = 9$. On the **Moscow Papyrus** written in 1850 BC, the volume of such a truncated square pyramid with side lengths x, y of the top and bottom faces, has already been given with the formula $h(x^2 + xy + y^2)/3$. Using this almost four-millennia year old formula and Lagrange, find the minimal volume

$$f(x, y) = 3x^2 + 3xy + 3y^2$$

under the constraint

$$g(x, y) = 3x + 2y = 14 \ .$$

You don't have to justify whether the solution is a minimum.



Photo: O. Knill, November 2017

Problem 10) (10 points)

When properly aired, sand becomes **liquid sand** and you can take a bath in a sand tank. Assume the force field acting on a body floating in it is

$$\vec{F} = \langle -y, x, z \rangle \ .$$

The flux $\int \int_S \vec{F} \cdot d\vec{S}$ of this vector field through the surface of the body is the uplift. What is the uplift of the football

$$x^2 + y^2 \leq \cos^2(z)$$

with $-\pi/2 \leq z \leq \pi/2$, with the outward orientation?



Source: Youtube, Mark Rober, November 28, 2017

Problem 11) (10 points)

Find the **flux** of the curl of the vector field

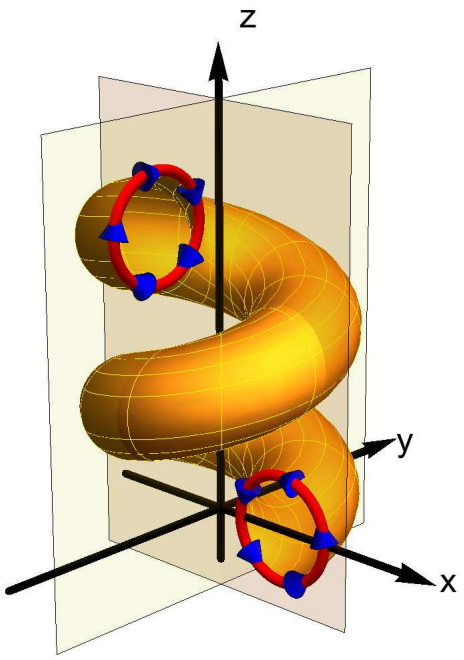
$$\vec{F}(x,y,z) = \langle -z, z+\sin(xyz), x-3\rangle + \langle x^5, y^7, z^4\rangle$$

through the **twisted surface** oriented inwards and parametrized by

$$\vec{r}(t,s) = \langle (3+2\cos(t))\cos(s), (3+2\cos(t))\sin(s), s+2\sin(t)\rangle$$

where $0 \le s \le 7\pi/2$ and $0 \le t \le 2\pi$.

Hint: This parametrization leads correctly already to a vector $\vec{r}_t \times \vec{r}_s$ pointing inwards. The boundary of the surface is made of two circles $\vec{r}(t,0)$ and $\vec{r}(t,7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).



Problem 12) (10 points)

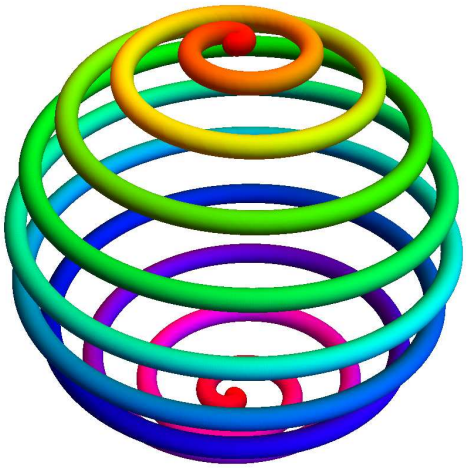
Find the line integral of the vector field

$$\vec{F}(x,y,z) = \langle yz + x^2, xz + y^2 + \sin(y), xy + \cos(z)\rangle$$

along the **spherical curve**

$$\vec{r}(t) = \langle \cos(20t)\sin(t), \sin(20t)\sin(t), \cos(t)\rangle \ ,$$

where $0 \le t \le \pi$.

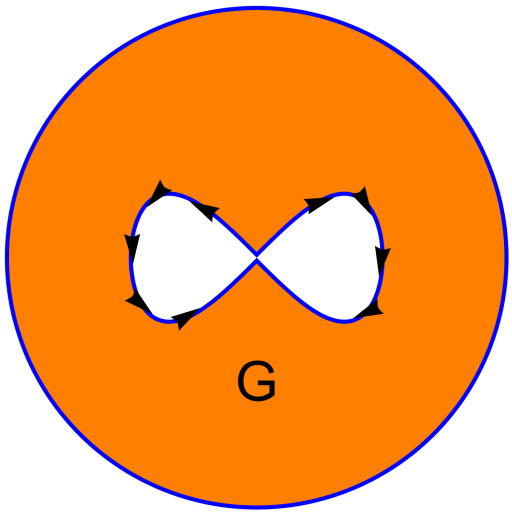


Problem 13) (10 points)

Look at the shaded region G bounded by a circle of radius 2 and an inner **figure eight lemniscate** with parametric equation

$$\vec{r}(t) = \langle \sin(t), \sin(t)\cos(t) \rangle$$

with $0 \leq t \leq 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.



12/15/2017 FINAL PRACTICE I **Math 21a, Fall 2017**

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5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The surface $-x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid.

2) T F The vector projection $\vec{\text{proj}}_{\vec{w}}(\vec{v})$ of a vector $\vec{v}$ onto a non-zero vector $\vec{w}$ is always non-zero.

3) T F The linearization of $f(x, y) = 5 + 7x + 3y$ at any point (a, b) is the function $L(x, y) = 5 + 7x + 3y$.

4) T F For any function $f(x, y, z)$, for any unit vector $\vec{u}$ and any point (x_0, y_0, z_0) we have $D_{\vec{u}} f(x_0, y_0, z_0) = -D_{-\vec{u}} f(x_0, y_0, z_0)$.

5) T F There is a vector field $\vec{F} = \langle P, Q, R \rangle$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.

6) T F The formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ holds if $\vec{v}, \vec{w}$ are vectors in space and α is the angle between them.

7) T F If $\vec{F} = \langle -2y, 2x \rangle$ and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise, then $\int_C \vec{F} \cdot d\vec{r} = 16\pi$.

8) T F The parametrization $\vec{r}(u, v) = \langle u, \sqrt{1 - u^2 - v^2}, v \rangle, u^2 + v^2 \leq 1$ describes a half sphere.

9) T F The vectors $\vec{v} = \langle 1, 0, 1 \rangle$ and $\vec{w} = \langle -1, 1, 1 \rangle$ are perpendicular.

10) T F If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) then $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.

11) T F The vector field $\vec{F} = \langle e^x, e^y, e^z \rangle$ satisfies $\text{grad}(\text{div}(\vec{F})) = \vec{F}$.

12) T F If $\vec{F}(x, y, z)$ has zero curl everywhere in space, then $\vec{F}$ is a gradient field.

13) T F If $\vec{r}(u, v)$ is a parametrization of the surface $g(x, y, z) = x^2 + e^{y^3} + z^2 = 5$ then for any u and v we have $\nabla g(\vec{r}(u, v)) \cdot \vec{r}_u(u, v) = 0$.

14) T F The equation $\text{grad}(\text{div}(\text{grad}(f))) = \langle 0, 0, 0 \rangle$ always holds.

15) T F There is a non-constant function $f(x, y, z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$ everywhere.

16) T F If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S oriented outwards is the volume of the enclosed solid.

17) T F If $f(x, y)$ is maximized at (a, b) under the constraint $g(x, y) = c$, then $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel.

18) T F The distance between a point P and the line L through two different points A, B is given by the formula $|\vec{PA} \times \vec{AB}|/|\vec{PA}|$.

19) T F The unit tangent vector $\vec{T}(t)$ is always perpendicular to the vector $\vec{T}\,'(t)$.

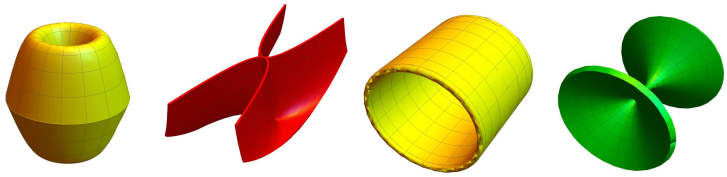
20) T F The vector field $\vec{F}(x, y, z) = \langle x^5, x^6, x^7 \rangle$ can not be the curl of another vector field.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface $\vec{r}(u,v) =$	A-D
$\langle u, u^2v, v^2\rangle$	
$\langle u^2\cos(v), u, u^2\sin(v)\rangle$	
$\langle \cos(v), \sin(u), \sin(v)\rangle$	
$\langle v\cos(u), v\sin(u), \sin(v)\rangle$	

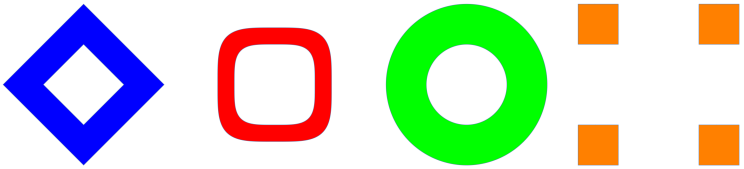
A B C D



b) (2 points) Match the following 2D region plots. There is an exact match.

Region	A-D
$1 \le x^2 \le 4, 1 \le y^2 \le 4$	
$1 \le x^2 + y^2 \le 4$	
$1 \le x^4 + y^4 \le 4$	
$1 \le \lvert x\rvert + \lvert y\rvert \le 2$	

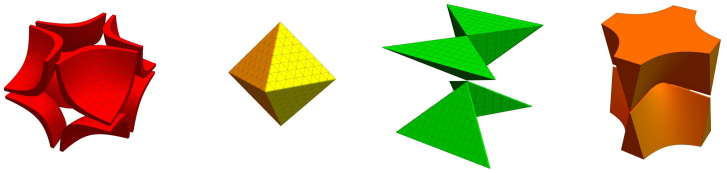
A B C D



c) (2 points) Match the following 3D regions. There is an exact match.

Solid	A-D
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert \le 2$	
$\lvert x\rvert < \lvert y\rvert < \lvert z\rvert$	
$1 < \lvert xyz\rvert < 2$	
$x^2y^2 < z^2$	

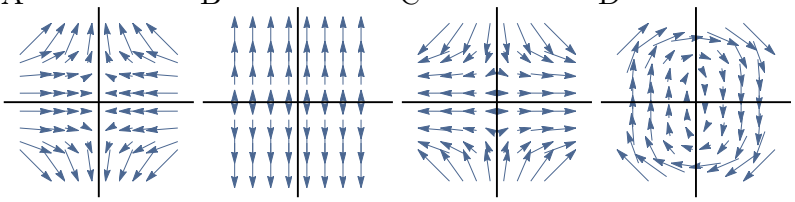
A B C D



d) (2 points) The following figures display vector fields. There is an exact match.

Field	A-D
$\vec{F}(x,y) = \langle 0, y\rangle$	
$\vec{F}(x,y) = \langle -x, y^3\rangle$	
$\vec{F}(x,y) = \langle y^3, -x\rangle$	
$\vec{F}(x,y) = \langle x, -y^3\rangle$	

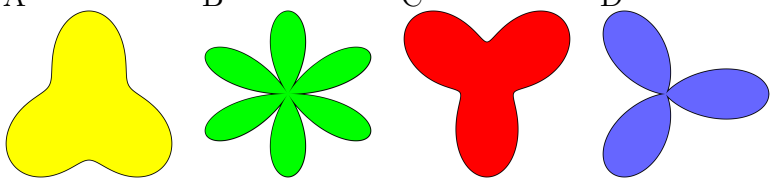
A B C D



e) (2 points) The following figures display polar regions. There is an exact match.

Polar region	A-D
$r \le 1 + \cos(3\theta)$	
$r \le 2 + \sin(3\theta)$	
$r \le 3 - \sin(3\theta)$	
$r \le \lvert\sin(3\theta)\rvert$	

A B C D



Problem 3) (10 points)

a) (6 points)

The concept of **boundary** plays an important role in integral theorems. In each of the following six rows, check exactly one entry which best describes the boundary.

The boundary of	solid	surface	curves	points	empty
$x^2 + y^2 + z^2 = 1$					
$x^2 + y^2 = 1, z = 0$					
$x^2 + y^2 + z^2 \leq 1, x = y = 0$					
$x^2 + y^2 + z^2 \leq 1$					
$x^2 + y^2 \leq 1, z = 0$					
$x^2 + y^2 = 1, z^2 \leq 1$					

b) (4 points) Match the following partial differential equations (PDEs) by picking 4 from the 5 given choices A-E.

PDE	Enter A-E
heat equation	
wave equation	
transport equation	
Burgers equation	

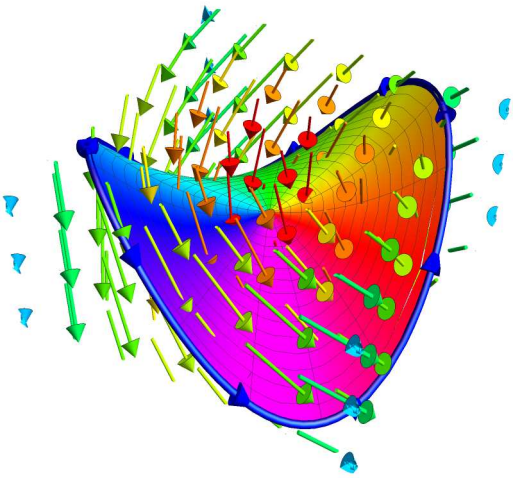
$u_{xx} = u_t$	A
$u_{xx} = u_t + uu_x$	B
$u_x = u_t$	C
$u_{xx} = -u_{tt}$	D
$u_{xx} = u_{tt}$	E

Problem 4) (10 points)

a) (3 points) A surface S is parameterized by

$$\vec{r}(u,v) = \langle u, v, uv \rangle \ ,$$

where $u^2 + v^2 \leq 1$. Find its surface area.



b) (3 points) Parametrize the boundary curve C matching the orientation $\vec{r}_u \times \vec{r}_v$ of S, then compute the line integral $\int_C \vec{F} \cdot \ d\vec{r}$ with $\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$.

c) (2 points) The coordinates of the surface S satisfy $xy - z = 0$. Find the tangent plane to S at $(2,1,2)$.

d) (2 points) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at the point $(2,1)$.

Problem 5) (10 points)

On November 17 2017, the **NASA Eagleworks paper** appeared, making the **EM drive** more probable. It might in future be used for deep space missions. The drive produces a thrust, apparently violating momentum conservation.

a) (5 points) Assume the drive flies in the gravitational field

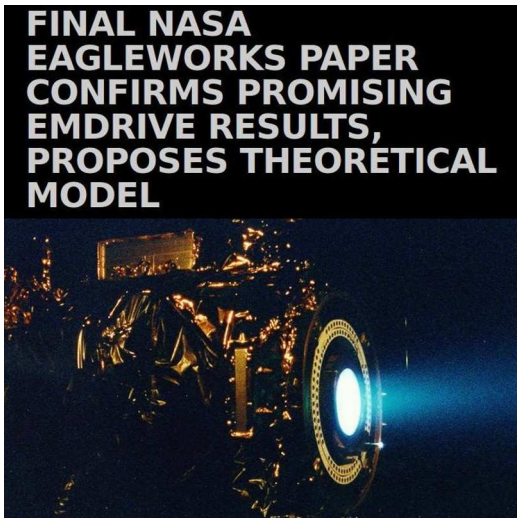
$$\vec{F}(x,y,z) = \langle x^7 + xy^2z^2, x^2yz^2, x^2y^2z \rangle$$

along the path

$$C : \vec{r}(t) = \langle t\cos(t), t\sin(t), t(5\pi - t) \rangle$$

with $0 \leq t \leq 5\pi$. Compute the work

$$\int_0^{5\pi} \vec{F} \ \cdot \ d\vec{r}.$$



b) (3 points) Compute $d = |\int_0^{5\pi} \vec{r}\,'(t)\ dt|$.

c) (2 points) If L is the arc length of C, circle the one box below which applies:

$d = L$	$d > L$	$d < L$

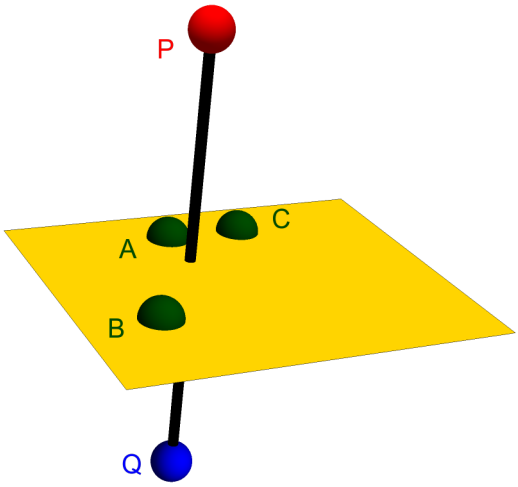
Problem 6) (10 points)

a) (2 points) Find the equation $ax+by+cz=d$ of the plane through $A=(1,1,1), B=(3,4,5), C=(4,4,2)$.

b) (3 points) Compute the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$.

c) (3 points) Determine the volume of the parallelepiped spanned by $\vec{AB}, \vec{AC}, \vec{AP}$ where $P=(1,3,4)$.

d) (2 points) Find the distance $|\vec{PQ}|$, where Q is the mirror image of P opposite of the plane. It is determined by the fact that the middle point $(P+Q)/2$ is on the plane and that $\vec{PQ}$ is perpendicular to the plane.



Problem 7) (10 points)

The triple scalar product is also written as

$$[\vec{u},\vec{v},\vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w}).$$

The **torsion** of a space curve is defined as

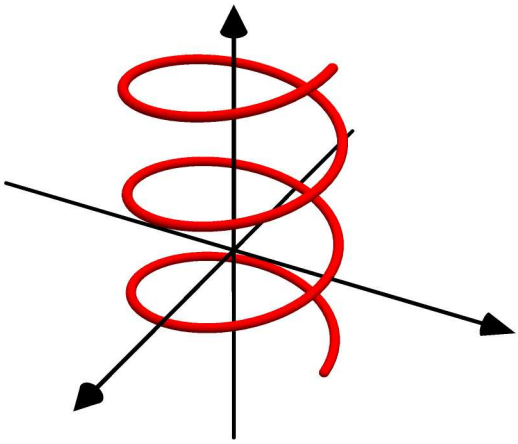
$$\frac{[\vec{r}\,',\vec{r}\,'',\vec{r}\,''']}{|\vec{r}\,'\times\vec{r}\,''|^2}.$$

a) (3 points) Compute $\vec{r}\,'(0), \vec{r}\,''(0), \vec{r}\,'''(0)$ for

$$\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle.$$

b) (4 points) Compute the torsion of the curve at the point $\vec{r}(0)$.

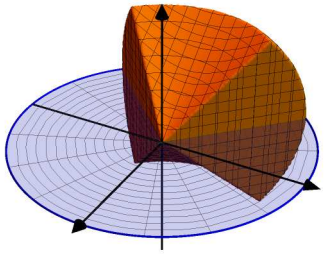
c) (3 points) Assume you have an arbitrary curve $\vec{r}(t)$ which is contained in the xy-plane. What is its torsion?



Problem 8) (10 points)

Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$



a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \, dxdydz.$$

b) (3 points) Let $\vec{F}$ be a vector field

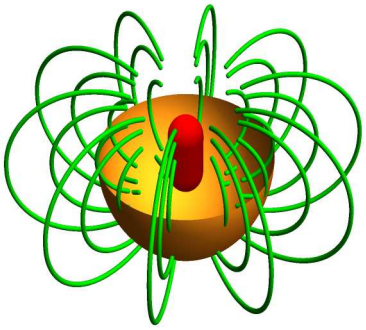
$$\vec{F} = \langle x^3, y^3, z^3 \rangle.$$

Find the flux of $\vec{F}$ through the boundary surface of E, oriented outwards.

Problem 9) (10 points)

The vector field

$$\vec{A}(x, y, z) = \frac{\langle -y, x, 0 \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$



is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

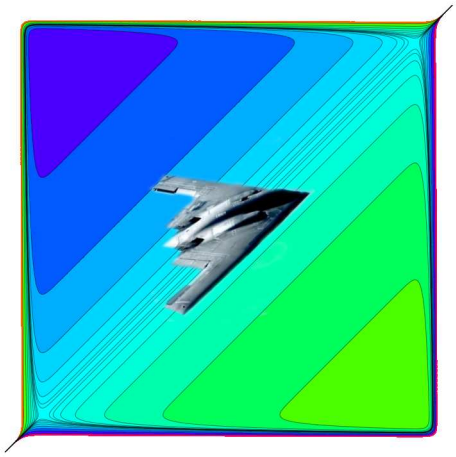
The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards.

Problem 10) (10 points)

a) (8 points) Classify the critical points of the **area 51** function

$$f(x,y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.



b) (2 points) Does the function have a global maximum or global minimum on the region $x^2+y^2 \leq 1$ including the boundary? Write "yes" or "no" with a brief explanation. There is no need to find the global extrema.

Problem 11) (10 points)

Using the Lagrange method, find the maximum and minimum of the elliptic curve function

$$f(x,y) = y^2 - x^3 - x^2 - x$$

on the circle $g(x,y) = x^2 + y^2 = 1$.



This problem is motivated from a real life application. To encrypt communication in **"WhatsApp"**, the elliptic curve 25519 given by $y^2 = x^3 + 486662x^2 + x$ over the prime $p = 2^{255} - 19 =$

57896044618658097711785492504343953926634992332820282019728792003956564819949

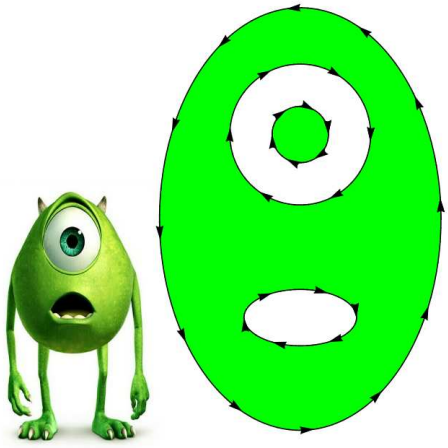
is used.

Problem 12) (10 points)

Given the scalar function $f(x,y) = x^5 + xy^4$, compute the line integral of

$$\vec{F}(x,y) = \langle 5y + 3y^2, 6xy + y^4 \rangle + \nabla f$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. The picture describes the orientations of the boundary curves perfectly and they are as they are! We warn you not to ask about this, or else we will bring in "Mike" from **Monsters, Inc.**



Problem 13) (10 points)

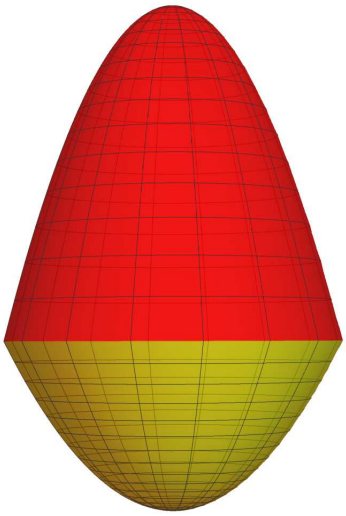
"ProtEgg" is a defense spell. It produces an egg shaped solid E enclosed by the surfaces

$$S : z = 2 - 2x^2 - 2y^2, z \geq 0 \ ,$$

where S is oriented upwards and

$$T : z = x^2 + y^2 - 1, z \leq 0 \ ,$$

where T is oriented downwards.



a) (4 points) Find the volume $\iiint_E 1 \, dxdydz$ of E.

b) (4 points) The spell uses a force field

$$\vec{F}(x,y,z) = \langle 0, 0, 2x^2 + 2y^2 + z \rangle \ .$$

S is parametrized by $\vec{r}(u,v) = \langle u, v, 2 - 2u^2 - 2v^2 \rangle$ with $u^2 + v^2 \leq 1$ oriented upwards. Compute the flux $\iint_S \vec{F} \cdot d\vec{S}$ without an integral theorem.

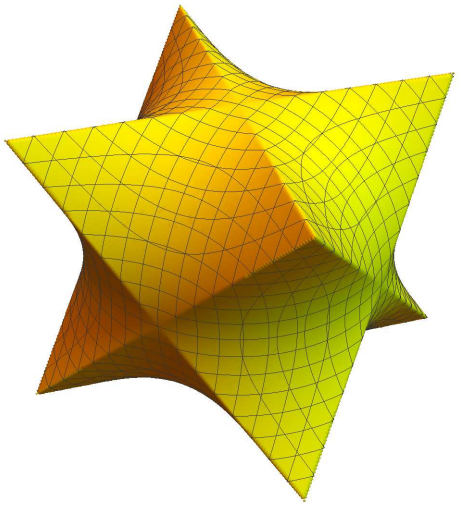
c) (2 points) The flux $\iint_T \vec{F} \cdot d\vec{S}$ can be determined using an integral theorem. What is the value of the flux? Check all that apply:

A	B	C	D
$\iint_S \vec{F} \cdot d\vec{S}$	$-\iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV + \iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV - \iint_S \vec{F} \cdot d\vec{S}$

Problem 14) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 \ .$$



Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

$$\vec{F}(x,y,z) = \langle 2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z \rangle$$

through the boundary surface S of R, assuming that S is oriented outwards.

12/15/2017 FINAL PRACTICE II **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The parametrization $\vec{r}(u,v) = \langle v^2, v^2\cos(u), v^2\sin(u)\rangle$ describes a cone.

2) T F If three vectors $\vec{u}, \vec{v}, \vec{w}$ satisfy $\vec{u}\cdot\vec{v} = 0$ and $\vec{v}\times\vec{w} = \vec{0}$, then $\vec{u}\cdot\vec{w} = 0$.

3) T F Let S be a surface bounding a solid E and $\vec{F}$ is a vector field in space which is incompressible, $\text{div}(\vec{F}) = 0$, then $\iint_S \vec{F}\cdot d\vec{S} = 0$.

4) T F If $\text{curl}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F}\cdot d\vec{S} = 0$ for any closed surface S.

5) T F If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ has zero curl everywhere.

6) T F If $\vec{F}, \vec{G}$ are two vector fields which have the same divergence then $\vec{F} - \vec{G}$ is constant.

7) T F The linearization of the constant function $f(x,y) = 3$ at $(x,y) = (1,1)$ is the function $L(x,y) = 0$.

8) T F The surface area of a surface S is $\int\int_S \langle x,y,z\rangle \cdot d\vec{S}$.

9) T F There is a non-constant vector field $\vec{F}(x,y,z)$ such that $\text{curl}(\vec{F}) = \text{curl}(\text{curl}(\vec{F}))$

10) T F If $\vec{F}$ is a vector field and E is the unit ball then $\iiint_E \text{div}(\text{curl}(\vec{F}))\, dV = 0$.

11) T F If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is zero.

12) T F The equation $\text{grad}(\text{div}(\text{grad}(f))) = \vec{0}$ always holds.

13) T F The vector $\vec{k}\times(\vec{j}\times\vec{i})$ is the zero vector, if $\vec{i} = \langle 1,0,0\rangle, \vec{j} = \langle 0,1,0\rangle$, and $\vec{k} = \langle 0,0,1\rangle$.

14) T F If f is minimized at (a,b) under the constraint $g = c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are perpendicular.

15) T F If A, B, C, D are four points in space such that the line through A, B intersects the line through C, D, then A, B, C, D lie on some plane.

16) T F The chain rule assures that for a vector field $\vec{F} = \langle P, Q, R\rangle$ the formula $\frac{\partial}{\partial u}\vec{F}(\vec{r}(u,v)) = \langle \nabla P\cdot\vec{r}_u, \nabla Q\cdot\vec{r}_u, \nabla R\cdot\vec{r}_u\rangle$ holds.

17) T F The vector field $\vec{F} = \langle \cos(y), 0, \sin(y)\rangle$ satisfies $\text{curl}(\vec{F}) = \vec{F}$. By the way, it is called the Cheng-Chiang field.

18) T F The unit tangent vector $\vec{T}(t)$, the normal vector $\vec{N}(t)$ and the binormal vector $\vec{B}(t)$ for a given curve $\vec{r}(t)$ span a cube of volume 1 at $t = 1$.

19) T F The vector field $\vec{F}(x,y,z) = \langle z,z,z\rangle$ can not be the curl of a vector field.

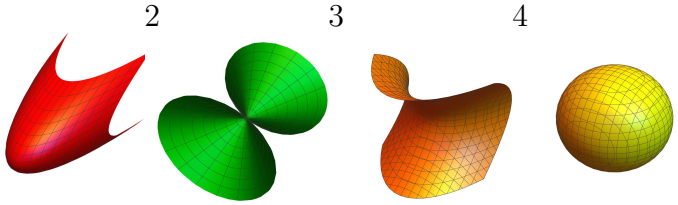
20) T F The expression $\text{curl}(\text{grad}(\text{div}(\text{grad}(\text{div}(\text{curl}(\vec{F}))))))$ is a well defined vector field in three dimensional space.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = \langle u, u^2 - v^2, v\rangle$	
$y - x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u, u\sin(v)\rangle$	
$x^2 + y^2 = 1 - z^2$	

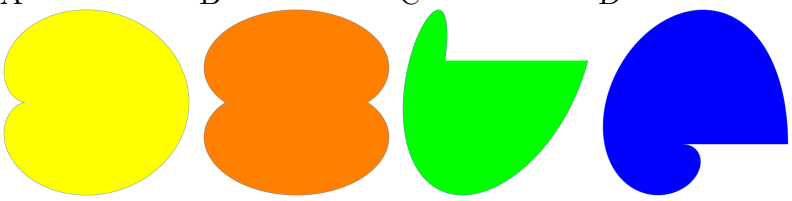
1 2 3 4



b) (2 points) Match the following regions given in polar coordinates (r, θ):

Region	A-D
$r < \theta^2$	
$r < 1 + \cos(\theta)$	
$r < 1 + \lvert\sin(\theta)\rvert$	
$r < (4\pi^2 - \theta^2)$	

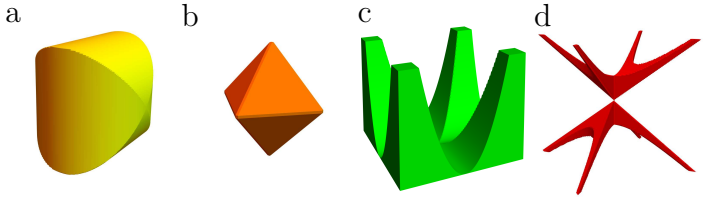
A B C D



c) (2 points) Match the regions. There is an exact match.

Solid	a-d
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert \leq 1$	
$x^2 + y^2 \leq 1, x^2 + z^2 \leq 1$	
$x \leq y^2, x \leq z^2$	
$0 \leq x^2 - y^2 \leq 4, 0 \leq y^2 - z^2 \leq 4$	

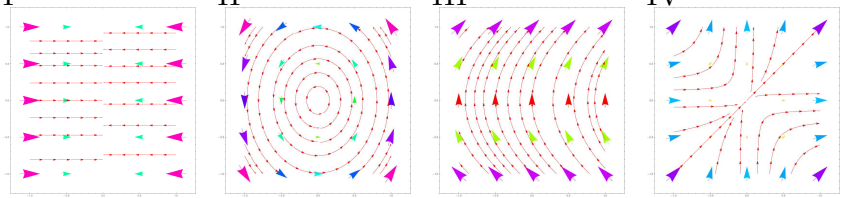
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle y, 1\rangle$	
$\vec{F}(x,y) = \langle -2y, 3x\rangle$	
$\vec{F}(x,y) = \langle -x, 0\rangle$	
$\vec{F}(x,y) = \langle x^2, y^2\rangle$	

I II III IV



e) (1 point) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at $(2,1)$.

f) (1 point) Write down the wave equation for a function $f(t,x)$:

Problem 3) (10 points)

a) (6 points) In the following, f is a function, $\vec{F}$ a vector field, I is an interval, G is a region in the two dimensional plane, S is a closed surface parametrized by $\vec{r}(u,v)$, C is a closed curve parametrized by $\vec{r}(t)$ and E is a solid. Fill the blanks using "volume", "area", "length" or "0", where a choice can appear a multiple times.

$$\boxed{} = \int\int_G \operatorname{curl}(\langle x, 0\rangle)\, dxdy$$

$$\boxed{} = \int\int_G \operatorname{curl}(\langle -y, 0\rangle)\, dxdy$$

$$\boxed{} = \int\int\int_E \operatorname{div}(\langle x, x, x\rangle)\, dxdydz$$

$$\boxed{} = \int\int_S |\vec{r}_u \times \vec{r}_v|\, dudv$$

$$\boxed{} = \int_C |\vec{r}\,'(t)|\, dt + \int_C \operatorname{grad}(f)\ \cdot d\vec{r}$$

$$\boxed{} = \int\int_S \operatorname{curl}(\vec{F})\ \cdot d\vec{S}$$

b) (4 points)
Complete the formulas of the following boxes

$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \boxed{}\, d\rho d\phi d\theta$ Integral for volume of unit ball $x^2+y^2+z^2 \le 1$.

$\int_0^{2\pi}\int_0^1 \boxed{}\, drd\theta$ Integral for area of unit disc $x^2+y^2 \le 1$

$g_x(x,y) = \dfrac{\boxed{}}{\overline{f_z(x,y,z)}}$ Implicit derivative of g satisfying $f(x,y,g(x,y)) = 0$

$|\vec{u} \cdot (\boxed{})|$ Volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

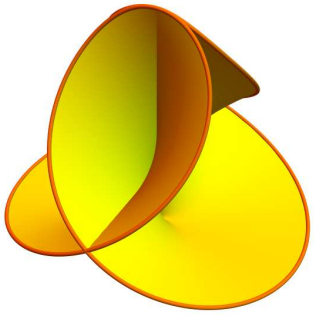
Problem 4) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), \frac{v^2}{2}\rangle \ ,$$

where (u,v) are in the domain $u^2+v^2 \leq 9$.

P.S. You do not have to worry that this cool surface has self-intersections.



Problem 5) (10 points)

On planet **Tatooine**, Luke Skywalker travels along a path C parametrized by

$$\vec{r}(t) = \langle t\cos(t), t\sin(t), 0\rangle$$

from $t=0$ to $t=2\pi$. What is the work done

$$\int_C \vec{F}\cdot d\vec{r}$$

by the "force"

$$\vec{F} = \langle x^2+y+z, y^3+x, z^5+x\rangle \ .$$

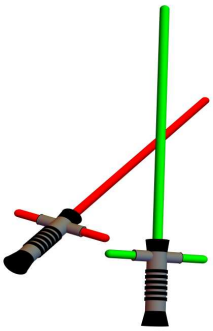


Problem 6) (10 points)

Tomorrow, on December 18, the "force awakens". There will be light sabre battles, without doubt.

a) (7 points) What is the distance between two light sabres given by cylinders of radius 1 around the line $\vec{r}(t) = \langle t, -t, t\rangle$ and the line connecting $(0,14,0)$ with $(3,5,6)$.

b) (3 points) A spark connects the two points of the sabre which are closest to each other. Find a vector in that direction.



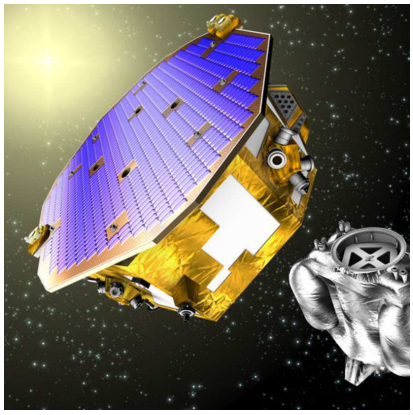
Sabre by Connor (mathematica project).

Problem 7) (10 points)

100 years ago, Einstein proposed gravitational waves. To measure them, the LISA pathfinder was launched on December 3, 2015. It carries two cubes to the Lagrangian point between Earth and Moon aiming to measure the waves. Assume a gravitational wave from a black hole merger produces a force leading to an acceleration

$$\vec{r}\,''(t) = \langle \sin(t), \cos(t), \sin(t) \rangle \ .$$

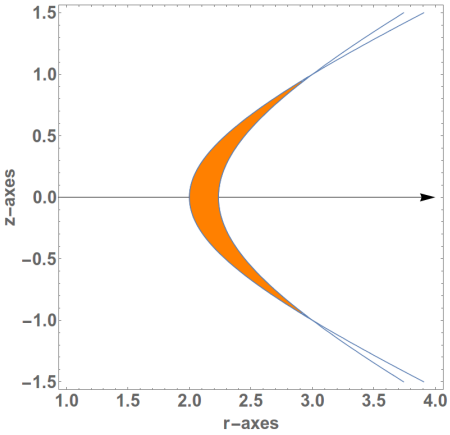
What is $\vec{r}(t)$ at time $t = \pi$ if $\vec{r}\,'(0) = \langle 2, 0, 0 \rangle$ and $\vec{r}(0) = \langle 1, 0, 3 \rangle$.



LISA proof of concept will be followed by eLISA in 2034.

Problem 8) (10 points)

a) (5 points) Find the integral $\int \int_G r \ drdz$, where G is the region enclosed by the curves $r^2 - 4z^2 = 5$ and $r^2 - 5z^2 = 4$ and contained in $r \geq 0$.



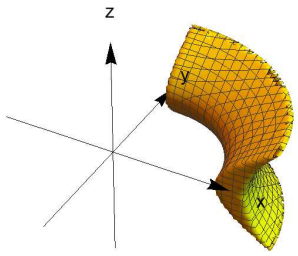
b) (5 points) The Galactic Empire builds a space craft E given as a solid in $x \geq 0, y \geq 0$, enclosed by

$$x^2 + y^2 - 4z^2 = 5$$

and

$$x^2 + y^2 - 5z^2 = 4 \ .$$

Find its volume. P.S. You can make use of problem a) to solve part b) as the problems are related.



Problem 9) (10 points)

In September 2015, the west side of the Harvard Science center honored the concept of curl by displaying paddle wheels. One of the wheel tips moves on an oriented curve $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$ bounding the disc parametrized by $\vec{r}(u,v) = \langle u\cos(v), 0, u\sin(v) \rangle$ with $0 \leq u \leq 1, 0 \leq v \leq 2\pi$. Let $\vec{F}$ be the wind vector field

$$\vec{F}(x,y,z) = \langle x^9 + y^7 + 3z, x^9 + y^9 + \sin(z), z^5 e^z \rangle \ .$$

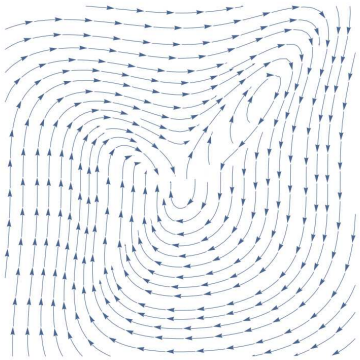
Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ measuring the energy gain during one rotation along the curve C parametrized by $\vec{r}(t)$.



Problem 10) (10 points)

The value of the line integral of the vector field $\vec{F}(u,v) = (2/\pi)\langle -uv^2 + v^3, uv - u^3 \rangle$ along a curve $\vec{r}(t) = \langle x + \cos(t), y + \sin(t) \rangle$ depends only on the center point (x,y) and is given by

$$f(x,y) = -3 - 6x^2 + 2y + 4xy - 6y^2 \ .$$



a) (7 points) Find all critical points (x,y) for the function $f(x,y)$ and analyze them using the second derivative test.
b) (3 points) Given that

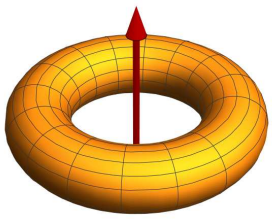
$$f(x,y) = -3 - (x-2y)^2 - 5x^2 - 2y^2 + 2y \ ,$$

decide whether there is a global maximum for f.

Problem 11) (10 points)

The moment of inertia $f(x,y)$ of a torus of mass 4 with smaller tube radius x and bigger center curve radius y is

$$f(x,y) = 3x^2 + 4y^2 \ .$$



a) (7 points) Find the parameters (x_0, y_0) for the torus which have minimal moment of inertia under the constraint that

$$g(x,y) = x + 4y = 13 \ .$$

b) (3 points) Write down the equation of the tangent line to the level curve of f which passes through (x_0, y_0).

Problem 12) (10 points)

A new **elliptical machine** has been designed to simulate running better. The leg of a runner moves on the curve parametrized by

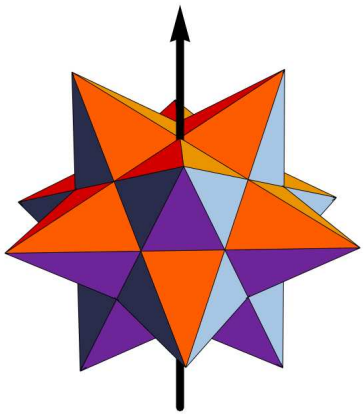
$$\vec{r}(t) = \langle 8\cos(t), 2\sin(t) + \sin(2t) + \cos(2t) \rangle$$

with $0 \leq t \leq 2\pi$. Find the area of the region enclosed by the curve.



Problem 13) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle y^5 + x, z^5 + y, x^5 + z \rangle$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = \langle x^3/3, y^3/3, 0 \rangle$$

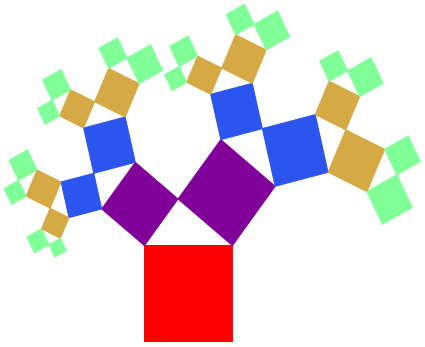
through S?

Problem 14) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = \langle -y + x^8, x - y^9 \rangle$$

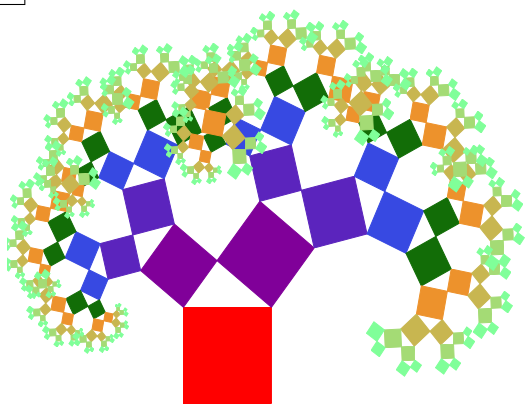
along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



Pythagoras tree theorem:

The generation n Pythagorean tree has area $n+1$.

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.



Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The parametrization $\vec{r}(u,v) = \langle \cos(u), \sin(u), v\rangle$ describes a cone.

2) T F The vectors $\vec{v} = \langle 2,1,5\rangle$ and $\vec{w} = \langle 2,1,-1\rangle$ are perpendicular.

3) T F Let E be a solid region with boundary surface S. If $\iint_S \vec{F}\cdot d\vec{S} = 0$, then $\mathrm{div}(\vec{F})(x,y,z) = 0$ everywhere inside E.

4) T F If $\mathrm{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F}\cdot d\vec{S} = 0$ for any closed surface S.

5) T F If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.

6) T F If $\vec{F}, \vec{G}$ are two vector fields for which $\vec{F} - \vec{G} = \mathrm{curl}(\vec{H})$, then $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{G}\cdot d\vec{S}$ for any closed surface S.

7) T F The linearization of $f(x,y) = x^2 + y^3 - x$ at $(2,1)$ is $L(x,y) = 4 + 4(x-2) + 3(y-1)$.

8) T F The volume of the solid E is $\int\int_S \langle x, 2x+z, x-y\rangle \cdot d\vec{S}$, where S is the surface of the solid E.

9) T F The vector field $\vec{F}(x,y,z) = \langle x+y, x-y, 3\rangle$ has zero curl and zero divergence everywhere.

10) T F If $\vec{F}$ is a vector field, then the flux of the vector field $\mathrm{curl}(\mathrm{curl}(F))$ through a sphere $x^2+y^2+z^2 = 1$ is zero.

11) T F If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.

12) T F The equation $\mathrm{div}(\mathrm{grad}(\mathrm{f})) = 0$ is an example of a partial differential equation for an unknown function $f(x,y,z)$.

13) T F The vector $(\vec{i}+\vec{j})\times(\vec{i}-\vec{j})$ is the zero vector if $\vec{i} = \langle 1,0,0\rangle$ and $\vec{j} = \langle 0,1,0\rangle$.

14) T F If f is maximal under the constraint $g = c$, then the angle between $\nabla f(x,y)$ and $\nabla g(x,y)$ is zero.

15) T F Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P,L) \geq d(P,\Sigma)$.

16) T F The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.

17) T F If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P,Q)$ is minimized.

18) T F If $\vec{B}(t)$ is the bi-normal vector to an ellipse $\vec{r}(t)$ contained in the plane $x+y+z = 1$, then $\vec{B}$ is parallel to $\langle 1,1,1\rangle$.

19) T F The parametrized surface $\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$ is everywhere perpendicular to the vector field $\vec{F}(x,y,z) = \langle x, y, x^2+y^2\rangle$.

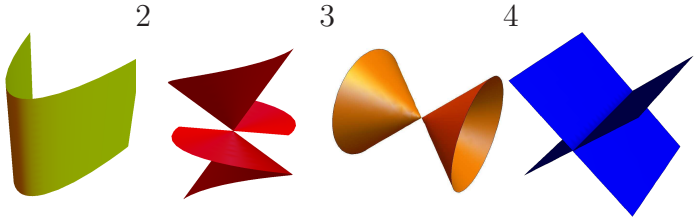
20) T F Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F} = \nabla f$. Then $\vec{r}\,'(t) = \vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), uv\rangle$	
$\vec{r}(u,v) = \langle u, u^2, v\rangle$	
$x^2 - y^2 = z^2$	

1 2 3 4



b) (2 points) Match the expressions. There is an exact match.

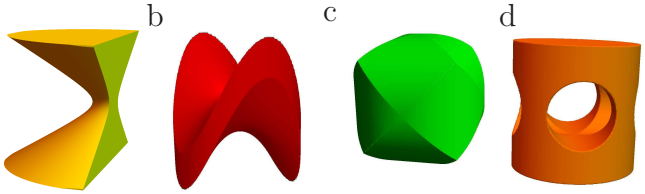
Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert\, dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$	
$\int_a^b \int_c^d \lvert\vec{r}_u \times \vec{r}_v\rvert\, dudv$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$0 \le x^2 - y^2 - z \le 1$	
$x^2 + y^2 - z^2 \le 1, x \ge 0$	
$x^2 + y^2 \le 3, y^2 + z^2 \ge 1, x^2 + z^2 \ge 1$	
$x^2 + y^2 \le 1, y^2 + z^2 \le 1, x^2 + z^2 \le 1$	

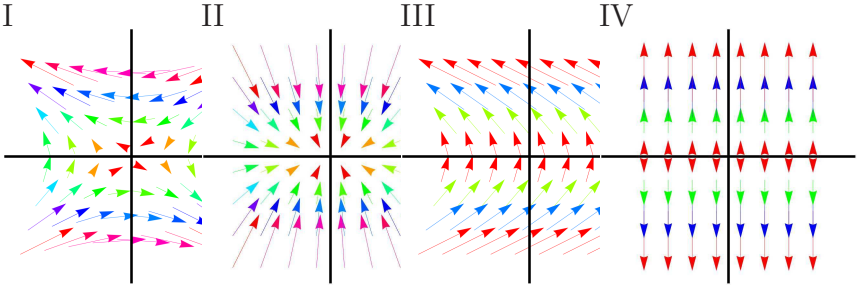
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle 0, 2y\rangle$	
$\vec{F}(x,y) = \langle -x, -2y\rangle$	
$\vec{F}(x,y) = \langle -2y, -x\rangle$	
$\vec{F}(x,y) = \langle -2y, 1\rangle$	

I II III IV



e) (2 points) Match the partial differential equations with formulas and functions $u(t,x)$. There is an exact match.

Equation	1-3	A-C
wave		
heat		
Laplace		

	Formulas
1	$u_{tt} = -u_{xx}$
2	$u_t = u_{xx}$
3	$u_{tt} = u_{xx}$

	Functions
A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

Problem 3) (10 points)

a) (2 points) Mark the statement or statements which have a zero answer.

Statement	Must be zero
The curl of the gradient $\nabla f(x,y,z)$ at $(1,1,1)$	
The divergence of the curl $\nabla \times \vec{F}(x,y,z)$ at $(1,1,1)$	
The flux of a gradient field $\nabla f(x,y,z)$ through a sphere	
The dot product of $\vec{F}(1,1,1)$ with the curl$(F)(1,1,1)$	
The divergence of a gradient field $\nabla f(x,y,z)$ at $(1,1,1)$	

b) (2 points) Two of the following statements do not make sense. Recall that "incompressible" means zero divergence everywhere and that "irrotational" means zero curl everywhere.

Statement	Makes no sense
The discriminant of the gradient field	
A conservative and incompressible vector field	
The flux of the gradient of a function through a surface	
The gradient of the curl of a vector field	

c) (2 points) Match the following formulas with the geometric object they describe. Fill in the blanks as needed.

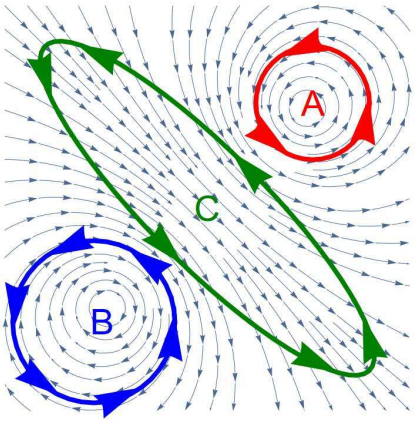
	Geometric object
A	unit tangent vector to a curve
B	unit normal vector to a surface
C	unit normal vector to a level curve
D	curvature of the curve
E	unit tangent vector to the surface

A-E	Formula
	$(\vec{r}_u \times \vec{r}_v)/\lvert\vec{r}_u \times \vec{r}_v\rvert$
	$\nabla f(x,y)/\lvert\nabla f(x,y)\rvert$
	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
	$(\vec{r}_u + \vec{r}_v)/\lvert\vec{r}_u + \vec{r}_v\rvert$
	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

d) (2 points) All three curves in the figure are oriented counter clockwise. Check whether in each of the three cases, the line integral is positive, negative or zero.

Check with a mark which applies. The line integral $\int_\gamma \vec{F} \cdot \vec{dr}$ is

γ	Positive	Negative	Zero
A			
B			
C			

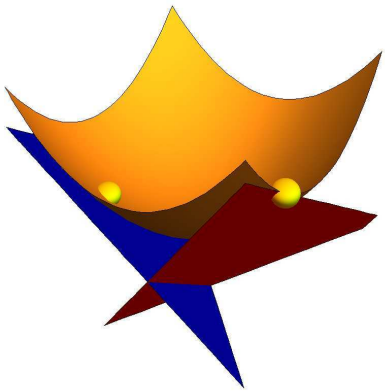


e) (2 points) Line or a plane? That is here the question.

U or V ?	Object
	$\vec{r}(0) + t\vec{B}(\vec{r}(0)) + s\vec{N}(\vec{r}(0))$
	$\vec{r}(0,0) + t\ \vec{r}_u \times \vec{r}_v$

	Dichotomy
U	is a normal plane to a curve
V	is a normal line to a surface

Problem 4) (10 points)

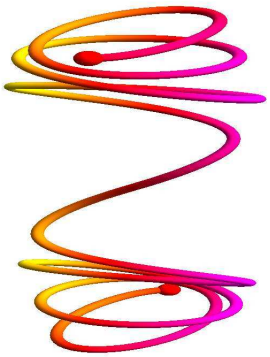


a) (6 points) Find the tangent planes at the points $A = (1, 1, 2)$ and $B = (-2, 1, 5)$ to the surface

$$x^2 + y^2 - z = 0 \, .$$

b) (4 points) Find the parametric equation of the line of intersection of these two tangent planes.

Problem 5) (10 points)



Compute the line integral of the vector field

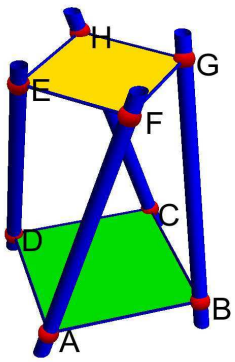
$$\vec{F}(x, y, z) = \langle 2x, 3y^3, 3z^2 \rangle$$

along the curve

$$\vec{r}(t) = \langle \sin(t)\cos(t^3), \cos(t)\sin(t^3), t \rangle$$

from $t = -\pi/2$ to $t = \pi/2$.

Problem 6) (10 points)



A monument has the form of a **twisted frustrum**. It is built from a base square platform $\{A = (-10, -10, 0), B = (10, -10, 0), C = (10, 10, 0), D = (-10, 10, 0)\}$ connected with an upper square $\{E = (-10, 0, 20), F = (0, -10, 20), G = (10, 0, 20), H = (0, 10, 20)\}$ using cylinders of radius 1. Find the distance between the pillar going through AF and the pillar going through BG.

Problem 7) (10 points)

The monkey saddle gets a cameo: find the volume of the solid contained in the cylinder

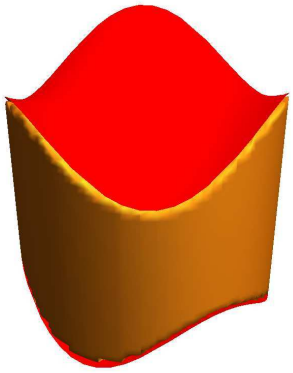
$$x^2 + y^2 < 1$$

above the surface

$$z = x^3 - 3xy^2 - 3$$

and below the surface

$$z = y^3 - 3yx^2 + 3 \, .$$

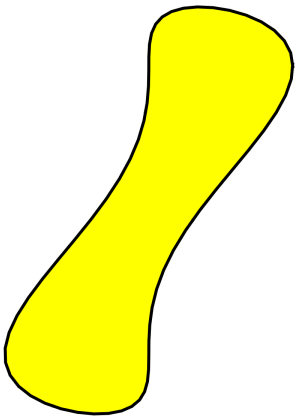


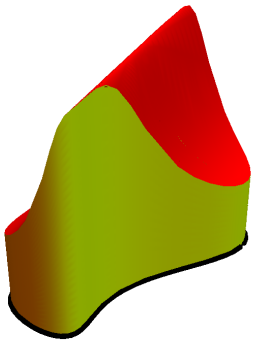
Problem 8) (10 points)

The volume of the solid above the region bound by the planar curve C : $\langle \cos^3(t), \sin(t) + \cos(t) \rangle$ and the graph of $f(x, y) = x^{-2/3}$ is given by the double integral

$$\int \int_G \frac{1}{\sqrt[3]{x^2}} \, dxdy \, .$$

Use the vector field $\vec{F} = \langle 0, 3x^{1/3} \rangle$ to compute this integral.





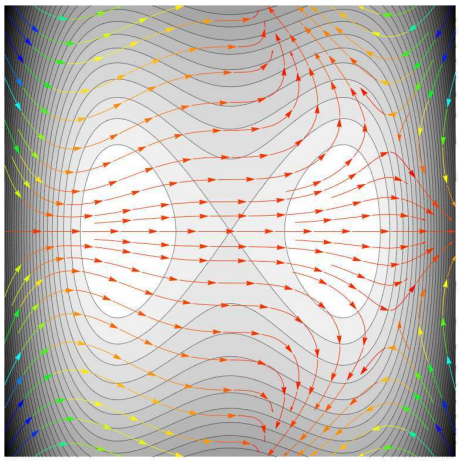
Problem 9) (10 points)

a) (8 points) The divergence of the vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle 1 - xy^2, 2 + 2yx^2 - yx^4 \rangle$$

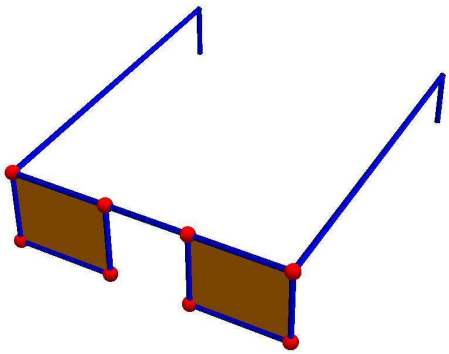
is the function $f(x, y) = P_x + Q_y$. Find and classify the critical points of $f(x, y)$.

b) (2 points) We have seen in general that the gradient field $\nabla f(x, y)$ is perpendicular to level curves $\{f(x, y) = c\}$ and that $\nabla f(x, y)$ is the zero vector at maxima or minima. Is the vector field $\vec{F}(x, y)$ zero at a maximum of $f(x, y) = \text{div}(F)(x, y)$?



Problem 10) (10 points)

Economists know a **constraint duality principle**: "maximizing a first quantity while fixing the second is equivalent to minimizing the second when fixing the first". Let's experiment with that:



a) (5 points) Use the Lagrange method to find the **reading glasses** with maximal glass area $f(x,y) = 2xy$ and fixed frame material $4x + 15y = 120$.

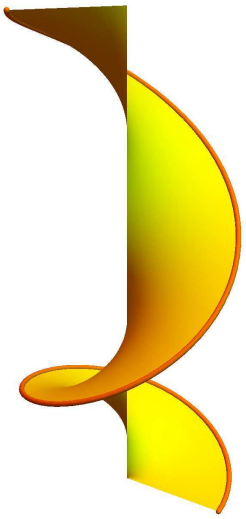
b) (5 points) Use again the Lagrange method to find the reading glasses with minimal frame material $f(x,y) = 4x + 15y$ and fixed glass area $g(x,y) = 2xy = 120$.

Problem 11) (10 points)

a) (4 points) Find the arc length of the **helical curve**

$$\vec{r}(t) = \langle \cos(t), \sin(t), \frac{2t^{3/2}}{3} \rangle \ ,$$

where t goes from 0 to 9.



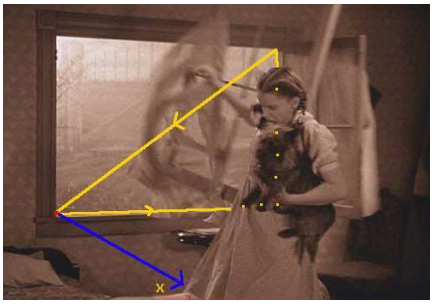
b) (3 points) Determine the angle between the velocity $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Write down the surface area integral for the surface $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), 2t^{3/2}/3 \rangle$ contained inside the cylinder $x^2 + y^2 \leq 1$ and between $0 \leq z \leq 18$ containing the previous curve in its boundary. You do not have to compute the integral but write down an expression of the form $\int_{\square}^{\square} \int_{\square}^{\square} f(s,t)\, dsdt$ with a function $f(s,t)$ you determine.

Problem 12) (10 points)

The tornado in the **wizard of Oz** induces the **force field**

$$\vec{F} = \langle \cos(x), -2x, y^3 + \sin(z^5) \rangle \ .$$



Dorothy's cardboard is picked up by the storm and pushed along the boundary of the triangle parametrized by

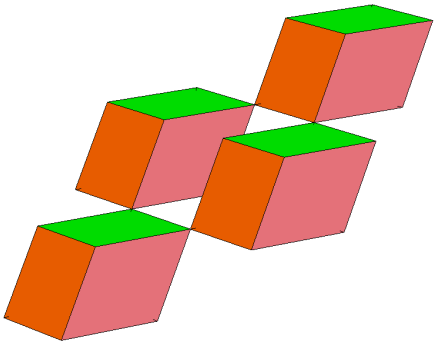
$$\vec{r}(u,v) = \langle 0, u, v \rangle$$

with $0 \leq u \leq 2$ and $0 \leq v \leq u/2$. Let C be the boundary of the triangle, oriented counter clockwise when looking from $(1,0,0)$ onto the window. Find the work $\int_C \vec{F} \cdot d\vec{r}$ which the tornado does onto the card board.

Problem 13) (10 points)

A solid E is the union of 4 congruent, non-intersecting parallelepipeds. One of them is spanned by the three vectors

$$\vec{u} = \langle 1,0,0 \rangle, \vec{v} = \langle 1,1,0 \rangle, \vec{w} = \langle 0,1,1 \rangle \ .$$



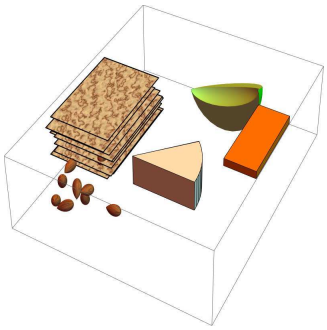
Find the flux of the vector field

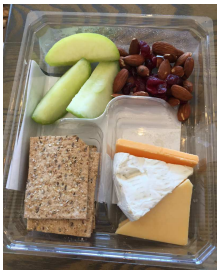
$$\vec{F} = \langle 4x + y^{2014}, z^{2014}, x^{2014} \rangle + \text{curl}(\langle -y^{2014}, x^{2014}, 3^{2014} \rangle)$$

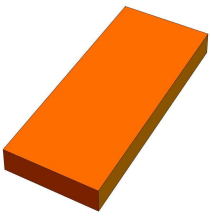
through the outwards oriented boundary surface of E.

Problem 14) (10 points) No justifications necessary

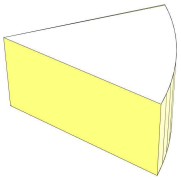
a) (5 points) **Cheese-fruit bistro boxes** can be rarely found in coffee shop these days because a "bistro monster" is eating them all. One of them contains apples, nuts, cheese and crackers. Lets match the objects and volume integrals:



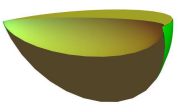




I



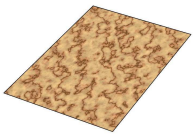
II



III



IV

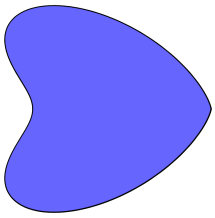


V

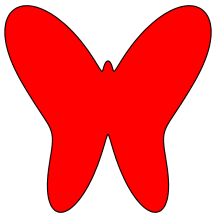
Enter I-V	volume formula
	$\int_1^2 \int_0^{\pi} \int_0^{\pi/5} \rho^2 \sin(\phi)\ d\theta d\phi d\rho$
	$\int_{-6}^{6} \int_{-10}^{10} \int_{-0.1}^{0.1} 1\ dz dx dy$
	$\int_0^{\pi/4} \int_0^{10} \int_{-5}^{5} r\ dz dr d\theta$
	$\int_{-1}^{1} \int_{-4}^{4} \int_{-8}^{8} 1\ dx dy dz$
	$\int_0^{\pi} \int_0^{2\pi} \int_0^{\cos(\phi/2)/4} \rho^2 \sin(\phi)\ d\rho d\theta d\phi$

b) (5 points) Biologist **Piet Gielis** once patented polar regions because they can be used to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when matching the following polar regions:

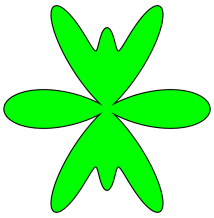
A



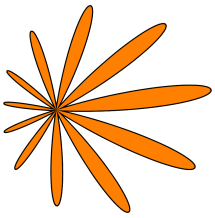
B



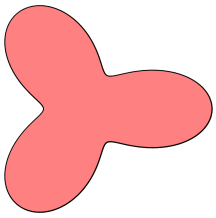
C



D



E



Enter A-E	polar region
	$r(t) \leq \lvert 2 + \cos(3t) \rvert$
	$r(t) \leq \lvert \cos(5t) - 5\cos(t) \rvert$
	$r(t) \leq \lvert 1 + \cos(t)\cos(7t) \rvert$
	$r(t) \leq \lvert 8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t) \rvert$
	$r(t) \leq \lvert \sin(11t) + \cos(t)/2 \rvert$

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F If $f(x,y,z)$ is a function then the line integral of curl(∇f)) around any closed circle is zero.

2) T F If E is the solid half-sphere $x^2+y^2+z^2 \leq 1, z<0$ then $\iiint_E x^4\, dxdydz$ is positive.

3) T F If the vector field $\vec{F}$ is incompressible and S and R are surfaces with the same boundary C and orientation then $\iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot d\vec{S}$.

4) T F $\vec{r}(u,v) = \langle \cos(u), \sin(u), 0\rangle + v\langle -\sin(u), \cos(u), 1\rangle$ parametrizes a surface in the one-sheeted hyperboloid $x^2+y^2-z^2=1$.

5) T F The equation $\text{div}(\text{grad} f) = |\text{grad} f|^2$ is an example of a partial differential equation for the unknown function $f(x,y,z)$.

6) T F The vector $(\vec{i}\times\vec{j})\times\vec{i}$ is the zero vector.

7) T F The length of the gradient $|\nabla f|$ is minimal at a local minimum of $f(x,y)$ if a local minimum of f exists.

8) T F The length of the gradient $|\nabla f|$ is maximal at a local maximum of $f(x,y)$ if a local maximum of f exists.

9) T F If $\vec{r}(t)$ parametrizes the curve obtained by intersecting $y=0$ with $x^2+y^2+z^2=1$, then the bi-normal vector $\vec{B}$ is tangent to the surface.

10) T F If the line integral of $\vec{F}$ along the closed loop $x^2+y^2=1, z=0$ is zero then the vector field is conservative.

11) T F The vectors $\vec{v}=\langle 1,0,0\rangle$ and $\vec{w}=\langle 1,1,0\rangle$ have the property that $\vec{v}\cdot\vec{w} = |\vec{v}\times\vec{w}|$.

12) T F The surface area of a sphere depends on the orientation of the sphere. It is positive if the normal vector points outwards and changes sign if the orientation is changed.

13) T F If $f(x,y,z) = (\text{div}(\vec{F})(x,y,z))$ has a maximum at $(0,0,0)$, then $\text{grad}(\text{div}(\vec{F}))(0,0,0) = \langle 0,0,0\,\rangle$.

14) T F The curl of a conservative vector field is zero.

15) T F The flux of the curl of $\vec{F}$ through a disc $x^2+y^2\leq 1, z=0$ is always zero.

16) T F If the integral $\int\int\int_G \text{div}(\vec{F}(x,y,z))\, dxdydz$ is zero for the ball $G=\{x^2+y^2+z^2\leq 1\}$, then the divergence is zero at $(0,0,0)$.

17) T F The value $\sqrt{101\cdot 10002}$ can by linear approximation be estimated as $1000+5\cdot 1+(1/20)\cdot 2$.

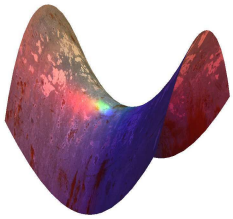
18) T F If $\vec{F} = \text{curl}(\vec{G})$ and $\text{div}(\vec{F}) = 0$ everywhere in space, then $\text{div}(\vec{G}) = 0$ everywhere in space.

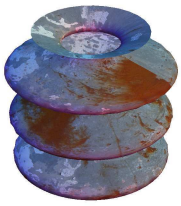
19) T F It is possible that $\vec{v}\cdot\vec{w}>0$ and $\vec{v}\times\vec{w}=\vec{0}$.

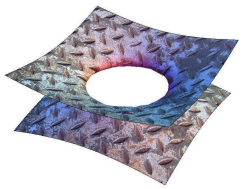
20) T F The directional derivative $D_{\vec{v}}(f)$ is defined as $\nabla f\times\vec{v}$.

Problem 2) (10 points) No justifications are necessary.

a) (3 points) The following surfaces are given either as a parametrization or implicitly in some coordinate system (Cartesian, cylindrical or spherical). Each surface matches exactly one definition.


A


B

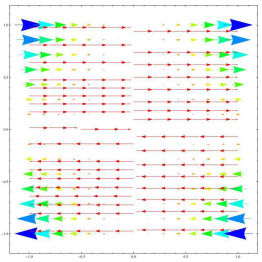

C

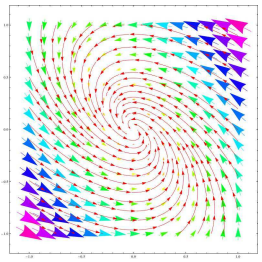

D

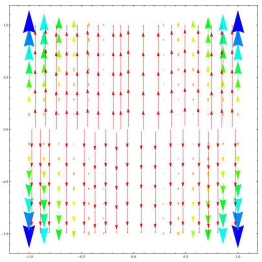
Enter A-D here	Function or parametrization
	$r = 3 + 2\sin(3z)$
	$\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$
	$x^4 - zy^4 + z^4 = 1$
	$r^2 - 8z^2 = 1$

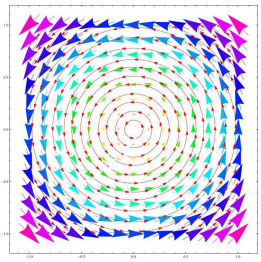
b) (3 points) The pictures display flow lines of vector fields in two dimensions. Match them.

Field	Enter 1-4
$\vec{F}(x, y) = \langle 0, x^2 y \rangle$	
$\vec{F}(x, y) = \langle x^2 y, 0 \rangle$	
$\vec{F}(x, y) = \langle -y - x, x \rangle$	
$\vec{F}(x, y) = \langle -y, x \rangle$	


1


2


3


4

c) (2 points) Match the following partial differential equations with functions $u(t, x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-C	1-3
Laplace		
wave		
heat		

A	$u(t, x) = t + t^2 - x^2$
B	$u(t, x) = t + t^2 + x^2$
C	$u(t, x) = x^2 + 2t$

1	$u_t(t, x) = u_{xx}(t, x)$
2	$u_{tt}(t, x) = -u_{xx}(t, x)$
3	$u_{tt}(t, x) = u_{xx}(t, x)$

d) (2 points) Two of the six expressions are **not** independent of the parametrization. Check them.

Velocity $\vec{r}\,'(t)$		Surface area $\int\int \lvert\vec{r}_u \times \vec{r}_v\rvert du dv$		Line integral $\int_a^b \vec{F} \cdot d\vec{r}$	
Arc length $\int \lvert\vec{r}\,'(t)\rvert\, dt$		Flux integral $\int\int_R \vec{F} \cdot d\vec{S}$		Normal vector $\vec{r}_u \times \vec{r}_v$	

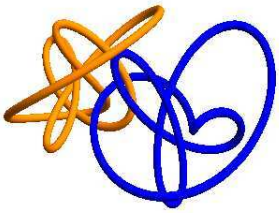
Problem 3) (10 points)

a) (4 points) The following objects are defined in three dimensional space.
Fill in either "surface", "curve", or "vector field" in each case.

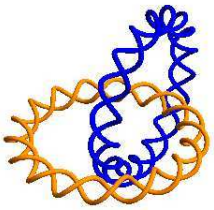
formula	surface, curve or field?
$x+y=1$	
$x+y=1, x-y=5$	
$\vec{F}(x,y,z)=\langle x, x+y, x-y\rangle$	
$\vec{r}(x,y)=\langle x, y, x-y\rangle$	
$\vec{r}(x)=\langle x, x, x^2-x\rangle$	

b) (2 points) Two closed curves $\vec{r}_1(t), \vec{r}_2(t)$ form a **link**. In our case, the curve $\vec{r}_2(t)$ is a copy of the other moved and turned around. Match three of them. Links and knots are relevant in biology: DNA strands can forms links or knots which need disentanglement.

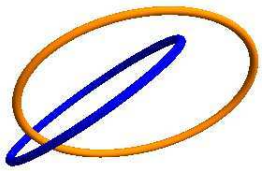
$\vec{r}_1(t)$	Enter A,B,C,D
$\langle 7\cos(t), 7\sin(t), 7\cos(t)\rangle$	
$\langle (7+\cos(17t))\cos(t), (7+\cos(17t))\sin(t), \sin(17t)\rangle$	
$\langle \cos(2t)+\sin(4t), \cos(4t)+\cos(3t), \cos(2t)+\sin(3t)\rangle$	



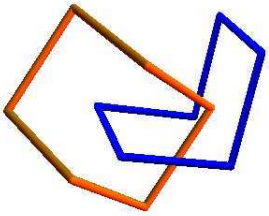
A



B



C

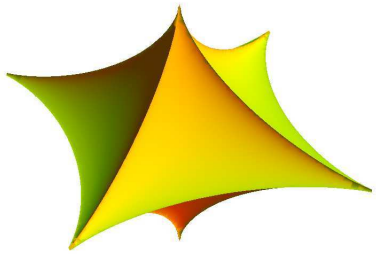


D

c) (4 points) Which of the following expressions are defined if $\vec{F}(x,y,z)$ is a vector field and $f(x,y,z)$ a scalar field in space. Is the result a scalar or vector field?

Formula	Defined	Not defined	Scalar	Vector
curl(grad(div($\vec{F}$)))				
curl(div(grad(f)))				
grad(div(curl($\vec{F}$)))				
grad(curl(div($\vec{F}$)))				
div(curl(grad(f)))				
div(grad(curl(f)))				

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the surface S given by

$$3x^{2/3} + 3y^{2/3} + 6z^{2/3} = 12$$

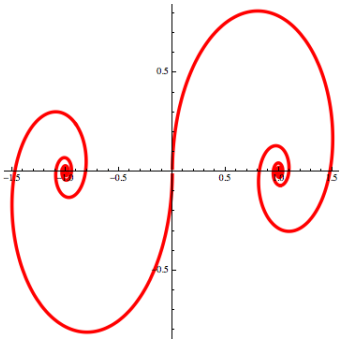
at the point $(1, 1, 1)$.

b) (5 points) When S is intersected with the plane $y = 1$, we get the curve

$$3x^{2/3} + 6z^{2/3} = 9\ .$$

Find the tangent line of the form $ax + bz = d$ for the tangent line at $(x, z) = (1, 1)$.

Problem 5) (10 points)



Find the line integral for the vector field

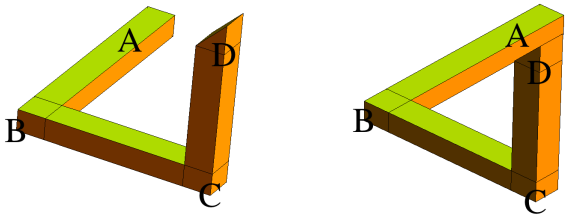
$$\vec{F}(x, y) = \langle x^6 + y + 3x^2y^3, y^7 + x + 3x^3y^2 \rangle$$

along the **ornamental curve**

$$\vec{r}(t) = \langle \frac{t}{|t|}(1 - \frac{1}{1+t^2}\cos(4t)), \frac{1}{1+t^2}\sin(4t) \rangle$$

from $t = -\infty$ to $t = \infty$. This curve connects the point $(-1, 0)$ with $(1, 0)$ along an **infinite epic journey**.

Problem 6) (10 points)



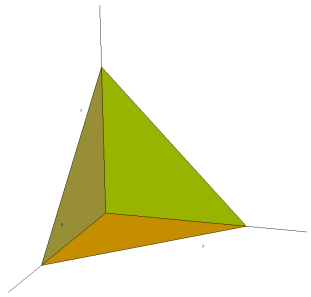
The **Penrose tribar** is a path in space connecting the points $A = (1, 0, 0), B = (0, 0, 0), C = (0, 0, 1), D = (0, 1, 1)$. While the distance between A and D is positive, we see an impossible triangle when the line of sight goes through A and D.

a) (2 points) Find the distance between the points A and D.

b) (4 points) Parametrize the line connecting A and D.

c) (4 points) Find the distance between the lines AB and CD.

Problem 7) (10 points)



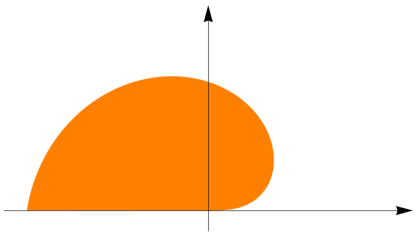
You invent a **3D printing process** in which materials of variable density can be printed. To try this out, we take a tetrahedral region E:

$$x + y + z \leq 1; x \geq 0, y \geq 0, z \geq 0$$

which has the density $f(x, y, z) = 24x$. Find the total mass

$$\int\int\int_E f(x, y, z)\, dxdydz\ .$$

Problem 8) (10 points)



When we integrate the function $f(x, y) = 2y/(\sqrt{x^2 + y^2}\arctan(y/x))$ over the **snail region** $r^2 \leq \theta \leq \pi$, we are led in polar coordinates to the integral

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} \frac{2r\sin(\theta)}{\theta}\, d\theta\, dr$$

Evaluate this integral.

Problem 9) (10 points)



Old Mc Donald wants to build a farm on a location, where the ground is as even as possible. Let $g(x, y) = y^2 + xy + x$ be the height of the ground. Find the point (x, y), where the **steepness**

$$f(x, y) = |\nabla g|^2$$

is minimal. Classify all critical points of f.

Problem 10) (10 points)



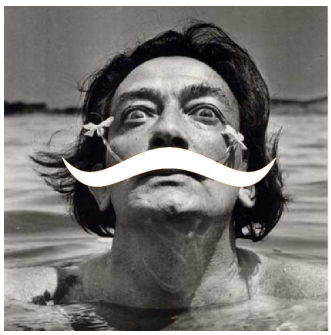
We build a **bike** which has as a frame a triangle of base length x and height y and a wheel which has radius y. Using Lagrange, find the bike which has maximal

$$f(x,y) = xy + 4\pi y^2$$

(which is twice the area) under the constraint

$$g(x,y) = x + 10\pi y = 3 \ .$$

Problem 11) (10 points)



Compute the area of the **moustache region** which is enclosed by the curve

$$\vec{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t) \rangle$$

with $0 \leq t \leq 2\pi$.

Hint. You can use without justification that integrating an odd 2π periodic function from 0 to 2π is zero.

Problem 12) (10 points)



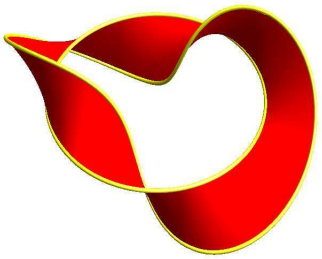
We enjoy the pre-holiday season in a local Harvard square coffee shop, where **coffee aroma** diffuses in the air. Find the flux of the air velocity field

$$\vec{F}(x,y,z) = \langle y^2, x^2, z^2 \rangle$$

leaving a coffee box

$$E : x^2 + y^2 \leq 1, x^2 + y^2 + z^2 \leq 4 \ .$$

Problem 13) (10 points)



Find the line integral of

$$\vec{F}(x,y,z) = \langle -y, x, e^{\sin z} \rangle$$

along the positively oriented boundary of the **ribbon** $\vec{r}(u,v)$ parametrized on $0 \le u \le 4\pi$ and $0 \le v \le 1/2$ with

$$\vec{r}(u,v) = \langle (1+v\cos(2u))\cos(u), (1+v\cos(2u))\sin(u), v\sin(2u) \rangle$$

for which a **good fairy** gives you the normal vector

$$\begin{aligned} \vec{r}_u \times \vec{r}_v \;\; = \;\; & \langle -\sin(u)(v\cos(4u) + 2(v+1)\cos(2u) - 3v + 2)/2 \,, \\ & \cos(u)(v\cos(4u) - 2(v-1)\cos(2u) - 3v - 2)/2 \,, \\ & -\cos(2u)(v\cos(2u) + 1) \rangle \,. \end{aligned}$$

Problem 14) (10 points)

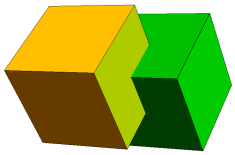
A computer can determine the volume of a solid enclosed by a triangulated surface by computing the fluxe of the vector field $\vec{F} = \langle 0, 0, z \rangle$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = \langle 0, 0, z \rangle$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\} \;\cup$$

$$\{1 \le x \le 3, 1 \le y \le 3, 1 \le z \le 3\} \,,$$

where $\cup$ is the union and the surface is oriented outwards.





Ceci n'est pas une pipe

Ceci c'est une vache

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F For any two vectors $\vec{u}$ and $\vec{v}$ we have $|\vec{u}| \leq |\vec{v}| + |\vec{v} - \vec{u}|$.

2) T F The grid curves of a parametric surface are always perpendicular to each other at any point on the surface.

3) T F The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ of $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 0, 1, 0 \rangle$ and $\vec{w} = \langle 2, 1, 1 \rangle$ is equal to 1.

4) T F For $f(x, y) = x^4 + y^4$ and $\vec{r}(t) = \langle t, t^2 \rangle$, we have $\frac{d}{dt} f(\vec{r}(t)) = \langle 4t^3, 4t^6 \rangle \cdot \langle 1, 2t \rangle$.

5) T F The flux of $\vec{F} = \langle x, 0, 0 \rangle$ through the outwardly-oriented boundary S of a parallelepiped spanned by edges $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle, \langle 1, 1, 3 \rangle$ is equal to 6.

6) T F The differential equation $u_x = u_t$ for a function $u(x, t)$ is called the heat equation.

7) T F If a function $f(x, y)$ has a critical point at $(0, 0)$ then $\text{div}(\text{grad}(f))(0, 0)$ is zero.

8) T F A function of two variables always has an odd number of critical points.

9) T F If $f(x, y)$ is a function of two variables and $(0, 0)$ is a maximum of $g(x) = f(x, 0)$ and as well as a maximum of $h(y) = f(0, y)$ then $(0, 0)$ is a maximum of f.

10) T F The divergence of a vector field $\vec{F}$ is always equal to the divergence of the curl of $\vec{F}$.

11) T F The flux of a vector field $\vec{F}$ of length $|\vec{F}| = 1$ through a triangular surface S can not be larger than the surface area of the triangle.

12) T F The arc length of the boundary of a surface is independent of the parametrization of the surface.

13) T F The vector field $\text{curl}(\vec{F})$ is at every point (x, y, z) perpendicular to $\vec{F}(x, y, z)$.

14) T F The scalar function $f = \text{div}(\vec{F})$ has the property that $\text{grad}(f(x, y, z))$ is perpendicular to $\vec{F}(x, y, z)$.

15) T F The Lagrange equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = x^2 + y^2 = 1$ have infinitely many solutions if $f = g$.

16) T F If a vector is perpendicular to itself, then it is the zero vector.

17) T F The gradient of the divergence of the curl of a vector field $\vec{F}$ is the vector field which assigns the zero vector to each point.

18) T F The identity $\text{Proj}_{\vec{v}}(\vec{w}) = \text{Proj}_{\vec{w}}(\vec{v})$ holds for all vectors $\vec{v}, \vec{w}$.

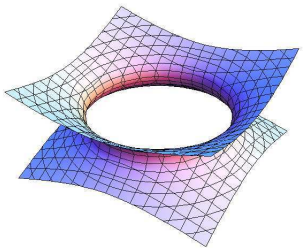
19) T F The function $f(x, y) = \sin(xy)$ is a solution to the Laplace equation $f_{xx} + f_{yy} = 0$.

20) T F The formula $\vec{r}(u, v) = \langle 2u, (9 + u^2)\cos(v), (9 + u^2)\sin(v) \rangle$ gives a parametrization of a one-sheeted hyperboloid.

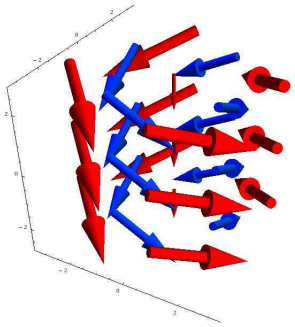
Problem 2) (10 points)

a) (6 points) Match the following objects.

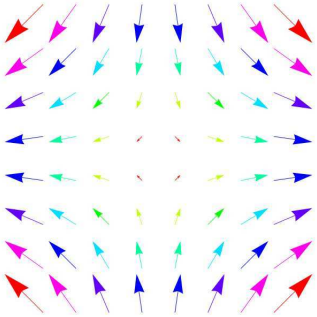
Formula	Enter 1-6
$\rho(\phi,\theta) \leq e^{\phi}$	
$\vec{F}(x,y,z) = \langle -y, x, -2 \rangle$	
$x^2 + y^2 - 5z^2 = 1$	
$z + x^2 - y^2 = 2$	
$\vec{F}(x,y) = \langle x, -y \rangle$	
$x^4 + 2y^4 \leq 3$	



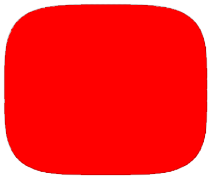
1



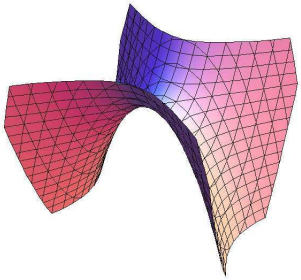
2



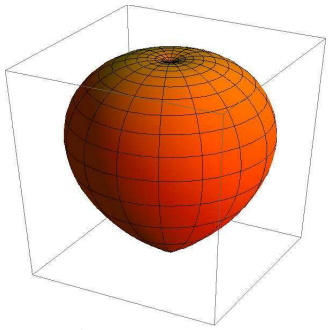
3



4



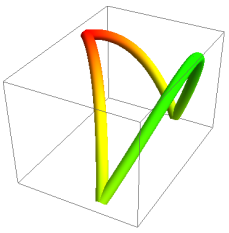
5



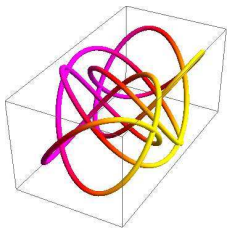
6

b) (4 points) A knot is a closed curve in space. Match the following knots

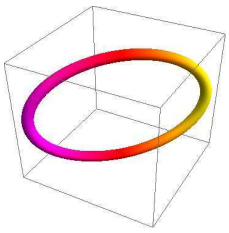
Formula	Enter A,B,C,D
$\vec{r}(t) = \langle \cos(5t), \cos(t) + \sin(5t), \cos(7t) \rangle$	
$\vec{r}(t) = \langle \lvert\cos(t)\rvert, \lvert\sin(t) + \cos(t)\rvert, \lvert\cos(2t)\rvert \rangle$	
$\vec{r}(t) = \langle (2 + \cos(\frac{3t}{2}))\cos(t), (2 + \cos(\frac{3t}{2}))\sin(t), \sin(\frac{3t}{2}) \rangle$	
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$	



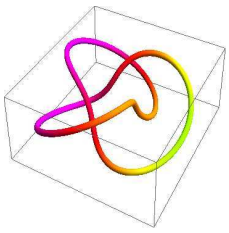
A



B



C



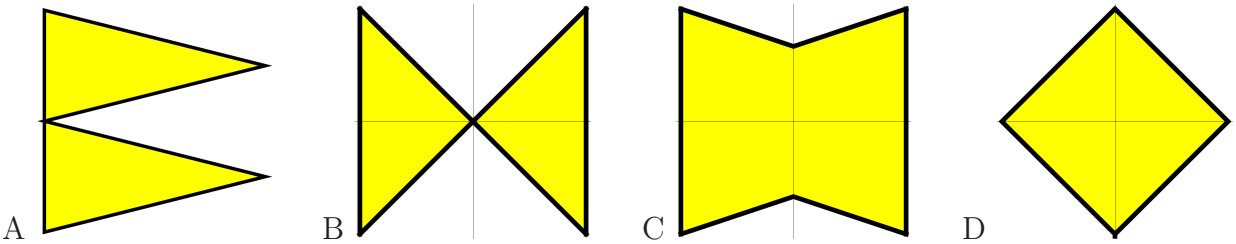
D

Problem 3) (10 points)

a) (4 points) It is Hobbit time. The following regions resemble ancient runes of the Anglo Saxons studied by JRR Tolkien. (A is "b", B is "d", C is "st", and D is "oe").

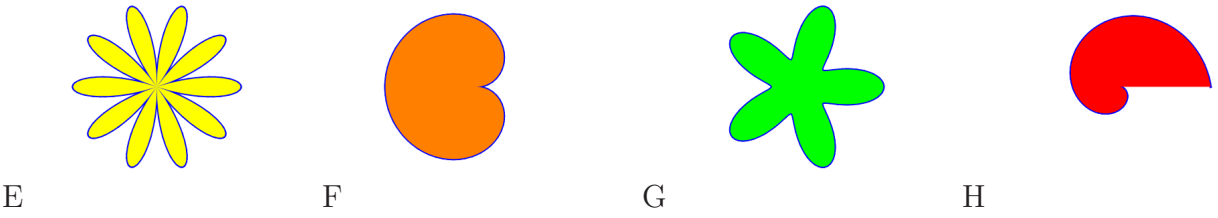
Integral	Enter A,B,C,D
$\int_{-1}^{1}\int_{-|x|}^{|x|} f(x,y)\,dydx$	
$\int_{-1}^{1}\int_{-1-|x|/2}^{1+|x|/2} f(x,y)\,dydx$	
$\int_{-1}^{1}\int_{0}^{||y+\frac{1}{2}|-|y-\frac{1}{2}||-|y|} f(x,y)\,dxdy$	
$\int_{-1}^{1}\int_{|y|-1}^{1-|y|} f(x,y)\,dxdy$	

[1]



b) (4 points) Matching polar regions

Formula	Enter E,F,G,H
$r \le \theta(2\pi - \theta)$	
$r \le \lvert\cos(5\theta)\rvert$	
$r \le 2 + \cos(5\theta)$	
$r \le 2\pi - \theta$	

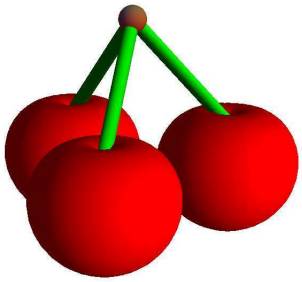


c) (2 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

Integral theorem	Grad	Curl	Div	Line integral	Flux integral
Divergence theorem					
Stokes' theorem					

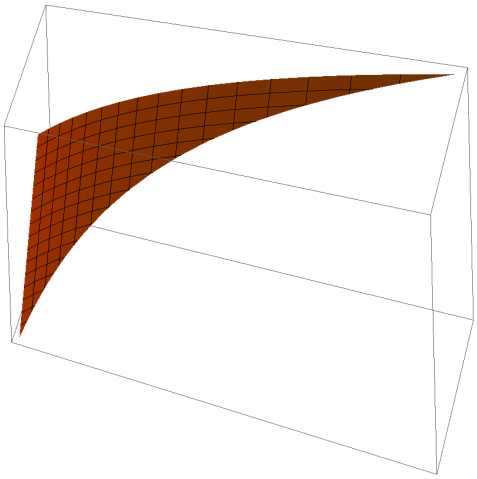
Problem 4) (10 points)

[1] More information on http://www.theshorterword.com/anglo-saxon-runes



Three cherries have centers at $A = (-1,-1,-1)$, $B = (1,0,-2)$ and $C = (0,1,-2)$ and are tied together at the origin $O = (0,0,0)$.
Find the distance between O and the plane through A, B, C. [2]

Problem 5) (10 points)

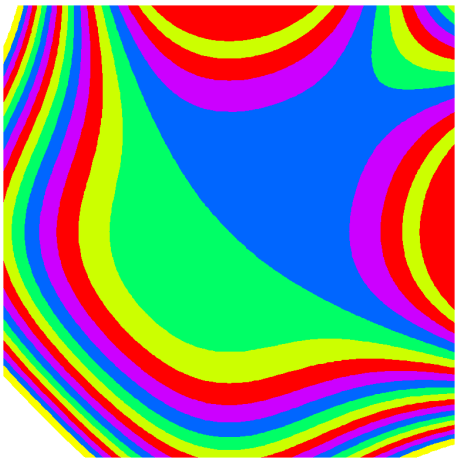


Find the surface area of the surface

$$\vec{r}(u,v) = \langle u^2 + v, u, v \rangle$$

for which $0 \leq v \leq 4$ and $\frac{v}{4} \leq u \leq 1$.

Problem 6) (10 points)

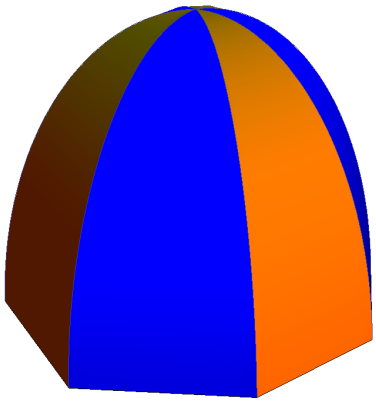


The function $f(x,y) = 2x^3 + 2y^3 - 3x^2y^2$ is called the "happy function" as you can see when you turn your head clockwise by $\pi/4$. Find and classify its extrema.

In one of the critical points, the discriminant D is zero. We want you nevertheless to decide whether this point is a "local maximum" a "local minimum" or "neither of them".

[2] Picture from Mathematica project by Alissa Zhang

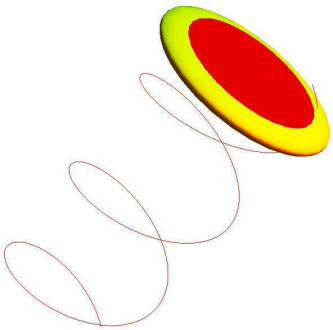
Problem 7) (10 points)



Archimedes computed volumes of solids which now bear his name. He showed that, as for the sphere, each "Archimedean globe" has volume equal to two thirds of the prism in which it is inscribed. Later it was discovered that also the surface area is two thirds of the surface area of a circumscribing prism. To find globes of minimal surface we are led to the problem:

Find values of r, h satisfying $g(r,h) = r^2 h = 3$ so that $f(r,h) = 3r^2 + 2rh$ is minimal.

Problem 8) (10 points)



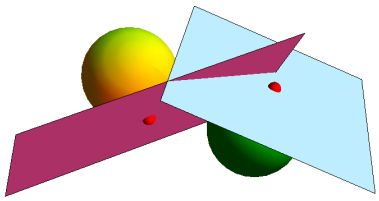
The frisbee was invented by Harvard students in 1845 when a student threw a cake plate to George Frisbie Hoar and shouted "Frisbie, catch!".
A point on the outer rim of a frisbee moves on a curve $\vec{r}(t)$ satisfying [3]

$$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t) \rangle \ .$$

We know that $\vec{r}(0) = \langle 0, 1, 0 \rangle$ and $\vec{r}\,'(0) = \langle 1, 0, 1 \rangle$. Find $\vec{r}(t)$ and the arc length of the curve $\vec{r}(t)$ from $t = 0$ to $t = 2\pi$.

Problem 9) (10 points)

[3] Some claim this incident happened at Yale, which is a fairy tale. **George Frisbie Hoar** graduated from Harvard in 1846 and became later a United States Senator. He fought against political corruption, and campaigned for the rights of African and Native Americans.



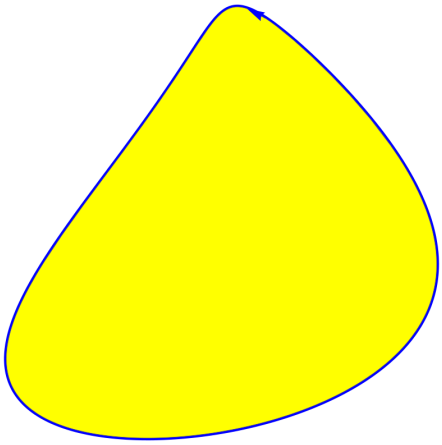
Find a parametrization of the line of intersection of the tangent plane at the point $(1, -1, 0)$ of the sphere

$$x^2 + y^2 + z^2 = 2$$

and the tangent plane to the point $(5, 1, 1)$ of the sphere

$$(x-5)^2 + y^2 + z^2 = 2\ .$$

Problem 10) (10 points)

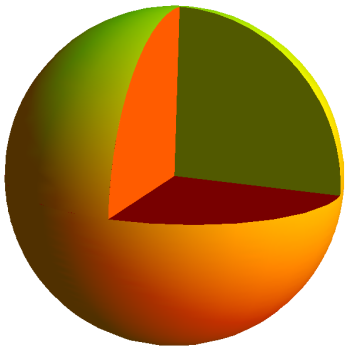


Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle 3\cos(t) - \sin(2t), 4\sin(t) + \cos(t) \rangle\ ,$$

where $0 \leq t \leq 2\pi$.

Problem 11) (10 points)

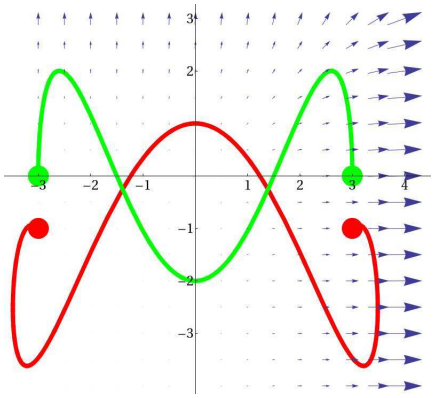


Find the flux of the vector field

$$\vec{F}(x, y, z) = \langle x^3 + x^2, y^3 - xy, z^3 - xz \rangle$$

through the boundary surface of E (oriented outwards), where the solid E is a unit sphere from which the first octant has been removed.

Problem 12) (10 points)



Assume the wind velocity on the Charles is

$$\vec{F}(x,y) = \langle e^x, e^y \rangle \ .$$

A sail boat takes the path

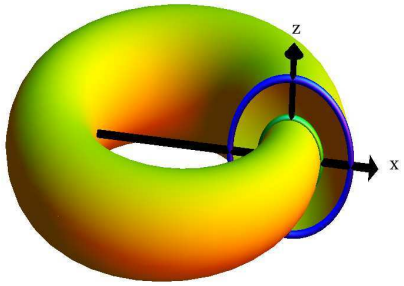
$$C_1 : \vec{r}(t) = \langle -3\cos(t) - \sin(2t), 2\sin(t) + 2\cos(4t) - 3 \rangle$$

from $(-3,-1)$ to $(3,-1)$. An other boat follows the path

$$C_2 : \vec{r}(t) = \langle -3\cos(t), 2\sin(3t) \rangle$$

from $(-3,0)$ to $(3,0)$. To find out which path needs more energy, compute both line integrals $\int_{C_1} \vec{F} \cdot \vec{dr}$ and $\int_{C_2} \vec{F} \cdot \vec{dr}$.

Problem 13) (10 points)



The "foot in the mouth" surface S seen in the picture is parametrized by

$$\vec{r}(u,v) = \langle \quad (4 + g(u,v)\cos(v))\cos(u) - 4, \\ (4 + g(u,v)\cos(v))\sin(u), \\ g(u,v)\sin(v) \rangle$$

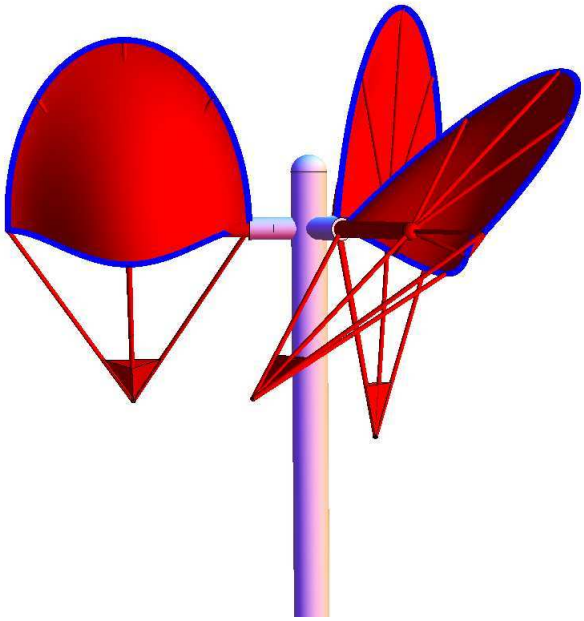
with $g(u,v) = (2 - \frac{u}{2\pi})$ and $0 \le u, v \le 2\pi$. It is oriented outwards. Its boundary consists of two circles in the xz-plane centered at the origin, with radius 1 and 2. Find the flux of the curl of

$$\vec{F}(x,y,z) = \langle z + yz, x, \sin(x^3 y) + y^2 + z^4 \rangle$$

through S.

Problem 14) (10 points)





Porter square in Cambridge features a moving art project. It proves that paddle wheels and curl are omni present. To build that model for google earth, we idealized one of the blades as the surface

$$\vec{r}(t,s) = \langle 2\cos(t)\sin(s), 4\sin(t)\sin(s), \cos^4(s)\rangle$$

with $t, s \in [0, \pi]$. The surface S is oriented so that the boundary consists of two parts. The surface boundary can be parametrized as

$$\vec{r}_1(t) = \langle 2\sin(t), 0, \cos^4(t)\rangle, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

and

$$\vec{r}_2(t) = \langle 2\cos(t), 4\sin(t), 0\rangle, 0 \leq t \leq \pi \ .$$

The wind force is given by the curl of the vector field $\vec{F}(x,y,z) = \langle y, z, 0\rangle$. Find the flux of curl$(\vec{F})$ through the surface S.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The tangent plane of the surface $z = f(x, y)$ at a local maximum of f is parallel to the xy-plane

2) T F For any smooth functions $f(x, y), x(t), y(t)$, we have $\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$.

3) T F At a local maximum of a function $f(x, y)$ we always have $f_{xx} \leq 0$ and $f_{yy} \leq 0$.

4) T F If $(0, 0)$ is a critical point for a function $f(x, y)$ as well as for a function $g(x, y)$ then $(0, 0)$ is a critical point of the function $f(x, y) + g(x, y)$.

5) T F The curves $\vec{r}(t) = \langle t, 2t\rangle$ and $\vec{s}(t) = \langle 2t, -t\rangle$ intersect at a right angle at $(0, 0)$.

6) T F The quadric $x - y^2 + z^2 = 5$ is a hyperbolic paraboloid.

7) T F If $\vec{u}, \vec{v}, \vec{w}$ are unit vectors, then the length of the vector projection of $\vec{u} \times \vec{v}$ onto $\vec{w}$ is the same as the length of the vector projection of $\vec{v} \times \vec{w}$ onto $\vec{u}$.

8) T F The partial differential equation $u_{tt} = u_{xx}$ is called the Clairaut equation.

9) T F $\iint_R \sqrt{1 - x^2 - y^2}\, dx\, dy = \frac{2\pi}{3}$, where R is the region $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy-plane.

10) T F There exists a vector field $\vec{F}(x, y, z)$ in space such that $\text{curl}(\vec{F}) = \langle 5x, -11y, 7z\rangle$.

11) T F Let S is the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ with normal pointing away from the center. Then the flux integral is $\iint_S \langle 0, 0, 1\rangle \cdot\, d\vec{S} = 2\pi$.

12) T F The points that satisfy $\theta = \pi/4$ and $\phi = \pi/4$ form a surface which is part of a cone.

13) T F The curvature of the curve $\vec{r}(t) = \langle t, t, t^2\rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle t^3, t^3, t^6\rangle$ at $t = 0$.

14) T F If $f(x, y, z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere (i.e. $\vec{F}$ is incompressible).

15) T F For any function $f(x, y, z)$ we have $\text{curl}(\text{curl}(\text{grad}(f))) = \vec{0}$.

16) T F For any vector field $\vec{F}$ and any curve $\vec{r}$ parametrized on $[a, b]$ we have $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$.

17) T F There exist vector fields $\vec{F}$ and $\vec{G}$ in space such that $\text{curl}(\vec{F}) = \text{grad}(\vec{G})$.

18) T F If $\vec{F}$ is a smooth vector field in space and S is a closed oriented surface, then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

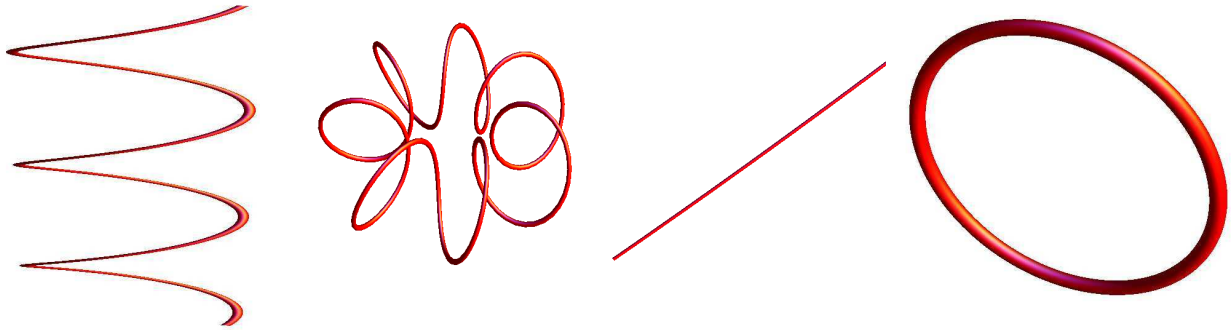
19) T F The solid enclosed by the surfaces $z = 2 - \sqrt{x^2 + y^2}$ and $z = \sqrt{x^2 + y^2}$ has the volume $\int_0^{2\pi} \int_0^1 \int_r^{2-r} r\, dz dr d\theta$.

20) T F If $\vec{r}\,''(t) = \langle 0, 0, \sin(t)\rangle$, $\vec{r}(0) = \langle 0, 1, 0\rangle$, $\vec{r}\,'(0) = \langle 1, 0, 0\rangle$, then $\vec{r}(t) = \langle t, 1 + t, t - \sin(t)\rangle$.

Problem 2) (6 points)

a) (4 points) Match the curves. There is an exact match.

1 2 3 4

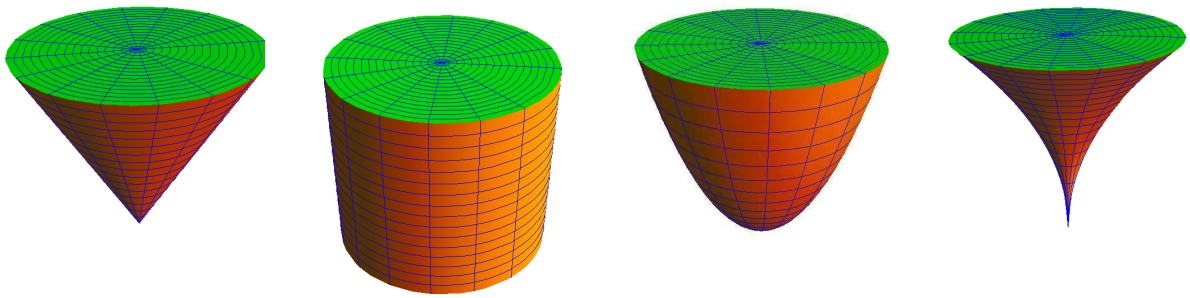


Enter 1-4	Object definition
	$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$
	$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$
	$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle t, t, t \rangle$

b) (4 points) Match the solids with the triple integrals. Also here, there is an exact match:

Enter A-D	3D integral computing volume
	$\int_0^{2\pi}\int_0^1\int_r^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_{r^2}^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_{\sqrt{r}}^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_0^1 r\ dzdrd\theta$

A B C D



c) (2 points) What was the name again?

Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = -g_{yy}$

Problem 3) (10 points)

a) (5 points) For the following quantities, decide whether they are vector fields or scalar fields (functions) or nonsense. Here $\vec{F} = \langle P, Q, R\rangle$ is a vector field in space, $f(x, y, z)$ is a scalar function and $\nabla = \langle \partial_x, \partial_y, \partial_z\rangle$. Recall that $\nabla \times \vec{F} = \text{curl}(\vec{F}), \nabla \cdot \vec{F} = \text{div}(\vec{F})$ and $\nabla f = \text{grad}(f)$.

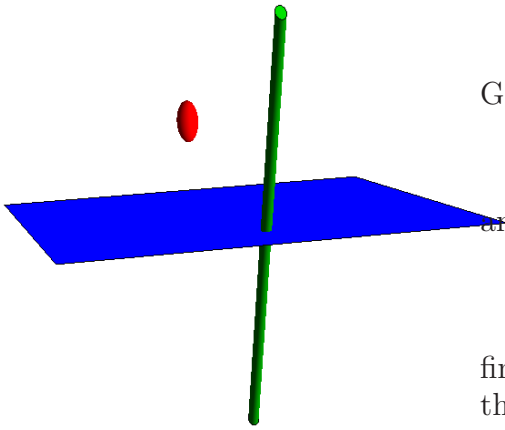
object	scalar	vector	not defined
$\nabla \vec{F}$			
$\nabla \cdot \vec{F}$			
$\nabla \times \vec{F}$			
$\nabla(\nabla \cdot \vec{F})$			
$\nabla \times (\nabla \times \vec{F})$			
$\nabla \times (\nabla \cdot \vec{F})$			
∇f			
$\nabla f \times \vec{F}$			
$\nabla f \cdot \vec{F}$			
$\nabla \times (\nabla f)$			

b) (5 points) Match the formulas for the position vector $\vec{r}(t)$ of a curve in space:

label	formula
A	$\vec{r}\,''(t)$
B	$\int_a^b \lvert\vec{r}\,'(t)\rvert\, dt$
C	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
D	$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$
E	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

expression	enter A-E
curvature	
unit normal vector	
unit tangent vector	
arc length	
acceleration	

Problem 4) (10 points)



Given a point $P = (4, 3, 1)$, a plane

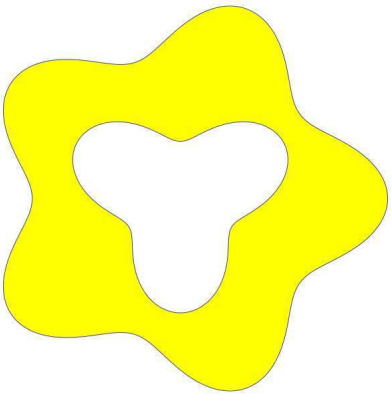
$$\Sigma : 3x + 4y - 12z = 0$$

and a line

$$L : \frac{x-1}{3} = \frac{y-2}{4} = \frac{z-1}{12},$$

find the sum $d(P, L) + d(P, \Sigma)$ of the distances of P to the line and plane.

Problem 5) (10 points)



a) (5 points) Find the double integral

$$\int_0^3 \int_y^3 \frac{\sin(2x)}{x}\, dx\, dy\ .$$

b) (5 points) What is the area of the polar region

$$3 + \sin(3\theta) \leq r \leq 6 + \cos(5\theta)\ ?$$

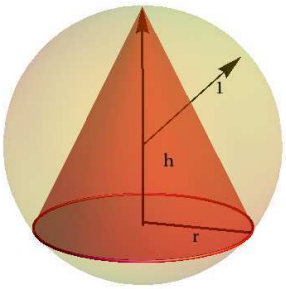
Problem 6) (10 points)

a) (8 points) Locate and classify all the local maxima, minima and saddle points of the function

$$f(x, y) = x^4 + y^4 - 8x^2 - 8y^2\ .$$

b) (2 points) Is there a global maximum or a global minimum of f? Explain.

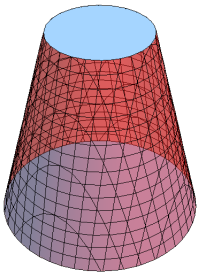
Problem 7) (10 points)



For which base radius r and height h does a cone inscribed into the unit sphere have maximal volume $f(r, h) = \pi r^2 h/3$? The constraint is given by Pythagoras as $g(r, h) = r^2 + (h-1)^2 = 1$. Use the Lagrange method.

Problem 8) (10 points)



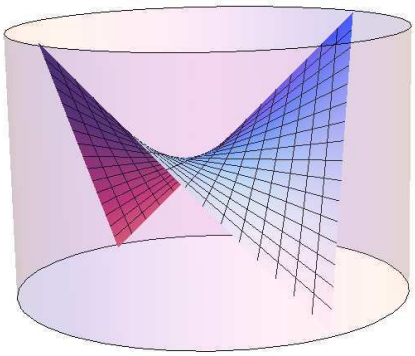


A bird's feeding cage E is part of a cone $x^2 + y^2 = 4(3 - z)^2$ with $1 < z < 2$. The cage is filled with different kind of seeds, the heavier have gone down and the density is $(3-z)$. We want to find the moment of inertia

$$\iiint_E (x^2 + y^2)(3 - z)\, dxdydz$$

so that we can know how much energy the feeding cage has if a squirrel spins it. You do not have to worry in this problem that squirrels are not birds.

Problem 9) (10 points)

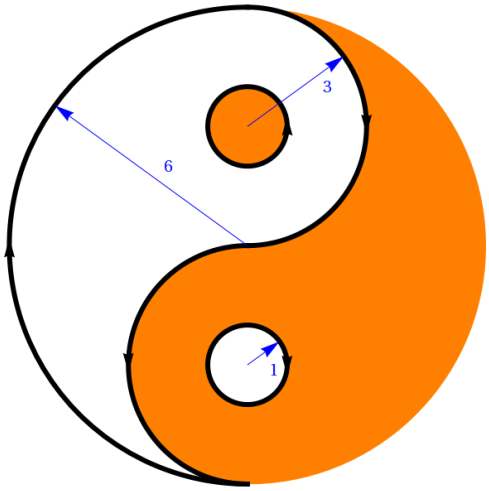


a) (4 points) Find the surface area of the surface

$$\vec{r}(s,t) = \langle s, -t, 2st \rangle$$

with $s^2 + t^2 \leq 9$.
b) (4 points) The coordinates of the surface satisfies $2xy + z = 0$. Find the tangent plane at $(1, 1, -2)$.
c) (2 points) What is the formula for the linearization of $f(x, y) = 2xy$ at the point $(1, 1)$.

Problem 10) (10 points)

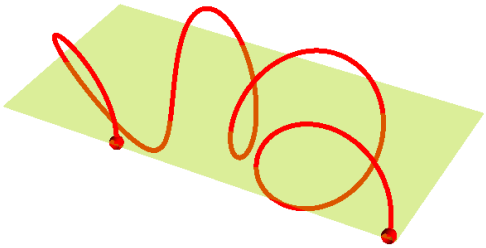


Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{F}(x, y) = \langle -y + \sin(e^x), x \rangle$$

around C. Notice that the Ying and the Yang have the same area.

Problem 11) (10 points)



Let C be the curve

$$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$$

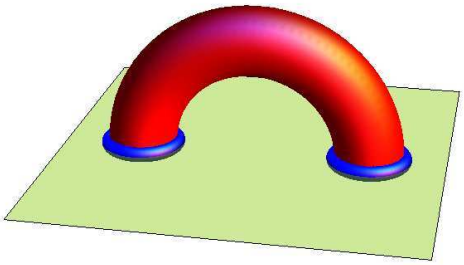
parametrized by $0 \leq t \leq \pi$ starting at $t = 0$ and ending at $t = \pi$. Calculate the line integral

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \,,$$

where $\vec{F}$ is the vector field

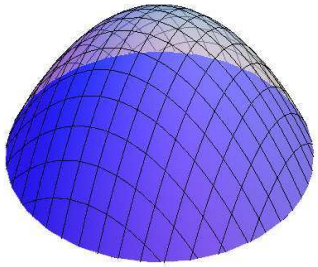
$$\vec{F}(x,y,z) = \langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2} \rangle \,.$$

Problem 12) (10 points)



Find the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x^2, 0 \rangle$ through a half torus surface S given by $(\sqrt{x^2+z^2}-3)^2 + y^2 = 1, z \geq 0$ which intersects the xy-plane $z = 0$ in two circles $C_1 : (x-3)^2 + y^2 = 1$ and $C_2 : (x+3)^2 + y^2 = 1$. The torus S is oriented outwards.

Problem 13) (10 points)



Find the flux of the vector field

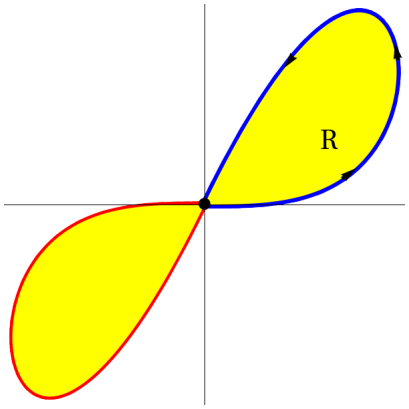
$$\vec{F}(x,y,z) = \langle x^3 z, y^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part S of the boundary of the solid

$$G : z + x^2 + y^2 \leq 1, \; z \geq 0 \,.$$

The paraboloid surface S is oriented upwards.

Problem 14) (10 points)



Find the area of the **propeller** shaped region enclosed by the figure 8 curve

$$\vec{r}(t) = \langle t - t^3, 2t^3 - 2t^5 \rangle \ ,$$

parametrized by $-1 \leq t \leq 1$. To find the total area compute the area of the region R enclosed by the right loop $0 \leq t \leq 1$ and multiply by 2.

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
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12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The functions $e^{x^2+y^3-y}$ and x^2+y^3-y have the same critical points.

2) T F The line $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$ hits the plane $x+y+z = 100$ at a right angle.

3) T F The quadric $x^2 - 2y^2 + z^2 = 5$ is a one sheeted hyperboloid.

4) T F The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.

5) T F The partial differential equation $u_x = u_{tt}$ is called the **Heat equation**.

6) T F The curvature of the curve $\vec{r}(t) = \langle \sin(2t), \cos(2t)/\sqrt{2}, \cos(2t)/\sqrt{2} \rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$ at $t = 0$.

7) T F The space curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.

8) T F If a smooth function $f(x,y)$ has a global maximum, then it has a global minimum.

9) T F If $L(x,y)$ is the linearization of $f(x,y)$ at (x_0, y_0) and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ on $f = c$ at the point $\vec{r}(t_0) = \vec{s}(t_0) = (x_0, y_0)$ so that $|\vec{r}\,'(t_0)| = |\vec{s}\,'(t_0)| = 1$, then $|d/dtL(\vec{s}(t))| = |d/dtf(\vec{r}(t))|$ at the time $t = t_0$.

10) T F If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.

11) T F The flux of the vector field $\vec{F} = \nabla f$ through the surface $f(x,y,z) = x^4 + y^4 + z^4 = 1$ is positive if the surface is oriented so that $\vec{r}_u \times \vec{r}_v$ points in the direction of the gradient of f.

12) T F If we extremize the function $f(x,y)$ under the constraint $g(x,y) = 1$, and the functions are the same $f = g$, all points on the constraint curve are extrema for f.

13) T F If a point (x_0, y_0) is a minimum of $f(x,y)$ under the constraint $g(x,y) = 1$, then it is also a local minimum of the function $f(x,y)$ without constraints.

14) T F If a vector field $\vec{F}(x,y)$ is a gradient field, then any line integral along any closed ellipse is zero.

15) T F The flux of an irrotational vector field is zero through any surface S in space.

16) T F The divergence of a gradient field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is everywhere zero.

17) T F The line integral of the vector field $\vec{F}(x,y,z) = \langle x, y, z \rangle$ along a circle in the $xy-$ plane is zero.

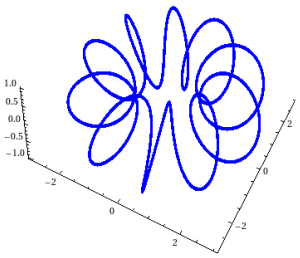
18) T F For any solid E, the moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ is always larger than the volume $\iiint_E 1 \, dxdydz$ of E.

19) T F The curvature of a parametrized curve satisfying $|\vec{r}\,'(t)| = 1$ is bounded above by the length $|\vec{r}\,''|$ of the acceleration.

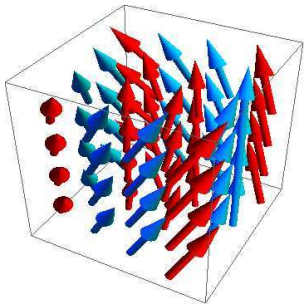
20) T F Given a vector field $\vec{F} = \langle P, Q, R \rangle$, the directional derivative of $\mathrm{div}(\vec{F}(x,y,z))$ in the direction $\vec{v} = \langle 1, 0, 0 \rangle$ is $P_{xx} + Q_{xy} + R_{xz}$.

Problem 2) (6 points)

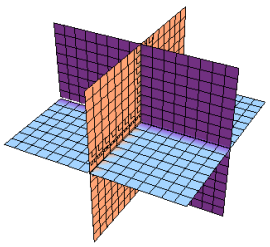
a) (6 points) Match the objects with their definitions



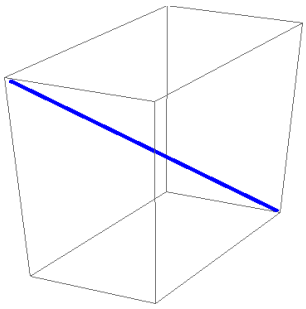
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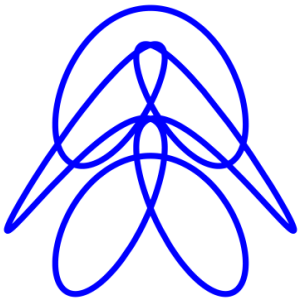
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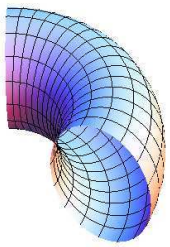
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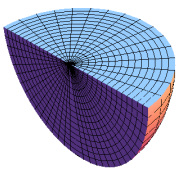


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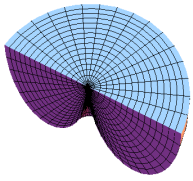
Enter 1-6	Object definition
	$\vec{r}(t) = \langle (2+\cos(10t))\cos(t), (2+\cos(10t))\sin(t), \sin(10t) \rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 2 \rangle$
	$\vec{r}(t,s) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
	$x^2y^2z^2 = 0$
	$(x-1)/5 = (y-2)/10 = (z-1)/3$
	$\vec{r}(t) = \langle \sin(t)+\cos(5t), \cos(t)+\cos(6t) \rangle$

b) (4 points) Match the solids with the triple integrals:

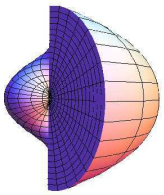
Enter A-D	3D integral computing volume
	$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1/\cos(\phi)} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{\sin(\phi)} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{1} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi-\theta} \rho^2\sin(\phi)\, d\rho d\phi d\theta$



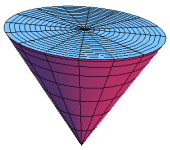
A



B



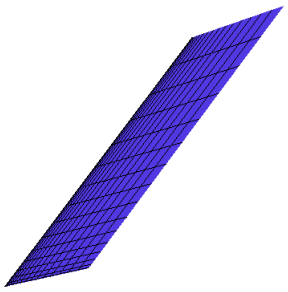
C

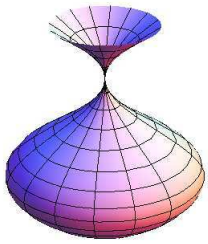


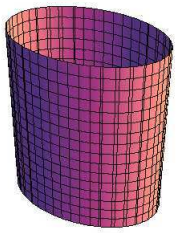
D

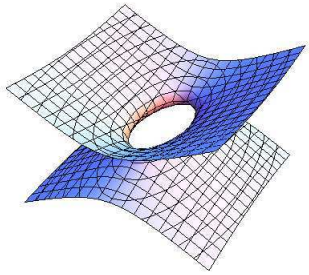
Problem 3) (10 points)

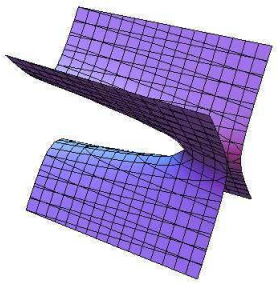
a) (6 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.

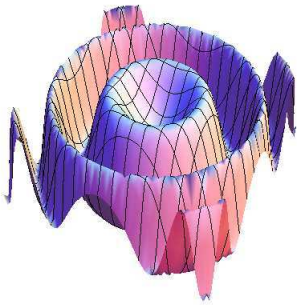
A 

B 

C 

D 

E 

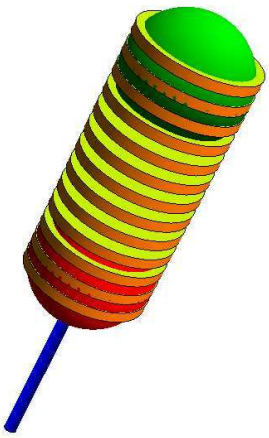
F 

Enter A-F here	Function or parametrization
	$\vec{r}(u,v) = \langle u^2, v^2, u^2+v^2\rangle$
	$\vec{r}(u,v) = \langle (1+\sin(u))\cos(v), (1+\sin(u))\sin(v), u\rangle$
	$4x^2+y^2-9z^2=1$
	$x-9y^2+4z^2=1$
	$\vec{r}(u,v) = \langle u, v, \sin(u^2+v^2)\rangle$
	$4x^2+9y^2=1$

b) (4 points) If the blank box is replaced by $\nabla f(5,6)$ the statement becomes true or false. Determine which case we have. The function $f(x,y)$ is an arbitrary nice function like for example $f(x,y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x,y) = f(5,6)$ and has the property that $\vec{r}(0) = \langle 5,6\rangle$.

True/False	Topic	Statement
	Linearization	$L(x,y) = f(5,6) +$ ☐ $\cdot\langle x-5, y-6\rangle$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t))\vert_{t=0} =$ ☐ $\cdot\vec{r}\,'(0)$
	Steepest descent	f decreases at $(5,6)$ most in the direction of ☐
	Estimation	$f(5+0.1, 5.99) \sim f(5,6) +$ ☐ $\cdot\langle 0.1, -0.01\rangle$
	Directional derivative	$D_{\vec{v}} f(5,6) =$ ☐ $\cdot\vec{v}$, $\vert\vec{v}\vert = 1$
	Level curve	of f through $(5,6)$ has the form ☐ $\cdot\langle x-5, y-6\rangle = 0$
	Vector projection	of $\nabla f(5,6)$ onto $\vec{v}$ is $\vec{v}(\vec{v}\cdot$ ☐ $)/\vert\vec{v}\vert^2$
	Tangent line	of $\vec{r}(t)$ at $(5,6)$ is parametrized by $\vec{R}(s) = \langle 5,6\rangle + s$ ☐

Problem 4) (10 points)



Two ice cream scoops given by spheres

$$x^2 + y^2 + (z+1)^2 = 1$$

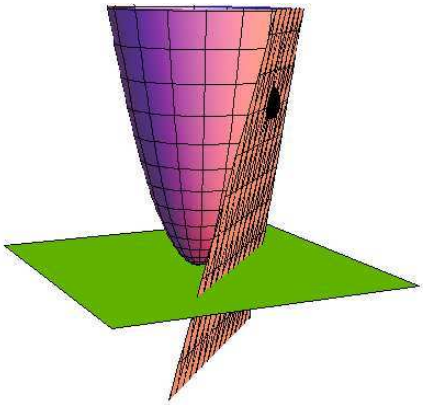
and

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 1$$

are enclosed by a cylinder which is tangent to both spheres. Find the equation of the cylinder.

Hint: consider the distance of a general point (x, y, z) to the line passing through the centers of the spheres.

Problem 5) (10 points)



Find a parametrization

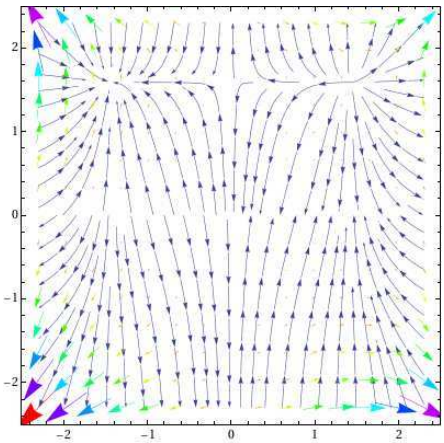
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for the line obtained by intersecting the tangent plane Σ to the surface

$$x^2 + y^2 - z = 0$$

at $(-1, -1, 2)$ with the xy-plane.

Problem 6) (10 points)



The vector field

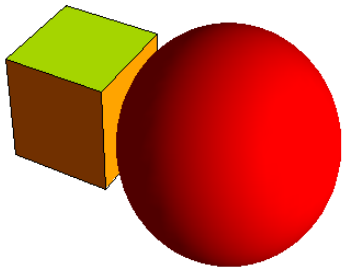
$$\vec{F}(x, y) = \langle P, Q \rangle = \langle y(x^4 - 2x^2), x(y^4 - 4y) \rangle$$

has the curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) \ .$$

Find and classify all critical points of f by deciding whether they are local maxima, local minima or saddle points. Is there a global maximum or global minimum of f?

Problem 7) (10 points)



We want to minimize the volume of the union of a **sphere** of radius x and a **cube** of side length y under the constraint that the sum of the two surface areas is equal to 4. Find the minimal value using the Lagrange method.

Remark: You do not have to show any derivations of the volume and surface area of the sphere.

Problem 8) (10 points)

A solid E in space is determined by the inequalities

$$0 \leq z \leq 9 \, ,$$

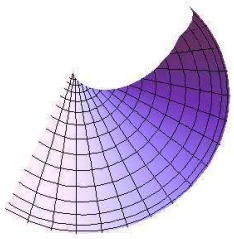
$$z^2 - x^2 - y^2 \geq 4$$

and

$$x^2 + y^2 \leq 1 \, .$$

Find the volume of E.

Problem 9) (10 points)



A surface S is parametrized by

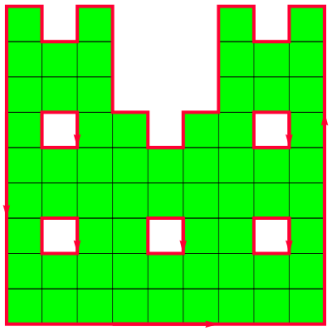
$$\vec{r}(u, v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle$$

where

$$0 \leq u \leq \sqrt{\pi}, u^2 \leq v \leq \pi \, .$$

Find its surface area.

Problem 10) (10 points)

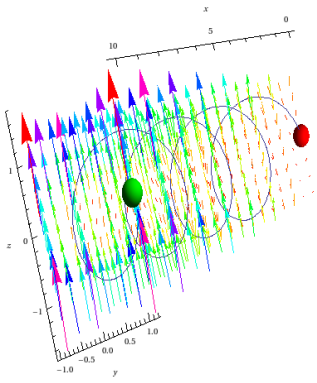


What is the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field

$$\vec{F}(x,y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

along the boundary C of the planar "castle region" shown in the picture? Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

Problem 11) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x,y,z) = \langle \cos(x), 2 + \cos(y), e^z + x(y^2 + z^2) \rangle$$

along the curve $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ with $0 \leq t \leq 3\pi$.

Hint: you might want to find a split $\vec{F} = \vec{G} + \vec{H}$ and compute line integrals of $\vec{G}$ and $\vec{H}$ separately.

Problem 12) (10 points)



A biker in the Harvard Hemenway gym pedals. Assume that the force of a foot is

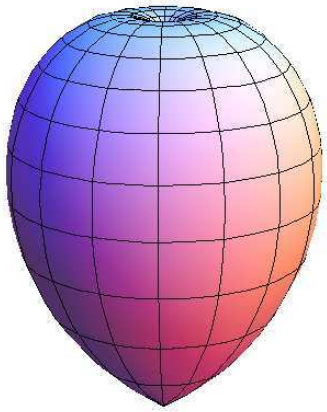
$$\vec{F} = \langle 0, 0, x^3 - x^2 + \sqrt{2 + \sin(z)} \rangle$$

and that one of the feet moves on a path $C : \vec{r}(t) = \langle 2\cos(t), 0, 2\sin(t) \rangle$. How much work

$$\int_C \vec{F} \cdot d\vec{r}$$

is done by this foot, when pedaling 10 times which means $0 \leq t \leq 20\pi$?

Problem 13) (10 points)



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x, y, z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y) \rangle \ .$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field $\vec{F}$ through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.

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14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v}+\vec{w}$ has length $1/3$.

2) [T] [F] For any three vectors, we have $|(\vec{u}\times\vec{v})\times\vec{w}| = |(\vec{v}\times\vec{w})\times\vec{u}|$.

3) [T] [F] Denote by $d(P,L)$ the distance from a point P to a line L in space. For any point P and any two lines L, K in space, we have $d(P,L)+d(P,K) \geq d(L,K)$.

4) [T] [F] For any three vectors $\vec{u},\vec{v},\vec{w}$, the relation $|\vec{u}\times(\vec{v}\times\vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.

5) [T] [F] If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature $1/2$ everywhere.

6) [T] [F] If the curvature of a space curve is constant 1 and the speed $|\vec{r}\,'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}\,''(t)| = 1$ everywhere.

7) [T] [F] If a vector field $\vec{F} = \langle P, Q\rangle$ has $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\text{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.

8) [T] [F] If the level curve $f(x,y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0,0)$ must be a critical point for which $D < 0$.

9) [T] [F] The surface $\vec{r}(u,v) = \langle u^3\cos(v), u^3\sin(v), u^3\rangle$ with $v \in [0, 2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.

10) [T] [F] There is a non-constant function $f(x,y,z)$ of three variables such that $\text{div}(\text{grad}(f)) = f$.

11) [T] [F] If $\text{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\text{div}(\vec{F}) = 0$ everywhere.

12) [T] [F] The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.

13) [T] [F] The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0,3,0)$ is $y = 3$.

14) [T] [F] Assume (x_0, y_0) is not a critical point of $f(x,y)$. It is possible that f increases at (x_0,y_0) most rapidly in the direction $\langle 1, 0\rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5\rangle$.

15) [T] [F] Assume $\vec{F}(x,y,z)$ is defined everywhere except on the z-axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z-axis, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for all curves C.

16) [T] [F] A point (x_0,y_0) is an extremum of $f(x,y)$ under the constraint $g(x,y) = 0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0,y_0) can not be a local maximum on the constraint curve.

17) [T] [F] The vector field $\vec{F}(x,y,z) = \langle x^2, y^2, z^2\rangle$ can be the curl of another vector field $\vec{G}$.

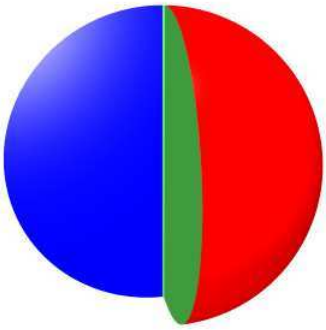
18) [T] [F] If $f(x,y)$ and $g(x,y)$ are two functions and $(2,3,3)$ is a critical point of the function $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$, then $(2,3)$ is a solution of the Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y) = 0$.

19) [T] [F] Assume $(0,0)$ is a global maximum of $f(x,y)$ on the disc $D = \{x^2+y^2 \leq 1\}$, then $\int\int_D f(x,y)\,dxdy \leq \pi f(0,0)$.

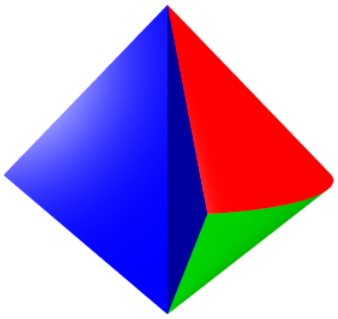
20) [T] [F] Let C be a curve parametrized by $\vec{r}(t), 0 \leq t \leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f\cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}\,''(t)}(f(\vec{r}(t))\,dt$.

Problem 2) (10 points)

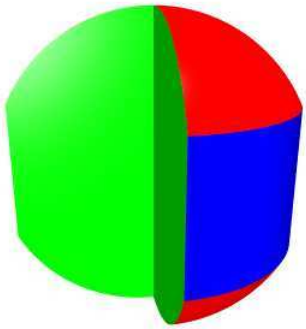
a) (4 points) Match the following triple integrals with the regions.



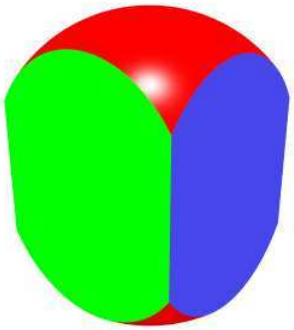
I



II



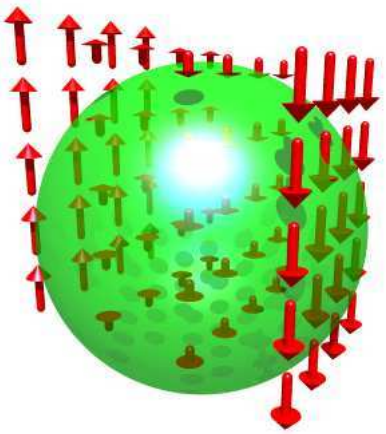
III



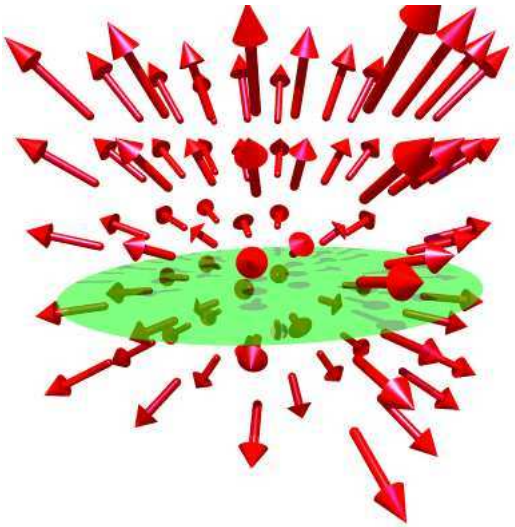
IV

Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2}\int_0^1\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$
	$\int_0^{3\pi/2}\int_0^1\int_{r-1}^{1-r} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$
	$\int_0^{3\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$
	$\int_{-1}^1\int_{-1}^1\int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x,y,z)\ dzdydx$

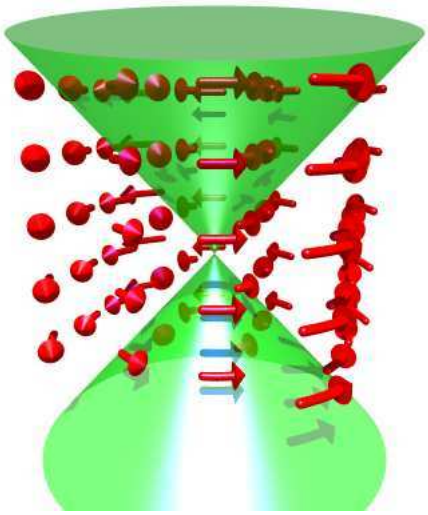
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



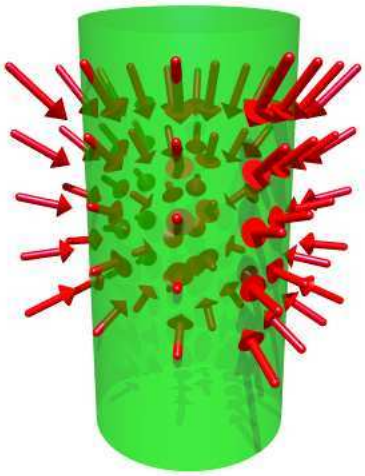
A



B



C

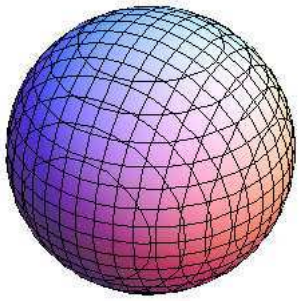


D

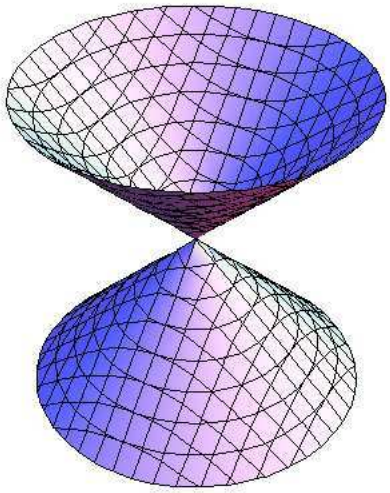
Enter A,B,C,D	Field	Surface	Flux zer
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$	$\vec{r}(u,v) = \langle u, v, 0\rangle$	
	$\vec{F}(x,y,z) = \langle 0, 0, y\rangle$	$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$	
	$\vec{F}(x,y,z) = \langle -x, -y, -z\rangle$	$\vec{r}(\theta,z) = \langle \cos(\theta), \sin(\theta), z\rangle$	
	$\vec{F}(x,y,z) = \langle -y, x, 0\rangle$	$\vec{r}(\theta,z) = \langle z\cos(\theta), z\sin(\theta), z\rangle$	

Problem 3) (10 points)

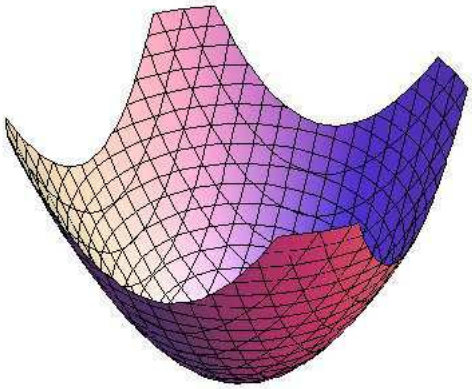
a) (6 points) Match the following level surfaces with functions $f(x, y, z)$ and also match the parametrization of part of the surface $f(x, y, z) = 0$.



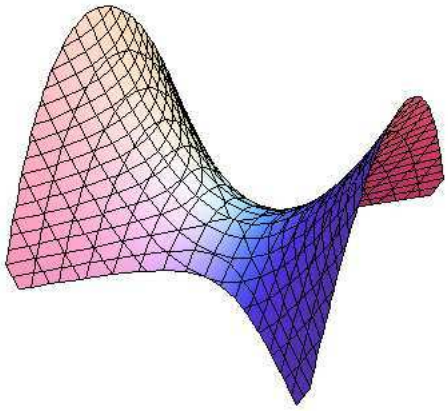
I



II



III



IV

Enter I,II,III,IV	$f(x, y, z) = 0$	Enter I,II,III,IV	parametrization
	$f(x, y, z) = -x^2 + y^2 + z$		$\langle u, v, u^2 - v^2 \rangle$
	$f(x, y, z) = x^2 + y^2 + z^2 - 1$		$\langle u, v, u^2 + v^2 \rangle$
	$f(x, y, z) = -x^2 - y^2 + z$		$\langle u, v, \sqrt{1 - u^2 - v^2} \rangle$
	$f(x, y, z) = -x^2 - y^2 + z^2$		$\langle s\cos(t), s\sin(t), s \rangle$

b) (2 points) We know that $\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0\rangle$, $\vec{r}(0) = \langle 2, 3, 4\rangle$ and $\vec{r}\,'(0) = \langle 0, 1, 1\rangle$. The expression $\langle \cos(t)+1, \sin(t)+3, t+4\rangle$ is equal to:

Check which applies	result
	the velocity $\vec{r}\,'(t)$
	the position $\vec{r}(t)$
	the curvature $\kappa(\vec{r}(t))$
	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of the partial differential equation $\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

Check which applies	PDE
	Transport equation
	Wave equation
	Heat equation
	Laplace equation

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2+y^2+(z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

Problem 5) (10 points)

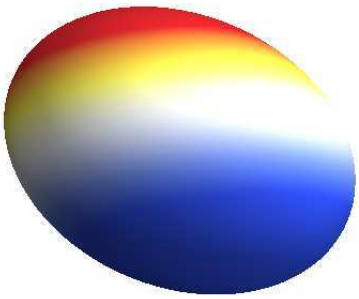
a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

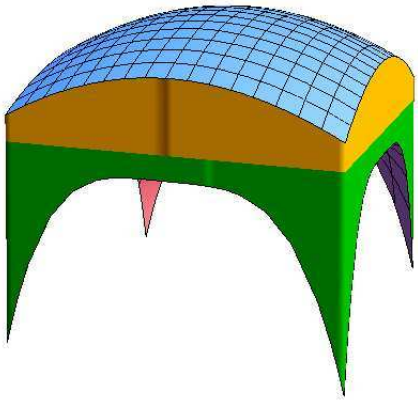
Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



Problem 7) (10 points)

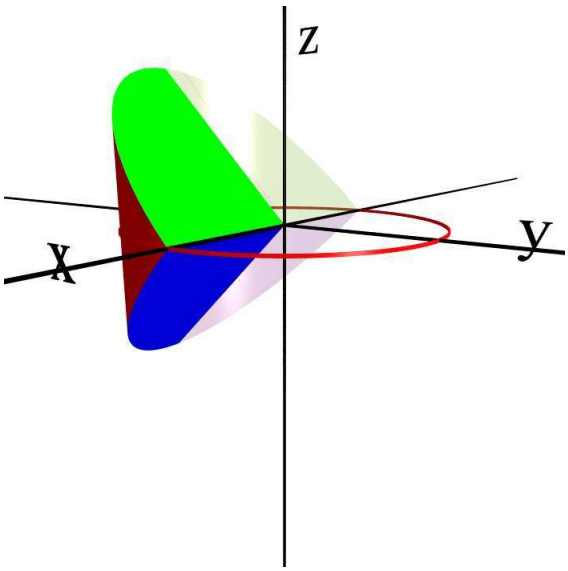
The thickness of the region enclosed by the two graphs $f_1(x,y) = 10 - 2x^2 - 2y^2$ and $f_2(x,y) = -x^4 - y^4 - 2$ is denoted by $f(x,y) = f_1(x,y) - f_2(x,y)$. Classify all critical points of f and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

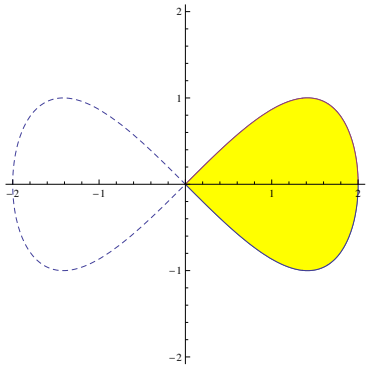
$$\vec{r}(u,v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle \ ,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Problem 10) (10 points)

Find the area $\int\int_R 1\, dxdy$ of the region R inside the right leaf of the **Gerono lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2\sin(t), 2\sin(t)\cos(t)\rangle \ .$$



Problem 11) (10 points)

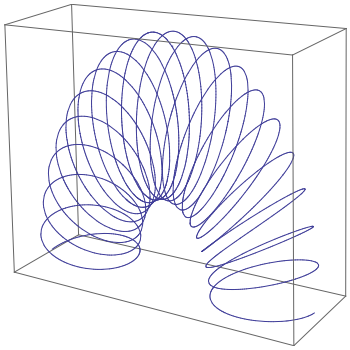
Find the line integral of the vector field

$\vec{F}(x,y,z) = \langle \cos(x+z), 2yze^{y^2z} + 7, \cos(x+z) + y^2e^{y^2z}\rangle$

along the **slinky** curve

$\vec{r}(t) = \langle \sin(40t), (2+\cos(40t))\cos(t), (2+\cos(40t))\sin(t)\rangle$

with $0 \le t \le \pi$.



Problem 12) (10 points)

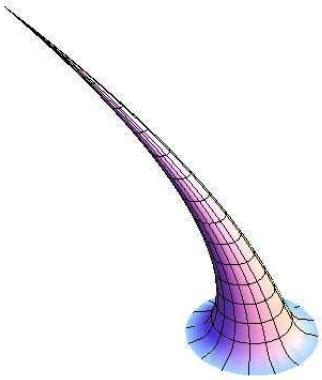
Find the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz\rangle$$

and S is the **thorn** surface parametrized by

$$\vec{r}(s,t) = \langle (1-s^{1/3})\cos(t) - 4s^2, (1-s^{1/3})\sin(t), 5s\rangle$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.

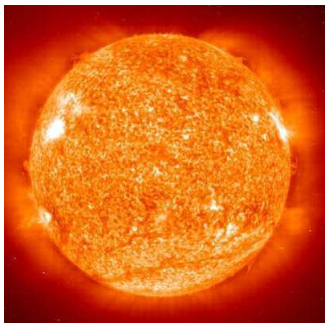


Problem 13) (10 points)

Assume the vector field

$\vec{F}(x,y,z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int\int_S \vec{F} \cdot d\vec{S}$.



Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u,v) = \langle \cos(u)\cos(\arctan(\sinh(v))), \sin(u)\cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle \ .$$

a) (3 points) Show that $|\vec{r}(u,v)| = 1$ verifying so that $\vec{r}(u,v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
b) (3 points) Show that $|\vec{r}_u(u,v)| = |\vec{r}_v(u,v)| = 1/\cosh(v)$ and that $\vec{r}_u(u,v) \cdot \vec{r}_v(u,v) = 0$.
c) (2 points) Use b) to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
d) (2 points) Use $\int 1/\cosh^2(x)\, dx = 2\arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π.

Hint for b): you can use the identity $\cos(\arctan(\sinh(v) = 1/\cosh(v)$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The function $f(x,y,z) = x^2 - y^2 - z^2$ increases in the direction $\langle -3,-1,2\rangle/\sqrt{14}$ at the point $(1,1,1)$.

2) [T] [F] The unit tangent vector of the curve $\vec{r}(t) = \langle 3t, 4t, t^2\rangle$ at time $t = 0$ is $\langle 3/5, 4/5, 0\rangle$.

3) [T] [F] There exist two nonzero vectors $\vec{a}$ and $\vec{b}$ such that the length of the vector projection of $\vec{a}$ to $\vec{a}\times\vec{b}$ is $\frac{1}{2}|\vec{b}|$.

4) [T] [F] The arc length of the curve $\vec{r}_1(t) = \langle e^{3t^3} - 1, t^6 + 2, \sin(2t^3)\rangle$, $0 \le t \le 1$ is larger than that of $\vec{r}_2(t) = \langle e^{3t} - 1, t^2 + 2, \sin(2t)\rangle$, $0 \le t \le 1$.

5) [T] [F] The tangent plane of the graph of $f(x,y) = \sin(x)+y^3$ at $(0,1,1)$ is $x+3y = 3$.

6) [T] [F] There exists a curve C on the level surface of $f(x,y,z) = x^3+e^{yz}+\cos(y) = 2$ such that the line integral $\int_C \nabla f \cdot d\vec{r} > 0$.

7) [T] [F] If Q is the point away from the plane $3x + 5y + z = 7$ and P is the point on the plane closest to Q, then $\vec{PQ}$ is parallel to $\langle 3,5,1\rangle$.

8) [T] [F] The vector field $\vec{F}(x,y,z) = \langle y^2 - xz + e^y, -yz, x^4 + y^2 - z^2\rangle$ is the curl of a vector field $\vec{G}$.

9) [T] [F] Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ and C be the unit circle oriented counterclockwise. Since $Q_x = P_y$ everywhere, Green implies $\int_C \vec{F} \cdot d\vec{r} = 0$.

10) [T] [F] By linear approximation of the function $f(x,y,z) = e^{x+y+z}$ we can estimate $f(0.1, 0.01, 0.001)$ as 1.111.

11) [T] [F] If $\vec{F}(x,y,z)$ is a vector field defined on $0 < x^2+y^2+z^2 < 4$ and $\text{curl}(\vec{F}) = 0$ everywhere on this solid, then $\vec{F} = \nabla f$ for some function f.

12) [T] [F] The tangent plane of the surface $x^2+y^4+z^6 = 6$ at $(2,1,1)$ is perpendicular to the line $\vec{r}(t) = \langle 1+2t, 3+2t, -4+3t\rangle$.

13) [T] [F] Given two curves $C_1 : \vec{r}_1(t) = \langle t, t^3\rangle, 0 \le t \le 1$ and $C_2 : \vec{r}_2(s) = \langle s, s^5\rangle, 0 \le s \le 1$ $f(x,y) = \sin(x^2y)$. Then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$.

14) [T] [F] If $f(x,y)$ has a global maximum, then the discriminant function $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ has a global maximum.

15) [T] [F] Let $\vec{F}(x,y,z) = \langle x,y,z\rangle$ and S the surface boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ oriented by outward normal vectors. Then $\int\int_S \vec{F} \cdot d\vec{S} = 0$.

16) [T] [F] Let $\vec{F}(x,y,z) = \langle x/3, y/3, z/3\rangle$ and S the unit sphere oriented by the outward normal vectors. Then $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ is the volume of the unit ball.

17) [T] [F] In three dimensional space there exist two nonzero vector fields $\vec{F}$ and $\vec{G}$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{G})$.

18) [T] [F] The vector field $\vec{F}(x,y,z) = \langle \cos(y), \cos(z), \cos(x)\rangle$ has the property that $\vec{F} = \text{curl}(\text{curl}(\vec{F}))$.

19) [T] [F] There exists a vector field $\vec{F}(x,y,z)$ defined on $\mathbf{R}^3$ such that every line integral $\int_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ over a closed curve C is equal to 0, but not every surface integral $\int\int_S \vec{F} \cdot dS$ over a closed surface S is equal to 0.

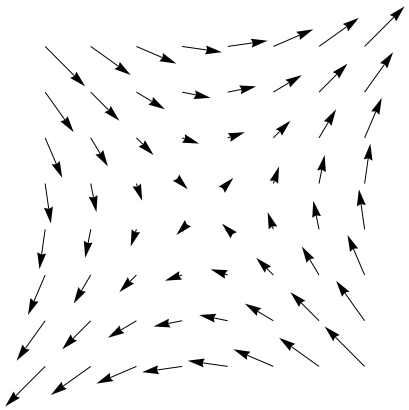
20) [T] [F] Whenever $\vec{F} = \nabla f$, for some function $f(x,y)$ defined on the annulus $\frac{1}{2} \le x^2 + y^2 \le 2$, then $\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the circle $x^2 + y^2 = 1$.

Problem 2) (10 points)

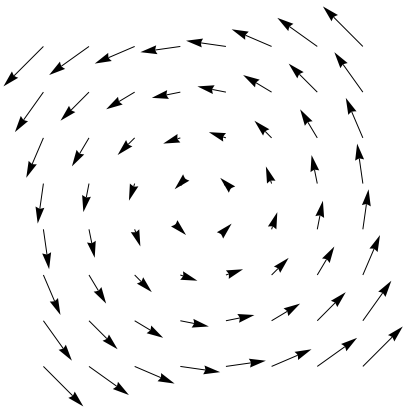
a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field $\vec{F}$ is **divergence free** if $\text{div}(\vec{F}) = 0$ everywhere in the plane. A field $\vec{F}$ is **irrotational**, if $\text{curl}(\vec{F}) = \vec{0}$ everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

field	enter I-IV	divergence free	irrotational
$\vec{F}(x,y) = \langle -y, x \rangle$			
$\vec{F}(x,y) = \langle y, x \rangle$			
$\vec{F}(x,y) = \langle -x-y, x-y \rangle$			
$\vec{F}(x,y) = \langle x+y, x+y \rangle$			

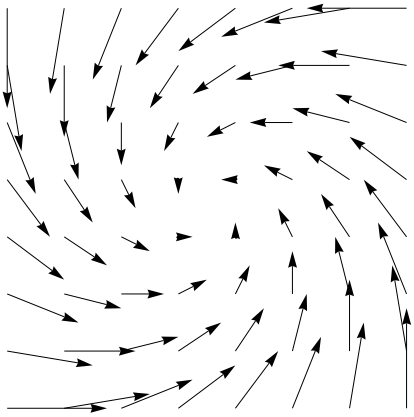
I



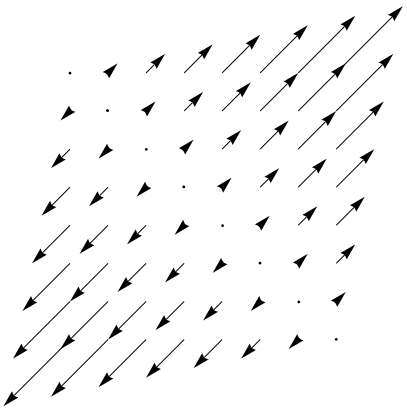
II



III



IV



b) (5 points) Match the following names of partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t,x) = t^2 + x^2$
B	$u(t,x) = t^2 - x^2$
C	$u(t,x) = \sin(x+t)$
D	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_x(t,x)$
2	$u_{tt}(t,x) = u_{xx}(t,x)$
3	$u_{tt}(t,x) = -u_{xx}(t,x)$
4	$u_t(t,x) = u_{xx}(t,x)$

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals $A - H$ in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\int\int_R x^2 - y^2 \; dxdy$	A
$\int\int_R 1 \; dxdy$	B
$\int\int\int_R 1 \; dxdydz$	C
$\int\int\int_R x^2 + z^2 \; dxdydz$	D

$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	F
$\int_a^b \lvert\vec{r}\,'(t)\rvert \; dt$	G
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \; dudv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x,y,z) = \langle -y, x, x \rangle$	$\langle 0, -1, 2 \rangle$	A
$\vec{F}(x,y,z) = \langle x^2, y, x \rangle$	$2x + 1$	B
$f(x,y,z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1 \rangle$	C
$f(x,y,z) = x^3 + 5y^2$	$10y$	D

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0, 1, -1), B = (4, 0, -1), C = (2, 1, 3), \text{ and } D = (2, 2, 0) .$$

a) (3 points) What is the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AD}$?

b) (3 points) Find the volume of the parallelepiped spanned by $\vec{AC}, \vec{AB}$ and $\vec{AD}$.

c) (4 points) Determine the distance between the two skew lines AB and CD.

Problem 5) (10 points)

a) (5 points) The curl of $\vec{F}(x, y) = \langle -e^{xy}, y \rangle$ is equal to a scalar function $f(x, y)$. Estimate $f(1.1, 0.001)$ by linear approximation.

b) (5 points) Using the same function as in a), the equation $f(x, y) = \text{curl}(\vec{F})(x, y) = 1$ defines y as a function $g(x)$ of x near $x = 1$. Find $g'(1)$.

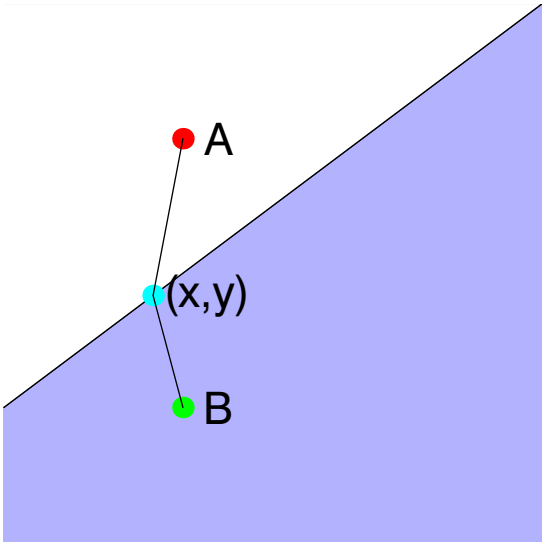
Problem 6) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from $A = (0, 2)$ to $B = (0, 0)$ along a straight line segment from A to (x, y) and swims through the lake $x - y \geq -1$ from (x, y) to a gold chest located at $B = (0, 0)$ again on a straight line segment. The effort from A to (x, y) is the square of the distance from A to (x, y). Her effort from (x, y) to B is 2 times the squared distance from (x, y) to B. Using the Lagrange method, find the choice of a drop point (x, y) on the lake shore that minimizes her effort.





Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}\,,$$

where C is the curve given by

$$\vec{r}(t) = \langle \frac{t\pi}{2}, 1-t, t^3 \rangle, 0 \le t \le 1$$

and

$$\vec{F}(x,y,z) = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz) \rangle\ .$$

Problem (9) (10 points)

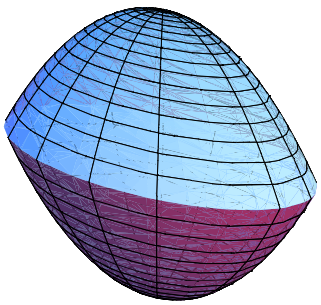
The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

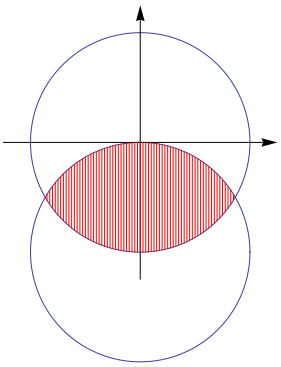
$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z+2)^2 = 4\ .$$

Find the surface area of the UFO.





Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\, dydxdz\ .$$

Problem 11) (10 points)

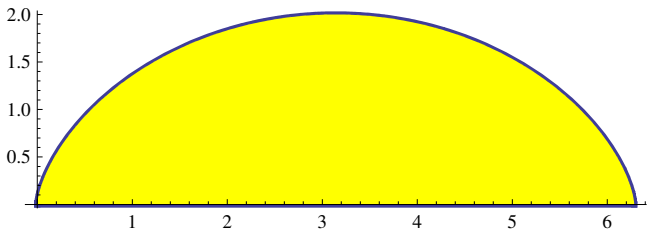
Let $\vec{F}(x,y,z) = \langle x+yz, xye^{-xz}, e^{-xz}\rangle$. Find

$$\int\int_S \vec{F}\cdot d\vec{S}\ ,$$

where S is the surface $z = 1-x^2-y^2, z \geq 0$ oriented so that the normal vector points upwards.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve $\vec{r}(t) = \langle t-\sin(t), 1-\cos(t)\rangle$ with $0 \leq t \leq 2\pi$ and the x-axes.



Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} \ ,$$

where $\vec{F}(x,y,z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 14) (10 points)

Let E be the rectangular solid $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$ and let S be the boundary of E. The surface S consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle \ ,$$

for which parameters a, b is the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S}$$

a global maximum?

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- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

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14		10
Total:		150

Problem 1) True/False questions (20 points)

1) T F The distance from $(1, 2, -1)$ to $(3, -2, 1)$ is $(-2, 4, -2)$.

2) T F The plane $y = 3$ is perpendicular to the xz plane.

3) T F All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.

4) T F The best linear approximation at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$

5) T F If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y)\, dy \right) dx = \int_0^1 \left(\int_y^1 f(x, y)\, dx \right) dy$.

6) T F Let $C = \{(x, y) \mid x^2 + y^2 = 1\}$ be the unit circle in the plane and $\vec{F}(x, y)$ a vector field satisfying $|\vec{F}| \leq 1$. Then $-2\pi \leq \int_C \vec{F} \cdot dr \leq 2\pi$.

7) T F Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ always point in different directions.

8) T F If all the second-order partial derivatives of $f(x, y)$ vanish at (x_0, y_0) then (x_0, y_0) is a critical point of f.

9) T F If $\vec{a}, \vec{b}$ are vectors, then $|\vec{a} \times \vec{b}|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

10) T F The distance between two points A, B in space is the length of the curve $\vec{r}(t) = A + t(B - A),\ t \in [0, 1]$.

11) T F The function $f(x, y) = xy$ has no critical point.

12) T F The length of a curve does not depend on the chosen parameterization.

13) T F There exists a non-zero function $f(x, y, z)$ and non-zero vector field $\vec{F}(x, y, z)$ so that $\vec{F} = \text{grad}(f)$ and $f = \text{div}\vec{F}$.

14) T F For any numbers a, b satisfying $|a| \neq |b|$, the vector $\langle a - b, a + b \rangle$ is perpendicular to $\langle a + b, b - a \rangle$.

15) T F The line integral of $\vec{F}(x, y) = \langle -y, x \rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R.

16) T F There is no surface for which both the parabola and the hyperbola appear as traces.

17) T F If $(u, v) \mapsto \vec{r}(u, v)$ is a parameterization for a surface, then $\vec{r}_u(u, v) + \vec{r}_v(u, v)$ is a vector which lies in the tangent plane to the surface.

18) T F When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi)\, d\rho d\phi d\theta$.

19) T F A surface in space for which all normal vectors are parallel to each other must be part of a plane.

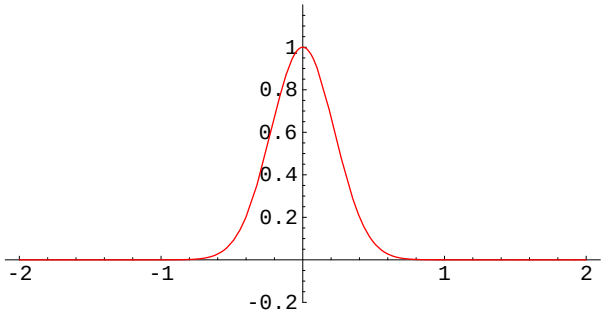
20) T F A vector field $\vec{F} = \langle P(x, y), Q(x, y) \rangle$ is conservative in the plane if and only if $P_y(x, y) = Q_x(x, y)$ for all points (x, y).

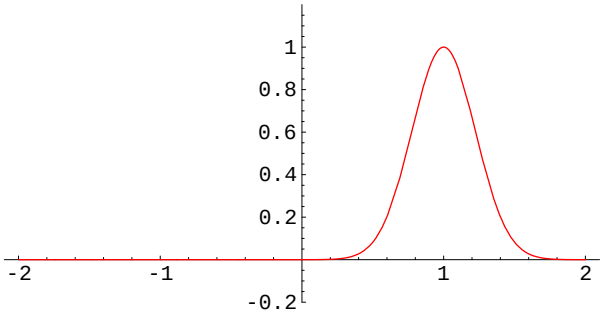
Problem 2) (10 points)

2 a) (5 points) Fill in names of the mathematicians: Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

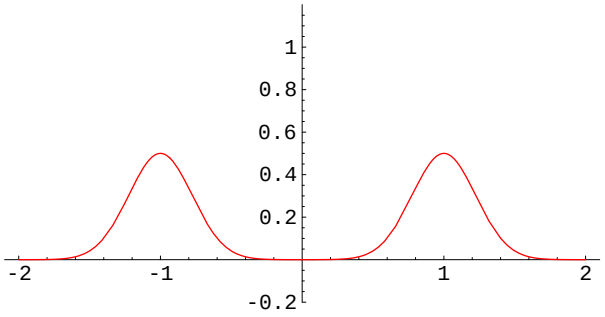
Formula	Name of the theorem
$\int_C \vec{F} \cdot \vec{dr} = \int\int_S \operatorname{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x,y) = f_{yx}(x,y)$	
$\int_C \vec{F} \cdot dr = \int\int_R \operatorname{curl}(\vec{F})\, dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int\int_S F \cdot dS = \int\int\int_E \operatorname{div}(F)\, dV$	
$\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$	

2 b) (5 points) We have a function $u(t,x)$ which is a solution to a partial differential equation. In all cases, we have $u(0,x) = e^{-x^2}$. The picture to the right shows this function $u(0,x)$. Which partial differential equation is involved, when you see the function $u(1,x)$ as a graph?

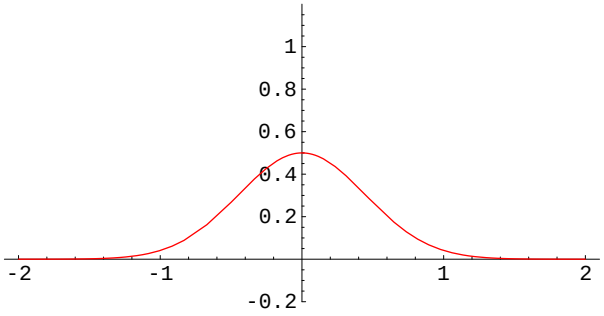




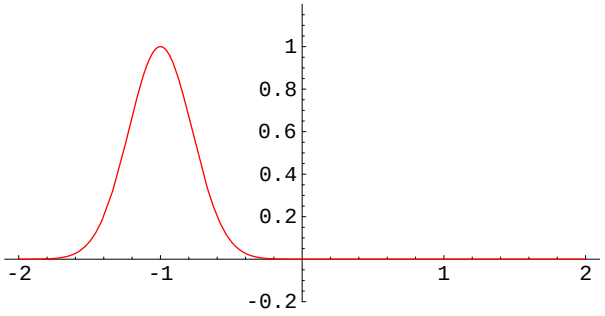
I



II



III



IV

Enter I,II,III,IV here	Equation
	$u_t(x,t) = u_x(x,t)$
	$u_t(x,t) = u_{xx}(x,t)$
	$u_{tt}(x,t) = u_{xx}(x,t)$
	$u_t(x,t) = -u_x(x,t)$

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $O = (0, 0, 0)$ to Σ.

c) Find the distance from the point P to the line through Q, R.

d) Find the volume of the parallelepiped with vertices O, P, Q, R.

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y.

a) Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$.

b) Estimate $g(1.01, 0.99)$ using linear approximation.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\vec{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u, v) in the disc $D = \{u^2 + v^2 \leq 1\}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x, y)$ which is the curl of $\vec{F}(x, y) = \langle -y^4/12 + y^3/6 - y, x^4/12 - x^3/6 \rangle$ on the disc $\{x^2 + y^2 \leq 4\}$.

a) Classify every critical point inside the disc $x^2 + y^2 < 4$.

b) Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of D.

Problem 7) (10 points)

a) Given two nonzero vectors $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ in space, write down a formula for the cosine of the angle between them. Find a nonzero vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 3, 2, 1 \rangle$. Describe geometrically the set of all $\vec{v}$, including zero, that are perpendicular to this vector $\vec{u}$.

b) Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P.

c) Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P.

d) Assume the level set of f is the graph of a function $g(x, y)$. Explain the relation between the gradient of g and the gradient of f. Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g?

Problem 8) (10 points)

Let R be the region inside the circle $x^2 + y^2 = 4$ and above the line $y = \sqrt{3}$. Evaluate

$$\int \int_R \frac{y}{x^2 + y^2} \, dA \, .$$

Problem 9) (10 points)

A region W in $\mathbf{R}^3$ is given by the relations

$$x^2 + y^2 \leq z^2 \leq 3(x^2 + y^2)$$
$$1 \leq x^2 + y^2 + z^2 \leq 4$$
$$x \geq 0$$

1. Sketch the region W.
2. Find the volume of the region W.

Problem 10) (10 points)

Consider the vector field

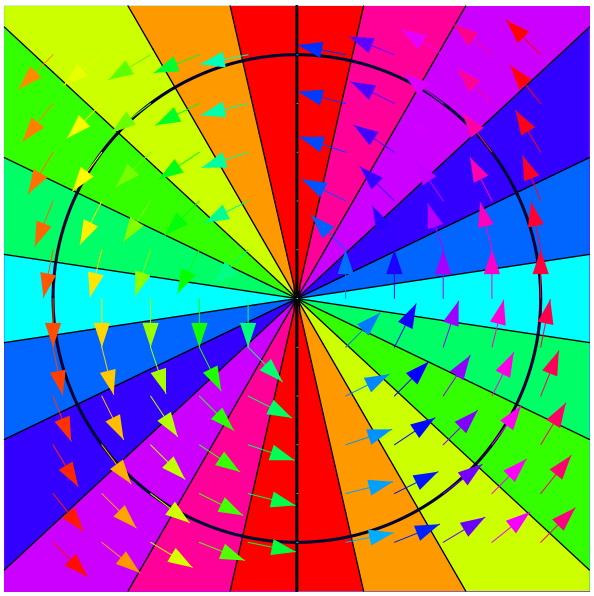
$$\vec{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$$

defined everywhere in the plane $\mathbf{R}^2$ except at the origin.

a) Let C be any closed curve which bounds a region D. Assume that $(0,0)$ is not contained in D and does not lie on C. Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0 \ .$$

b) Let C be the unit circle oriented counterclockwise. What is $\int_C \vec{F} \cdot d\vec{r}$? Explain why your answer shows that there is no function f for which $\vec{F}(x,y) = \nabla f(x,y)$ everywhere.



Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function $f(x,y,z) = xyz$ over the solid in space described by the inequalities $0 \leq z \leq \sqrt{(1-x^2-y^2)}, x^2+y^2 \leq 1, x-y \geq 0, y \geq 0$.

Problem 12) (10 points)

Let $\vec{F}(x,y)$ be a vector field in the plane given by the formula

$$\vec{F}(x,y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4+1}} \rangle \ .$$

If C is the path which goes from from $(-1,0)$ to $(1,0)$ along the semi circle $x^2+y^2=1$, $y \geq 0$, evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Problem 13) (10 points)

In appropriate units, the charge density $\sigma(x,y,z)$ in a region in space is given by $\sigma = \nabla \cdot \vec{E} = \mathrm{div}(\vec{E})$, where $\vec{E}$ is the electric field. Consider the cube of side lengths 1 given by $0 \leq x,y,z \leq 1$. What is the total charge in this cube if

$$\vec{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3+y^3+z^3), z(1-z)e^{\sqrt{x+y}} \rangle \ .$$

(The total charge is the integral of the charge density over the cube.)

Problem 14) (10 points)

a) By calculating the integral $\int \int_S \vec{F} \cdot \vec{dS}$ directly, find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through the sphere $x^2+y^2+z^2=9$, where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through the sphere $x^2+y^2+z^2 = 9$ using the divergence theorem.

c) Explain in words without invoking any integral theorem, why the flux integral of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through any sphere with positive radius centered at $(0,0,0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

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10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) [T] [F] For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v})$ is parallel to $\vec{w}$.

2) [T] [F] The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

3) [T] [F] If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

4) [T] [F] The curve $\vec{r}(t) = (1-t)A + tB, t \in [0,1]$ connects the point A with the point B.

5) [T] [F] For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

6) [T] [F] The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1+\cos^2(t)}\, dt$.

7) [T] [F] Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

8) [T] [F] The function $f(x,y,z) = x^2 - y^2 - z^2$ decreases in the direction $(2,-2,-2)/\sqrt{8}$ at the point $(1,1,1)$.

9) [T] [F] Assume $\vec{F}$ is a vector field satisfying $|\vec{F}(x,y,z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0,1]$, the line integral $\int_C \vec{F} \cdot \vec{dr}$ is less or equal than the arc length of C.

10) [T] [F] Let $\vec{F}$ be a vector field which coincides with the unit normal vector $\vec{N}$ for each point on a curve C. Then $\int_C \vec{F} \cdot \vec{dr} = 0$.

11) [T] [F] If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a,b,c)$, where a, b, c are constants.

12) [T] [F] If a nonempty quadric surface $g(x,y,z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.

13) [T] [F] If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z), then $\text{curl}(\vec{F}) = (0,0,0)$ for all (x,y,z).

14) [T] [F] If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

15) [T] [F] The divergence of the gradient of any $f(x,y,z)$ is always zero.

16) [T] [F] For every vector field $\vec{F}$ the identity $\text{grad}(\text{div}(\vec{F})) = \vec{0}$ holds.

17) [T] [F] For every function f, one has $\text{div}(\text{curl}(\text{grad}(f))) = 0$.

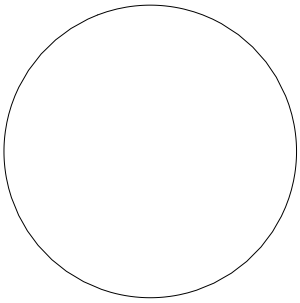
18) [T] [F] If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

19) [T] [F] The flux of the vector field $\vec{F}(x,y,z) = (y+z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

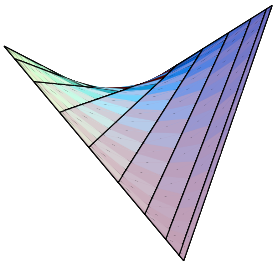
20) [T] [F] For every function $f(x,y,z)$, there exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = f$.

Problem 2) (10 points)

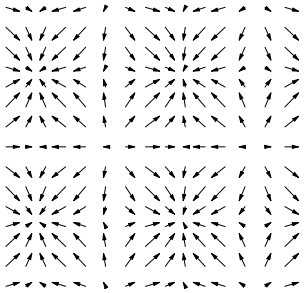
Problem 2a) (5 points) Match the equations with the objects. No justifications are needed.



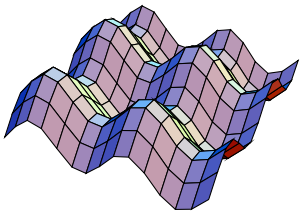
I



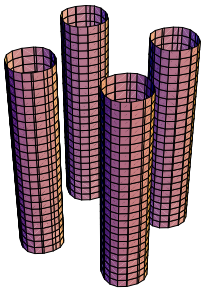
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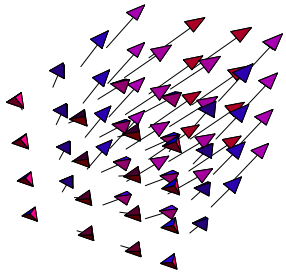
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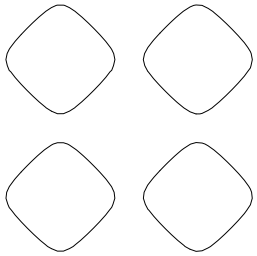
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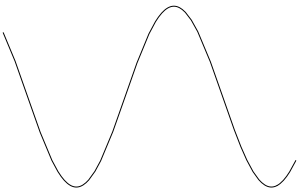
V



VI



VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x,y,z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$
	$\vec{r}(u,v) = \langle \cos(u), \sin(v), \cos(u)\sin(v) \rangle$
	$\vec{F}(x,y,z) = \langle \cos(x), \sin(x), 1 \rangle$
	$z = f(x,y) = \cos(x) + \sin(y)$
	$g(x,y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x,y) = \langle \cos(x), \sin(y) \rangle$

Problem 2b) (5 points) Mark with a cross in the column below "irrotational" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x,y,z) = (0,0,0)$ for all points (x,y,z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z)). No justifications are needed.

Vectorfield	irrotational $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x,y,z) = \langle -5, 5, 3\rangle$		
$\vec{F}(x,y,z) = \langle x, y, z\rangle$		
$\vec{F}(x,y,z) = \langle -y, x, z\rangle$		
$\vec{F}(x,y,z) = \langle x^2+y^2, xyz, x-y+z\rangle$		
$\vec{F}(x,y,z) = \langle x-2yz, y-2zx, z-2xy\rangle$		

Problem 3) (10 points)

a) (2 points) What is the area of the triangle A, B, P, where $A = (1,1,1), B = (1,2,3)$ and $P = (3,2,4)$?

b) (2 points) Find the distance between the point the point P and the line L passing through the points A with B.

Let E be a general parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

c) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

d) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

e) (3 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Problem 4) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x,y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2} \; .$$

There is a lake in that area as you can see in the photo.

a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \; \vec{dr}$.



Photo of the lake in the Swiss alps near the Simplon mountain pass.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\, dxdy.$$

Problem 7) (10 points)

In this problem we evaluate $\int\int_D \frac{(x-y)^4}{(x+y)^4}\, dxdy$, where D is the triangular region bounded by the x and y axis and the line $x + y = 1$.

a) (3 points) Find the region R in the uv-plane which is transformed into D by the change of variables $u = x - y, v = x + y$. (It is enough to draw a carefully labeled picture of R.)

b) (3 points) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $(x, y) = (\frac{u+v}{2}, \frac{v-u}{2})$.

c) (4 points) Evaluate $\int\int_D \frac{(x-y)^4}{(x+y)^4}\, dxdy$ using the above defined change of variables.

Hint. The general topic of change of variables does not appear this semester. You can solve the problem nevertheless, when given the formula $\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ for the integration factor (analogous to r when changing to polar coordinates, or $\rho^2 \sin(\phi)$ when going to spherical coordinates). The integral in c) becomes then $\int\int_R u^4/v^4 \, dudv$. The region R is the triangle bounded by the edges $(0,0), (1,1), (-1,1)$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x,y) = -(x^4 - 8x^2 + y^2 + 1)$.
b) (3 points) Classify the critical points.
c) (2 points) Locate the local and absolute maxima of f.
d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = \langle t, \sin t, t^2 \cos t \rangle$ for $0 \leq t \leq \pi$. Let $f(x,y,z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Problem 11) (10 points)

A cylindrical building $x^2 + (y-1)^2 = 1$ is intersected with the paraboloid $z = 4 - x^2 - y^2$.

a) Parametrize the intersection curve and set up an integral for its arc length.

b) Find a parametrization of the surface obtained by intersecting the paraboloid with the solid cylinder $x^2 + (y-1)^2 \leq 1$ and set up an integral for its surface area.

Problem 12) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x,y) = \langle y^2, x^2 \rangle$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0,0)$, $(1,0)$ and $(1,1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Green's theorem.

Problem 13) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x,y,z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle$ with $t \in [0, 2\pi]$.

Problem 14) (10 points)

Let S be the graph of the function $f(x,y) = 2 - x^2 - y^2$ which lies above the disk $\{(x,y) \mid x^2 + y^2 \leq 1 \}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int \int_S \vec{F} \cdot d\vec{S}$ of the vector field

$$\vec{F} = \langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \rangle$$

through S using the divergence theorem.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) T F The projection vector $\text{proj}_{\vec{v}}(\vec{w})$ is parallel to $\vec{w}$.

2) T F Any parametrized surface S is a graph of a function $f(x, y)$.

3) T F If the directional derivatives $D_{\vec{v}}(f)(1,1)$ and $D_{\vec{w}}(f)(1,1)$ are both 0 for $\vec{v} = \langle 1, 1\rangle/\sqrt{2}$ and $\vec{w} = \langle 1, -1\rangle/\sqrt{2}$, then $(1, 1)$ is a critical point.

4) T F The linearization $L(x, y)$ of $f(x, y) = x + y + 4$ at $(0, 0)$ satisfies $L(x, y) = f(x, y)$.

5) T F For any function $f(x, y)$ of two variables, the line integral of the vector field $\vec{F} = \nabla f$ on a level curve $\{f = c\}$ is always zero.

6) T F If $\vec{F}$ is a vector field of unit vectors defined in $1/2 \leq x^2 + y^2 \leq 2$ and $\vec{F}$ is tangent to the unit circle C, then $\int_C \vec{F} \cdot d\vec{r}$ is either equal to 2π or -2π.

7) T F If a curve C intersects a surface S at a right angle, then at the point of intersection, the tangent vector to the curve is parallel to the normal vector of the surface.

8) T F The curvature of the curve $\vec{r}(t) = \langle \cos(3t), \sin(6t)\rangle$ at the point $\vec{r}(0)$ is smaller than the curvature of the curve $\vec{r}(t) = \langle \cos(30t), \sin(60t)\rangle$ at the point $\vec{r}(0)$.

9) T F At every point (x, y, z) on the hyperboloid $x^2 + 2y^2 - z^2 = 1$, the vector $\langle x, 2y, -z\rangle$ is normal to the hyperboloid.

10) T F The set $\{\phi = \pi/2, \theta = \pi\}$ in spherical coordinates is the negative x axis.

11) T F The integral $\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi)\ d\phi\ d\theta\ d\rho$ is equal to the volume of the unit ball.

12) T F Four points A, B, C, D are located in a single common plane if $(B - A) \cdot ((C - A) \times (D - A)) = 0$.

13) T F If a function $f(x, y)$ has a local maximum at $(0, 0)$, then the discriminant D is negative.

14) T F The integral $\int_0^x \int_y^1 f(x, y)\ dx\ dy$ represents a double integral over a bounded region in the plane.

15) T F The following identity is true: $\int_0^3 \int_0^{2x} x^2\ dydx = \int_0^6 \int_{y/2}^3 x^2\ dx\ dy$

16) T F The integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ over the surface S of a cube is zero for all vector fields $\vec{F}$.

17) T F A vector field $\vec{F}$ defined on three space which is incompressible ($\text{div}(\vec{F}) = 0$) and irrotational ($\text{curl}(\vec{F}) = 0$) can be written as $\vec{F} = \nabla f$ with $\Delta f = \nabla^2 f = 0$.

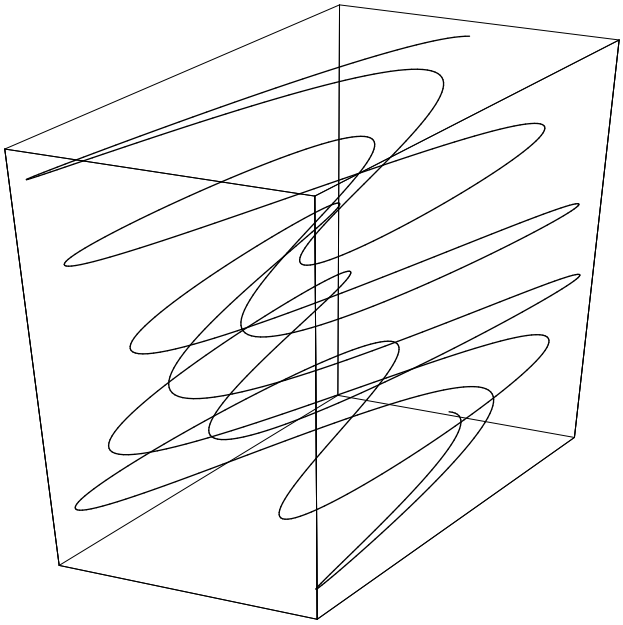
18) T F If a vector field $\vec{F}$ is defined at all points of three-space except the origin, and $\text{curl}(\vec{F}) = \vec{0}$ everywhere, then the line integral of $\vec{F}$ around the circle $x^2 + y^2 = 1$ in the xy-plane is equal to zero.

19) T F The identity $\text{curl}(\text{grad}(\text{div}(\vec{F}))) = \vec{0}$ is true for all vector fields $\vec{F}(x, y, z)$.

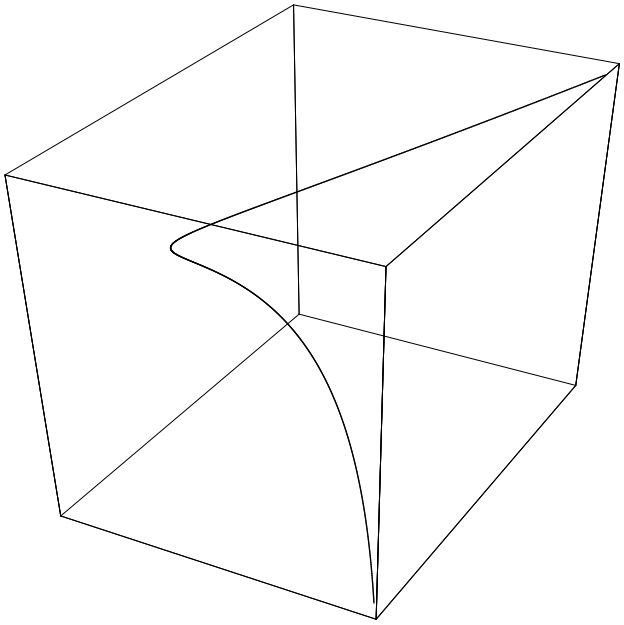
20) T F If $\vec{F} = \text{curl}(\vec{G})$, where $\vec{G} = \langle e^{e^x}, 5^x z^5, \sin y\rangle$, then $\text{div}(\vec{F}(x, y, z)) > 0$ for all (x, y, z).

Problem 2) (10 points)

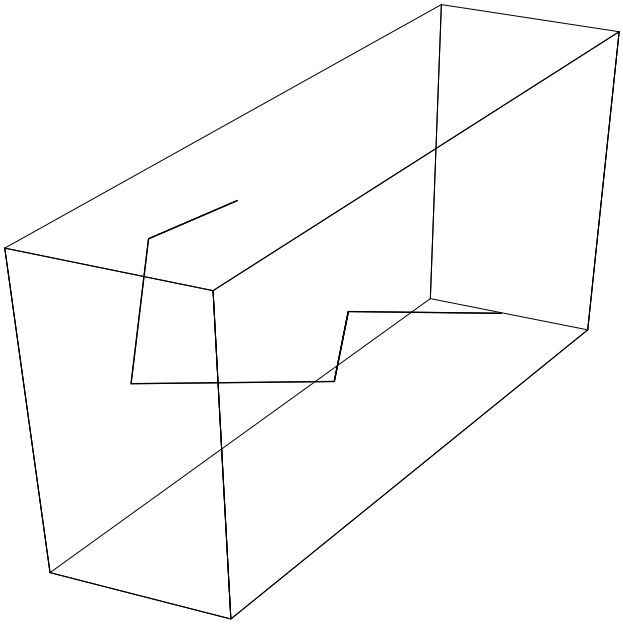
Match the equations with the space curves. No justifications are needed.



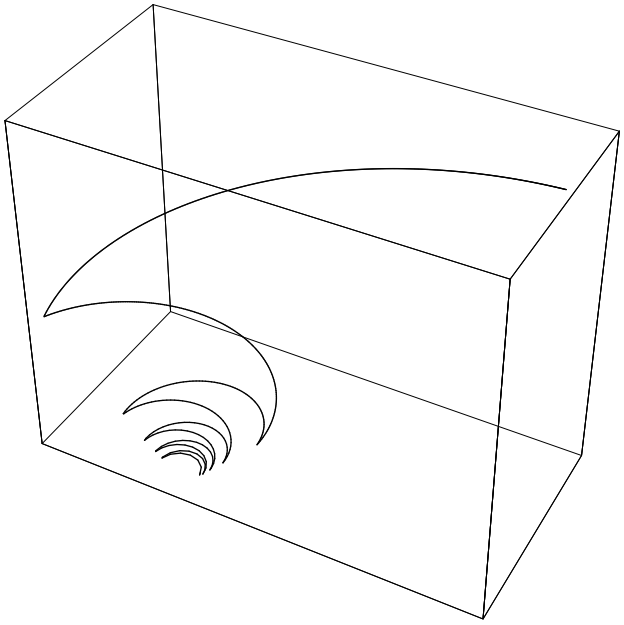
I



II



III

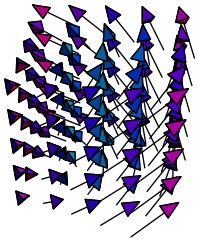


IV

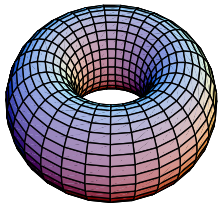
Enter I,II,III,IV here	Equation
	$\vec{r}(t) = \langle t^2, t^3 - t, t \rangle$
	$\vec{r}(t) = \langle \lvert 1 - \lvert t \rvert \rvert, \lvert t - \lvert t - 1 \rvert \rvert, t \rangle$
	$\vec{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$
	$\vec{r}(t) = \langle t\sin(1/t), t\lvert\cos(1/t)\rvert, t \rangle$

Problem 3) (10 points)

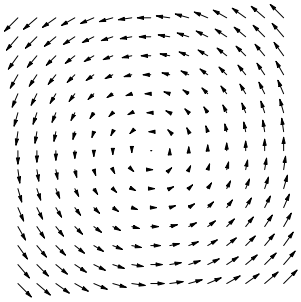
Match the equations with the objects. No justifications are needed.



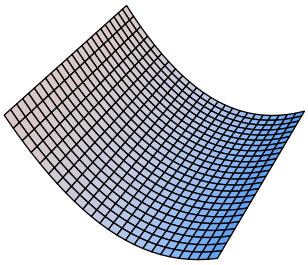
I



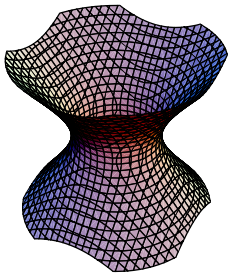
II



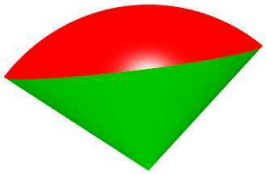
III



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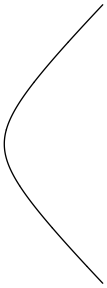
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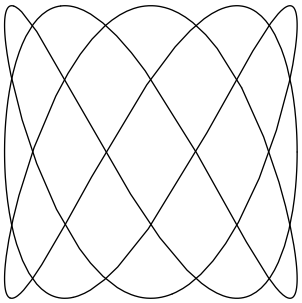


VI



VII





VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$
	$x^2 + y^2 - z^2 = 1$
	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
	$x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0$
	$z = f(x,y) = x^2 - y$
	$g(x,y) = x^2 - y^2 = 1$
	$\vec{F}(x,y) = \langle -y, x \rangle$

Problem 4) (10 points)

a) Find an equation for the plane Σ passing through the points $\vec{r}(0), \vec{r}(1), \vec{r}(2)$, where

$\vec{r}(t) = \langle t^2, t^4, t \rangle$.

b) Find the distance between the point $\vec{r}(-1)$ and the plane Σ found in a).

Problem 5) (10 points)

A vector field $\vec{F}(x, y)$ in the plane is given by $\vec{F}(x, y) = \langle x^2+5, y^2-1 \rangle$. Find all the critical points of $|\vec{F}(x, y)|$ and classify them. At which point or points is $|\vec{F}(x, y)|$ minimal?

Problem 6) (10 points)

A house is situated at the point $(0, 0)$ in the middle of a mountainous region. The altitude at each point (x, y) is given by the equation $f(x, y) = 4x^2y + y^3$. There is a pathway in the shape of an ellipse around the house, on which the (x, y) coordinates satisfy $2x^2 + y^2 = 6$. Find the highest and lowest points in the closed region bounded by the path.

Problem 7) (10 points)

We are given a function $f(x, y)$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$ as well as the following data points. Evaluate $\frac{\partial^2 f}{\partial \theta^2}$ at the point $r = 2,\ \theta = \frac{\pi}{2}$.

(x, y)	$(0, 2)$	$(2, 0)$	$(\pi, 2)$	$(2, \pi)$	$(0, 0)$
$f(x, y)$	2004	2005	2002	2003	2006
$f_x(x, y)$	3	4	6	2	0
$f_y(x, y)$	2	3	4	5	0
$f_{xx}(x, y)$	6	5	4	0	0
$f_{xy}(x, y)$	0	1	0	2	0
$f_{yy}(x, y)$	2	0	2	2	0

Problem 8) (10 points)

a) (4 points) Where does the tangent plane at $(1, 1, 1)$ to the surface $z = e^{x-y}$ intersect the z axis?

b) (4 points) Estimate $f(x,y,z) = 1 + \log(1 + x + 2y + z) + 2\sqrt{1+z}$ at the point $(0.02, -0.001, 0.01)$.

c) (2 points) $f(x,y,z) = 0$ defines z as a function $g(x,y)$ of x and y. Find the partial derivative $g_x(x,y)$ at the point $(x,y) = (0,0)$.

Problem 9) (10 points)

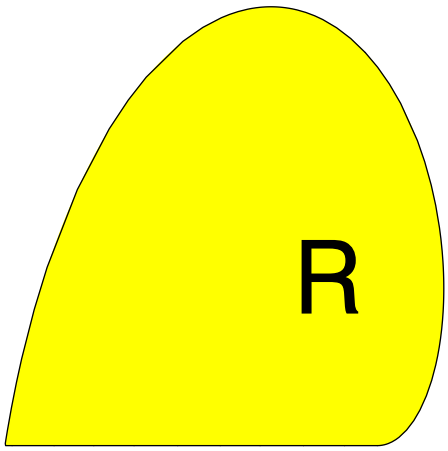
For each of the following quantities, set up a double or triple integral using any coordinate system you like. You do not have to evaluate the integrals, but the bounds of each single integral must be specified explicitly.

1. (3 points) The volume of the tetrahedron with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$.
2. (4 points) The surface area of the piece of the paraboloid $z = x^2 + y^2$ lying in the region $z = x^2 + y^2$, where $0 \le z \le 1$.
3. (3 points) The volume of the solid bounded by the planes $z = -1$, $z = 1$ and the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Problem 10) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \le \theta$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi - \sqrt{x^2+y^2})}\, dx\, dy\ .$$



Problem 11) (10 points)

A car drives up a freeway ramp C which is parametrized by

$$\vec{r}(t) = \langle \cos(t), 2\sin(t), t \rangle, \qquad 0 \le t \le 3\pi\ .$$

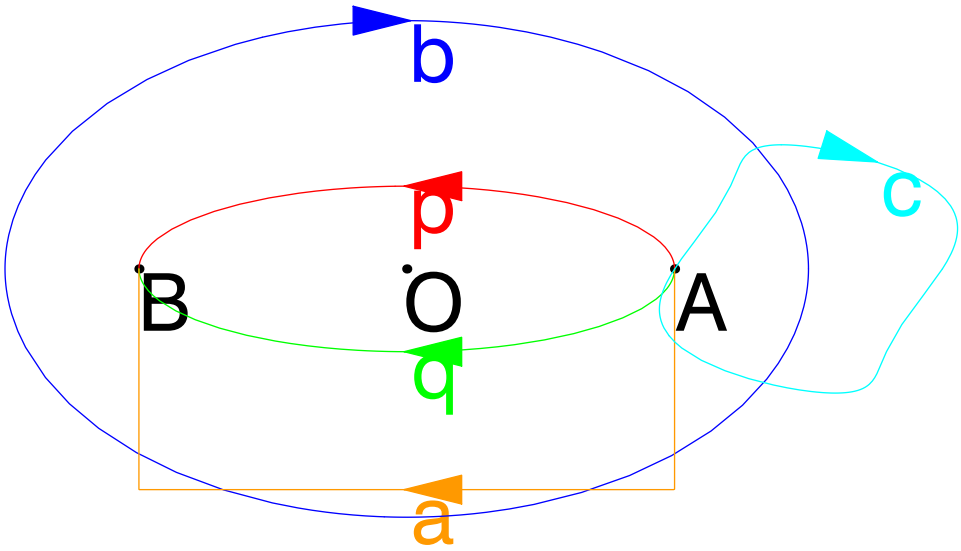
a) (3 points) Set up an integral which gives the length of the ramp. You do need not need to evaluate it.

b) (3 points) Find the unit tangent vector $\vec{T}$ to the curve at the point where $t = 0$.

c) (4 points) Suppose the wind pattern in the area is such that the wind exerts a force $\vec{F} = \langle 4x^2, y, 0 \rangle$ on the car at the position (x, y, z). What is the total work gain as the drives up the ramp? In other words, what is the line integral $\int_C \vec{F}\, d\vec{r}$.

Problem 12) (10 points)

Suppose $\vec{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. Suppose the line integral of $\vec{F}$ along the path p from A to B is 5 and the line integral of $\vec{F}$ along the path q from A to B is -4. Find the line integral of $\vec{F}$ along the following three paths:



a) (3 points) The path a from A to B going clockwise below the origin.

b) (4 points) The closed path b encircling the origin in a clockwise direction.

c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Problem 13) (10 points)

Let S be the surface which bounds the region enclosed by the paraboloid $z = x^2 + y^2 - 1$ and the xy plane. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle x + e^{\sin(z)}, z, -y \rangle$.

a) (5 points) Find the flux of $\vec{F}$ through the surface S.

b) (5 points) Find the flux of $\vec{F}$ through the part of the surface S that belongs to the paraboloid, oriented so that the normal vector points downwards.

Problem 14) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle 4z + \cos(\cos x), y^2, x + 2y \rangle$.

a) (4 points) Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$, for $0 \leq t \leq 2\pi$. Find the line integral of $\vec{F}$ along C.

b) (6 points) Let S be the hemisphere of the unit sphere defined by $y \leq 0$. Find the flux of the curl of $\vec{F}$ out of S. In other words, find

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \, .$$

For part b), the surface S is oriented so that the normal vector has a positive y-component.

9/27/2017 FIRST HOURLY **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 and 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The velocity vector of $\vec{r}(t) = \langle t, t, t \rangle$ at time $t = 2$ is twice the velocity vector at time $t = 1$.

Solution:
The velocity vector is the same at every time t.

2) [T] [F] The curvature of the curve $\vec{r}(t) = \langle 2t, 2t, 2t \rangle$ is always $1/2$.

Solution:
The curvature is zero because the curve is a line.

3) [T] [F] If $\vec{u} \times \vec{v} = \vec{0}$, then $\text{Proj}_{\vec{u}}(\vec{v}) \times \text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

Solution:
The projection vectors are parallel to the vectors onto which one has projected. Since the original vectors were parallel also the projections are parallel.

4) [T] [F] If $\vec{u} \times \vec{v} = \vec{v} \times \vec{w}$, then $\vec{v} \cdot (\vec{u} \times \vec{w}) = 0$.

Solution:
The assumption assumes that the three vectors u, v, w are in the same plane. The normal vector is also $u \times w$. But this is perpendicular to v.

5) [T] [F] There is a point not at the origin with Cartesian coordinates $(x, y, z) = (a, b, c)$ and spherical coordinates $(\rho, \theta, \phi) = (a, b, c)$.

Solution:
The assumption $a = \rho$ implies that only the first coordinate can be nonzero. If $z = 0$, then we have by assumption $\phi = 0$, but $\phi = 0$ and $z = 0$ implies $\rho = 0$ which is not possible.

6) [T] [F] The two planes $2x + 2y - z = 4$ and $-4x - 4y + 2z = 3$ intersect in a line.

Solution:
They are parallel and are not the same. So, they do not intersect at all.

7) [T] [F] If the distance between two points P and Q is zero, then $P = Q$.

Solution:
If P and Q were different, then the points would be different.

8) [T] [F] If the distance between two lines L and M is zero, then $L = M$.

Solution:
Lines can intersect without being equal. Their distance is then zero.

9) [T] [F] The arc length of a circle with constant curvature κ is $2\pi\kappa$.

Solution:
It is $2\pi/r$.

10) [T] [F] The surface $x^2 + y^2 + 4y = -z^2$ is a two-sheeted hyperboloid.

Solution:
Complete the square to get $x^2 - (y-2)^2 - z^2 = -4$

11) [T] [F] There are two vectors $\vec{v}, \vec{w}$ in $\mathbf{R}^3$ of length 1 for which the dot product is 2.

Solution:
Cauchy-Schwarz forbids it

12) [T] [F] If the acceleration of a curve $\vec{r}(t)$ is zero at all times and the velocity is non-zero at time $t = 0$, then the curve is a line.

13) [T] [F] The lines $\vec{r}(t) = \langle 3t, 4t, 5t\rangle$ and $\vec{s}(t) = \langle -4t, 3t, 0\rangle$ intersect perpendicularly.

Solution:
The velocity vectors are perpendicular at the intersection point $(0, 0, 0)$.

14) [T] [F] The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axis.

Solution:
It is the south pole.

15) [T] [F] Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|\vec{u} \cdot \vec{v}||\vec{w}| = |\vec{u}||\vec{v} \cdot \vec{w}|$.

Solution:
Take $\vec{u} = \langle 1, 0, 0\rangle, \vec{v} = \vec{w} = \langle 0, 1, 0\rangle$.

16) [T] [F] The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a cylinder.

Solution:
Multiply both by ρ to get $z = \rho^2$ which is a sphere.

17) [T] [F] The arc length of the curve $\langle \sin(t), \cos(t), t\rangle$ from $t = 0$ to $t = 2\pi$ is larger than 2π.

Solution:
It is a helix whose projection is a circle.

18) [T] [F] The surface parametrized by $\vec{r}(u, v) = \langle v\sin(u), v\cos(u), 0\rangle$ with $0 \leq u < 2\pi$, $v \geq 0$ is a plane.

Solution:
Indeed it is the plane $z = 0$.

19) | T | | F | It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

Solution:
One is bounded, the other not.

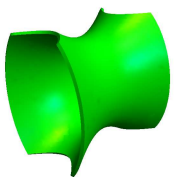
20) | T | | F | For any two points P, Q and vectors $\vec{v}, \vec{w}$, the mid point $M = (P+Q)/2$ has the same distance to the two lines $\vec{r}_1(t) = P + t\vec{v}$ and $\vec{r}_2(t) = Q + t\vec{w}$.

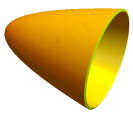
Solution:
This is not obvious and even the author of this problem (Oliver) got it first wrong. Yes, it is true that the mid point is on the plane equidistant to the two lines but the distance to the two lines can be different. Its best look at an extreme case. Let the first line be the z axes and the second line the line $\langle t, 2, 0\rangle$. Now if $P = (0, 0, 2)$ and $Q = (1000, 2, 0)$, then the midpoint is $(500, 1, 1)$

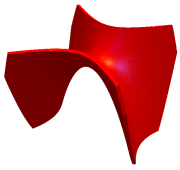
Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the surfaces with their equations $g(x, y, z) = 1$. Enter O, if there is no match.

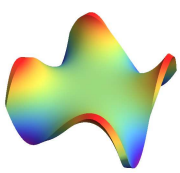

I

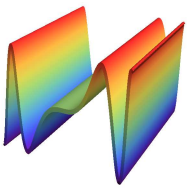

II

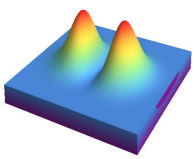

III

Function $g(x, y, z) =$	Enter O,I,II or III
$2x - y^2 - z^2$	
$2x^2 - y^2 + z^2$	
$2x - y$	
$2x^2 - y^2 - z$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

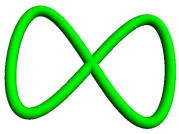

I


II


III

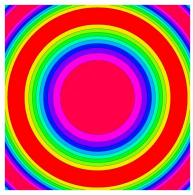
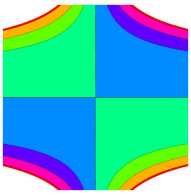
Function $f(x, y) =$	Enter O,I,II or III
$xy(x^2 - y^2)$	
$\sin(x^3)$	
$\sin(y^4)$	
$x^2 \exp(-x^2 - y^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

I  II  III 

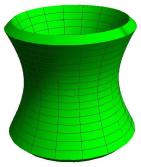
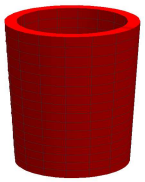
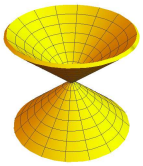
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, \sin(4t), \cos(4t)\rangle$	
$\langle \cos(t), \cos(t), \sin(2t)\rangle$	
$\langle 3t, 1-t, 5t\rangle$	
$\langle t\sin(t), t\cos(t), t\rangle$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Function $g(x,y) =$	Enter O, I,II or III
$\sin(x^2+y^2)$	
$\sin(x)-y$	
$\lvert x\rvert + y$	
xy^2	

e) (2 points) Match the quadrics. Enter O if there is no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2+y^2-z^2=1$	
$x^2+y^2+z^2=1$	
$x^2+y^2=1$	
$x^2+y^2=z^2$	

Solution:
a) 2,1,0,3
b) 1,2,0,3
c) 3,2,0,1
d) 2,0,1,3
e) 1,0,2,3

Problem 3) (10 points)	(Only answers are needed)

a) (3 points) Write the equations of a surface in Cartesian, Cylindrical and Spherical coordinates. The first row gives an example:

Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
$z = 1$	$z = 1$	$\rho\cos(\phi) = 1$
		$\rho\sin(\phi) = 1$
	$r\cos(\theta) = 1$	
$x^2 + y^2 + z^2 = 1$		

b) (3 points) Assume $\vec{u}, \vec{v}$ are unit vectors which are perpendicular. Check one box in each row:

The value	is larger than 0	is smaller than 0	is equal to 0
$\vec{u} \cdot \vec{v}$			
$\lvert\vec{u} \times \vec{v}\rvert$			
$\vec{u} \cdot (\vec{v} \times \vec{u})$			

c) (2 points) Complete the following table which uses the vectors $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle, \vec{k} = \langle 0, 0, 1\rangle, -\vec{i}, -\vec{j}, -\vec{k}$ or $\vec{0}$ in one of the first 3 boxes. Enter an scalar in each of the 3 boxes at the bottom.

$\vec{i} \times \vec{i} =$		$\vec{i} \times \vec{j} =$		$\vec{k} \times \vec{j} =$	
$\vec{j} \cdot \vec{i} =$		$\vec{j} \cdot \vec{j} =$		$\vec{j} \cdot \vec{k} =$	

d) (2 points) Complete the following table about the **TNB frame**. In each case, enter either $\vec{T}, \vec{N}, \vec{B}$ or $\vec{0}$ in each of the 6 boxes. Every correct row gives a point:

$\text{Proj}_{\vec{T}}(\vec{T}) =$		$\text{Proj}_{\vec{T}}(\vec{N}) =$		$\text{Proj}_{\vec{B}}(\vec{T}) =$	
$\text{Proj}_{\vec{N}}(\vec{T}) =$		$\text{Proj}_{\vec{N}}(\vec{N}) =$		$\text{Proj}_{\vec{N}}(\vec{B}) =$	

Solution:
a) The first row describes a cylinder. We have $\boxed{x^2+y^2=1}$, $\boxed{r=1}$.
The second row is the plane $\boxed{x=1}$ or $\boxed{\rho\sin(\phi)\cos(\theta)=1}$.
The third row is a sphere or $\boxed{r^2+z^2=1}$ or $\boxed{\rho=1}$.
b) Perpendicular vectors have a zero dot product $\boxed{0}$. The length of the cross product of two perpendicular vectors is the product of the lengths. It is $\boxed{\text{positive}}$. The triple scalar product is $\boxed{0}$ as two vectors are the same.
c) The first row is $\boxed{0}$,$\boxed{\vec{k}}$,$\boxed{-\vec{i}}$.
The second row is $\boxed{0}$,$\boxed{1}$,$\boxed{0}$.
d) It is useful here to note the following general facts: The projection of a vector onto a vector perpendicular is the zero vector.
The projection of a vector on itself is the vector itself.
We have used also that the vectors $\vec{T},\vec{N},\vec{B}$ are perpendicular.
In the first row we get $\boxed{\vec{T}}$,$\boxed{\vec{0}}$,$\boxed{\vec{0}}$.
In the second row we get $\boxed{\vec{0}}$,$\boxed{\vec{N}}$,$\boxed{\vec{0}}$.

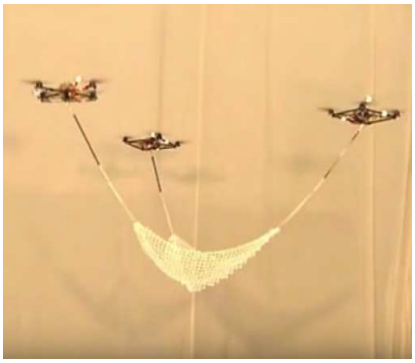
Problem 4) (10 points)

In a TED talk of 2013, **Raffaello D'Andrea** and a team demonstrated **"quadrotor athletes"**.

Assume the three robots are at positions

$$A=(2,3,1), B=(2,3,3) \text{ and } C=(5,4,2).$$

What is the area of the triangle they span?



A figure from an article by J. Sidman and A. St John in the Notices of the AMS, October 2017.

Solution:
We compute the area of the parallel epiped as $\vec{BC}\times\vec{AC}=\langle -2,6,0\rangle$ which has length $\sqrt{40}$. The area of the triangle is $\boxed{\sqrt{10}}$.

Problem 5) (10 points)

The six-legged **Gough-Stewart platform** has applications in flight simulators, robotics, crane technology, underwater research, telescopes and orthopedic surgery. The bottom positions of the legs are

$$A_1 = (5, -2, 0), A_2 = (5, 1, 0), A_3 = (-3, 5, 0),$$

$$A_4 = (-5, 3, 0), A_5 = (-5, -3, 0), A_6 = (-3, -5, 0).$$

The top positions of the legs are

$$B_1 = (-5, 2, 6), B_2 = (-5, -1, 6), B_3 = (3, -5, 6),$$

$$B_4 = (5, -3, 6), B_5 = (5, 3, 6), B_6 = (3, 5, 6).$$

a) (5 points) What is the distance between B_1 and the plane containing $A_1, \ldots, A_6$?
b) (5 points) What is the distance between B_1 and the line through A_1 and A_2?



Picture: Jessica Sidman and Audry St. John in the Notices of the AMS

Solution:
a) The distance 6 can be read off directly as all the points A_k are in the $z = 0$ plane and all the B_k are at $z = 6$.
It is also possible to use the **distance formula**: find the normal vector $\langle 0, 0, 1\rangle$, then form $|\vec{B_1A_1} \cdot \vec{n}|/|n| = 6/1 = 6$ which has an interpretation "Volume"/"area".
b) We can use the **distance formula between a point and a line**: It is the area of the parallelogram spanned by $\vec{v} = \vec{A_1A_2} = \langle 0, 3, 0\rangle$ and $\vec{w} = \vec{A_1B_1} = \langle -10, -4, 6\rangle$, divided by the length of $\vec{v}$. This is is $|\langle 18, 0, 30\rangle|/|\langle 0, 3, 0\rangle| = \sqrt{18^2 + 30^2}/3 = \boxed{2\sqrt{34}}$.

Problem 6) (10 points)

Even though Saturn is much larger than the Earth, its gravitational force is only 7 percent larger than here on Earth. When **Cassini** plunged into Saturn, it felt an acceleration

$$\vec{r}\,''(t) = \langle \pi \sin(\pi t), 0, -10 - 2t\rangle \ .$$

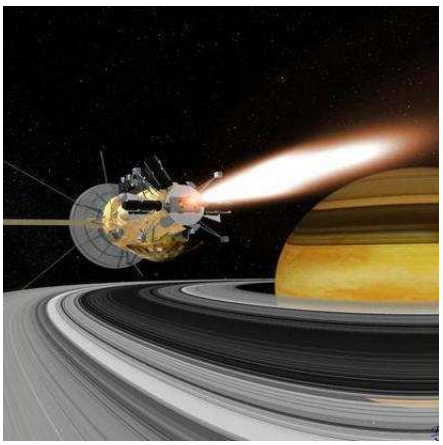
We know the initial velocity

$$\vec{r}\,'(0) = \langle 2, 5, 1\rangle$$

and initial position

$$\vec{r}(0) = \langle 0, 0, 1000\rangle \ .$$

Where is the spacecraft at $t = 1$?



Picture: NASA

Solution:
Integrate:

$$\vec{r}\,'(t) = \langle -\cos(\pi t), 0, -10t - t^2 \rangle + \langle C_1, C_2, C_3 \rangle \ .$$

Fixing the constants at $t = 0$ gives

$$\vec{r}\,'(t) = \langle -\cos(\pi t) + 3, 5, 1 - 10t - t^2 \rangle \ .$$

Integrate again to get

$$\vec{r}\,(t) = \langle -\sin(\pi t)/\pi + 3t + C_1, 5t + C_2, t - 5t^2 - t^3/3 + C_3 \rangle \ .$$

Fix the constants to get

$$\vec{r}\,(t) = \langle -\sin(\pi t)/\pi + 3t, 5t, 1000 + t - 5t^2 - t^3/3 \rangle \ .$$

At $t = 1$, we are at the point $(3, 5, 996 - 1/3)$ which is $\boxed{(3, 5, 2987/3)}$.

Problem 7) (10 points)

Let's look at the two planes

$$x + y + z = 1$$

and

$$x + y - z = 1 \ .$$

a) (4 points) Find the plane $ax+by+cz = d$ through $P = (1, 0, 0)$ which is perpendicular to both.

b) (2 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in both planes.

c) (4 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in the first plane but not the second.

Solution:
a) To get the normal vector, take the cross product between the normal vector $\langle 1, 1, 1 \rangle$ and $\langle 1, 1, -1 \rangle$ of the two planes. This is $\langle -2, 2, 0 \rangle$. The equation of the plane is $-2x + 2y = d$, where d is a constant. Plugging in a point gives $\boxed{-2x + 2y = 2}$.
b) The parametrization is $\vec{r}\,(t) = \langle 1, 0, 0 \rangle + t\langle -2, 2, 0 \rangle$ which is $\boxed{\vec{r}(t) = \langle t + 1, -t, 0 \rangle}$.
c) Pick a point Q on the first plane but not on the second. A possibility is $Q = (0, 0, 1)$. The vector $\vec{PQ}$ is $\langle -1, 0, 1 \rangle$. The parametrization is $\vec{r}\,(t) = \langle 1, 0, 0 \rangle + t\langle -1, 0, 1 \rangle$ which is $\boxed{\vec{r}(t) = \langle 1 - t, 0, t \rangle}$. Of course, there would be other parametrizations, depending on the point chosen. A commonly seen solution is also $\langle 1 + t, 0, -t \rangle$.

Problem 8) (10 points)

The world was supposed to end on September 23 due to the mysterious planetary system HD 7924. But here you sit and have to take the first hourly. A moon on HD 7924 moves on an epicycle

$$\vec{r}(t) = \langle 10\cos(t), 10\sin(t), 0\rangle + \langle 2\cos(5t), 2\sin(5t), 0\rangle \ .$$

a) (2 points) Find the velocity $\vec{r}\,'(0)$ at $t = 0$ and the speed $|\vec{r}\,'(0)|$ at $t = 0$.

b) (2 points) Find the acceleration $\vec{r}\,''(0)$ at $t = 0$.

c) (3 points) Find $\kappa(0) = |\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$.

d) (3 points) Inhabitants from HD 7924 beam you the hint $|\vec{r}\,'(t)|^2 = 400\cos^2(2t)$. Use this to find the arc length from $t = 0$ to $t = 2\pi$.

Will the world end on September 23?

30 Updated on September 20, 2017 at 12:37 PM, Posted on September 20, 2017 at 11:03 AM



Viral videos and theories claim an apocalypse will begin on Sept. 23. This photo is an artist's impression of a view from the HD 7924 planetary system looking back toward our sun. (Karen Teramura; BJ Fulton, University of Hawaii, Institute for Astronomy.)

Fun fact: the guy who came up with the date September 23 has revised his estimate to October 15. There is still hope to avoid the second midterm ...

Solution:
a) The velocity is

$$\vec{r}\,'(t) = \langle -10\sin(t), 10\cos(t), 0\rangle + \langle -10\sin(5t), 10\cos(5t), 0\rangle$$

Evaluated at $t = 0$, this is $\langle 0, 20, 0\rangle$. The length of this vector is $\boxed{20}$.
b) Differenticating again gives

$$\vec{r}\,''(t) = \langle -10\cos(t), -10\sin(t), 0\rangle + \langle -100\cos(5t), -100\sin(5t), 0\rangle$$

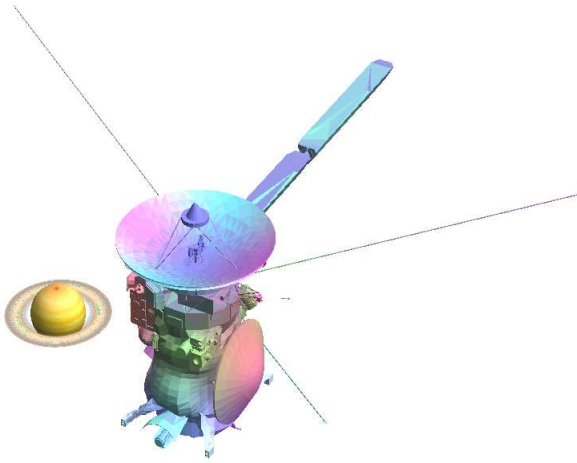
At $t = 0$, this is $\boxed{\langle -60, 0, 0\rangle}$.
c) The cross product is $\langle 0, 0, 1200\rangle$. It length is 1200. Divided by 20^3 gives $\boxed{3/20}$.
d) Since the speed is $20|\cos(t)|$, we get

$$20\int_0^{2\pi} |\cos(t)|\ dt = 20 \cdot 2 \int_{-\pi/2}^{\pi/2} \cos(t)\ dt = 20 \cdot 2 \cdot 2$$

which is $\boxed{80}$. A common mistake was to integrate $\cos(t)$ blindly which gives zero. An arc length is always positive if a curve is not just degenerated to a point.

Problem 9) (10 points)

Two weeks ago, in a grand finale, the Cassini space craft plunged into the atmosphere of **Saturn**. To build a model of the situation we have to parametrize various parts on the probe which were used both for measurement and communication.
You don't have to specify the parameter bounds but give the parametrizations for each of the 5 objects:



Picture: by Mathematica using a printable

3D STL models provided by NASA

a) (2 points) Saturn is a sphere $(x-1)^2+y^2+z^2=16$.

$$\vec{r}(\theta,\phi)=\langle \quad , \quad , \quad \rangle.$$

b) (2 points) The rings are given by $z=0, r^2=x^2+y^2\leq 25$.

$$\vec{r}(r,\theta)=\langle \quad , \quad , \quad \rangle.$$

c) (2 points) The satellite dish $(x-50)^2+(y-70)^2=z$ beams pictures back to earth.

$$\vec{r}(x,y)=\langle \quad , \quad , \quad \rangle.$$

d) (2 points) There is also a satellite antenna of the form $(x-50)^2+z^2=1/100$.

$$\vec{r}(\theta,y)=\langle \quad , \quad , \quad \rangle.$$

e) (2 points) There is also a device of the form $(x-50)^2+z^2-(y-70)^2=1$.

$$\vec{r}(\theta,y)=\langle \quad , \quad , \quad \rangle.$$

Solution:
a) Translate the **sphere** of radius 4 by $\langle 1, 0, 0\rangle$:
$\boxed{\langle 1 + 4\cos(\theta)\sin(\phi), 4\sin(\theta)\sin(\phi), 4\cos(\phi)\rangle}$.
b) This is the xy-plane parametrized using polar coordinates:
$\boxed{\langle r\cos(\theta), r\sin(\theta), 0\rangle.}$.
c) This is a **paraboloid**. We can either translate the paraboloid by $\langle 50, 70, 0\rangle$:
$\boxed{\langle 50 + \sqrt{z}\cos(\theta), 70 + \sqrt{z}\sin(\theta), z\rangle}$.
An other possibility is to use the standard x and y coordinates and get $\boxed{\langle x, y, (x-50)^2 + (y-70)^2\rangle}$.
d) This is a **hyperbolic paraboloid** translated by $\langle 50, 70, 0\rangle$. We can write $\langle 50, 70, 0\rangle + \langle\sqrt{1+y^2}\cos(\theta), y, \sqrt{1+y^2}\sin(\theta)\rangle = \boxed{\langle 50 + \sqrt{1+y^2}\cos(\theta), 70 + y, \sqrt{1+y^2}\sin(\theta)\rangle}$.
In that solution y is a translated y-coordinate. An other possibility is
$\langle 50 + \sqrt{1 + (y-70)^2}\cos(\theta), y, \sqrt{1 + (y-70)^2}\sin(\theta)\rangle$

9/27/2017 FIRST HOURLY PRACTICE 1 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] If $|\vec{r}\,'(t)| = 1$ for all t, then the curvature satisfies $\kappa(t) = |\vec{r}\,''(t)|$.

Solution:
It follows from the formula (given below in the test) and the fact that $\vec{r}\,''(t)$ is then perpendicular to $\vec{r}\,'(t)$.

2) [T] [F] Let $\vec{r}(t)$ be a parametric curve. Suppose that at the point $\vec{r}(t)$ the unit tangent vector is $\langle 0, 1, 0\rangle$ and the binormal vector is $\langle 0, 0, 1\rangle$. Then the unit normal vector at the point is $\langle 1, 0, 0\rangle$.

Solution:
The sign is wrong. It is $\langle -1, 0, 0\rangle$.

3) [T] [F] If $|\vec{v} + \vec{w}|^2 = 2\vec{v} \cdot \vec{w}$, then $\vec{v} = 2\vec{w}$.

Solution:
The statement actually implies that $\vec{v} = \vec{w} = \langle 0, 0, 0\rangle$.

4) [T] [F] The distance between two points P and Q is smaller than or equal to $|\vec{OP}| + |\vec{OQ}|$, where $O = (0, 0, 0)$ is the origin.

Solution:
This is the triangle inequality. If you make a detour over a third point, the distance gets bigger.

5) [T] [F] The line $\vec{r}(t) = \langle 5 + 2t, 2 + t, 3 + t\rangle$ is located on the plane $x - y - z = 0$.

Solution:
Just plug in $x = 5 + 2t, y = 2 + t, z = 3 + t$ into the equation.

6) [T] [F] The arc length of a circle with constant curvature 20 is $\pi/10$.

Solution:
The radius of the circle must be 1/20. Multiply this by 2π to get the arc length.

7) [T] [F] The surface $x^2 - y^2 + 4y = z^2 + 2z$ is an elliptic paraboloid.

Solution:
Complete the square. It is a hyperbolic paraboloid.

8) [T] [F] For any two vectors, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| + |\vec{w}|$.

Solution:
It is not plus on the right hand side, but times.

9) [T] [F] The acceleration of $\vec{r}(t) = \langle t, t, t \rangle$ is $\langle 0, 0, 0 \rangle$ everywhere.

Solution:
Indeed, the second derivative is $\langle 0, 0, 0 \rangle$.

10) [T] [F] If $\vec{v} = \vec{PQ} = \langle 2, 1, 1 \rangle$ then $|\vec{v}|$ is larger than the distance between P and Q.

Solution:
It is equal.

11) [T] [F] If $\vec{v} \cdot \vec{w} > 0$, then the angle between $\vec{v}$ and $\vec{w}$ is larger than $90°$.

Solution:
Use the cos formula. Positive means actually that the angle is acute.

12) [T] [F] If $\langle 5, 6, 4 \rangle \times \vec{x} = \vec{x}$ then $\vec{x} = \langle 0, 0, 0 \rangle$.

Solution:
Yes, since $\vec{x}$ must be perpendicular to $\vec{x}$.

13) [T] [F] The curvature of $\vec{r}(t) = \langle t-1, 1-t, t\rangle$ is 0 at the point $t = 1$.

Solution:
It is a line. So, the curvature is zero.

14) [T] [F] The line $\vec{r}(t) = t\langle 7, 7, 1\rangle$ hits the plane $-x - y = 7z$ in a right angle.

Solution:
The normal vector is $-1, -1, 7$.

15) [T] [F] The surface given in spherical coordinates as $\sin(\phi) = 1/\rho$ is a cylinder.

Solution:
It is on the equator

16) [T] [F] Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u}\cdot\vec{v})(\vec{w}\cdot\vec{u})| \leq |\vec{u}|^2|\vec{v}||\vec{w}|$.

Solution:
Use the cos identity for the length of the dot product

17) [T] [F] The surface given in spherical coordinates as $\rho^2 = 1$ is a sphere.

Solution:
Look at the traces

18) [T] [F] The arc length of the curve $\langle 5\sin(t), 1, 5\cos(t)\rangle$ from $t = 0$ to $t = 2\pi$ is equal to 10π.

Solution:
It is a circle

19) [T] [F] The surface parametrized by $\vec{r}(u,v) = \langle u^5 - v^5, u^5 + v^5, u^5 \rangle$ is a plane.

Solution:
$x + y = 2z$

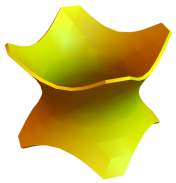
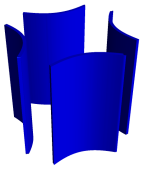
20) [T] [F] If the non-zero cross product of a vector $\vec{v}$ with a vector $\vec{w}$ is parallel to $\vec{v}$, then the dot product between $\vec{v}$ and $\vec{w}$ is zero.

Solution:
The cross product is perpendicular to $\vec{v}$. If it is also parallel, then this implies that one of the vectors $\vec{v}$ or $\vec{w}$ is the zero vector.

[ ] Total

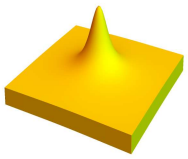
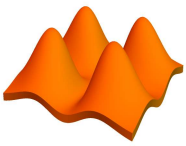
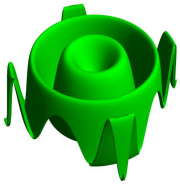
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.

I  II  III 

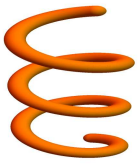
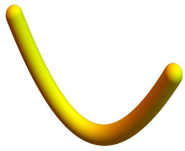
Function $g(x, y, z) = 1$	Enter O,I,II or III
$x^2 + y^2 - z^2$	
x^2y^2	
$x - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

I  II  III 

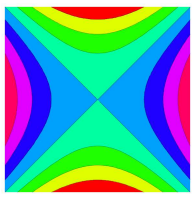
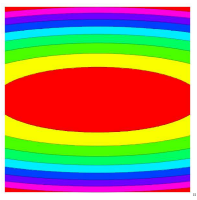
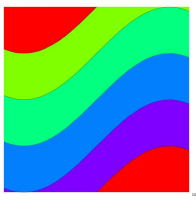
Function $f(x, y) =$	Enter O,I,II or III
$1/(1 + x^2 + y^2)$	
$x^2y^2e^{-x^2-y^2}$	
$\sin(x^2 + y^2)$	
$\cos(y)$	

c) (2 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

I  II  III 

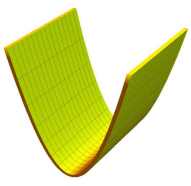
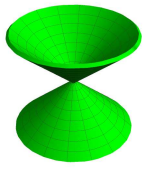
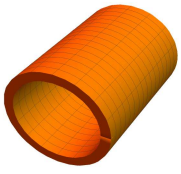
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, t\sin(3t), t\cos(3t)\rangle$	
$\langle \cos(t), \sin(t), \cos(t)\rangle$	
$\langle \sin(t), \cos(t), t\rangle$	
$\langle t, t, t^2\rangle$	

d) (2 points) Match functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Equation	Enter O, I,II or III
$\sin(x) - y$	
$x^2 - y^2$	
$10x^2 + y^2$	
$x^2 + 10y^2$	

e) (2 points) Match the quadrics. Enter O if no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2 = z$	
$x^2 + y^2 = 1$	
$xy = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

Solution:
a) I,III,II,O
b) I,II,III,O
c) I,O,II,III
d) III,I,O,II
e) I,III,O,O,II

Problem 3) (10 points)	(Only answers are needed)

a) (4 points) Mark what applies for any two vectors $\vec{v}$ and $\vec{w}$ in space.

Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0 \rangle$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				
$(\vec{v} \times \vec{w}) \cdot \vec{v}$				
$(\text{proj}_{\vec{v}} \vec{w}) \times \vec{w}$				
$(\text{proj}_{\vec{w}} \vec{v}) \cdot \vec{w}$				

b) (3 points) **Conic sections** (parabolas, ellipses or hyperbolas) can be seen as intersections of a two dimensional cone

$$x^2 + y^2 = z^2$$

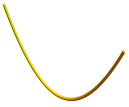
with a 2D plane. Identify the quadrics in the following three cases:

Intersect with plane	Enter A-D
$z = 1$	
$z = \sqrt{2}x$	
$x = y + 1$	

A 

B 

C 

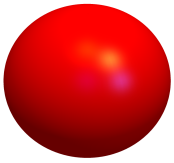
D 

c) (3 points) Three dimensional cone is given by the equation

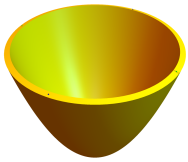
$$x^2 + y^2 + z^2 = w^2$$

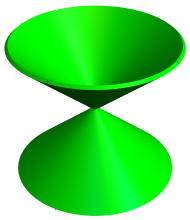
in four dimensional space. If we intersect it with a three dimensional space, we get quadrics. We want you to identify a few quadrics. In the pictures the quadrics might be turned or scaled. You get a point for every right answer, meaning that you can miss one and still have full credit.

Intersect with the 3D plane	Enter A-D or O
$w = 0$	
$w = z + 1$	
$y = z$	
$w = 1$	

A 

B 

C 

D 

Solution:
a)

Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0 \rangle$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				*
$(\vec{v} \times \vec{w}) \cdot \vec{v}$	*			
$(\text{proj}_{\vec{v}}\vec{w}) \times \vec{w}$				*
$(\text{proj}_{\vec{w}}\vec{v}) \cdot \vec{w}$		*		

b) A,B,C
c) O,C,D,A.

Problem 4) (10 points)



We are going into the **furniture design business**. Our first task is to construct a chair. Find the distance between the lines spanned by the parallel arm rests AB and CD, where

$$A = (1,0,2), B = (4,1,2)$$

and

$$C = (3,3,3), D = (6,4,3)\ .$$

Solution:
Since the two lines are parallel, we can not just use blindly the distance formula for two lines. But we can take one point on the first line and compute the distance to the other. The distance is therefore given by

$$|\overrightarrow{AB} \times \overrightarrow{AC}|/|\overrightarrow{AB}| \ .$$

As $\overrightarrow{AB} = \langle 3, 1, 0\rangle$ and $\overrightarrow{AC} = \langle 2, 3, 1\rangle$, we have $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, -3, 7\rangle$. So $|\overrightarrow{AB} \times \overrightarrow{AC}|/|\overrightarrow{AB}| = |\langle 1, -3, 7\rangle|/|\langle 3, 1, 0\rangle| = \boxed{\sqrt{59}/\sqrt{10}}$.

Problem 5) (10 points)

a) (6 points) The first derivative of parametrized curve is called **"velocity"**. You have also learned the terms **"acceleration"** and maybe **"jerk"** for the second and third derivative. Less well known are **"snap"**, **"crackle"**, **"pop"** for the fourth, fifth and sixth derivatives. Since we could not yet find the seventh derivative named, let's call it the **"Harvard"**. Compute the **"Harvard"** of the curve

$$\vec{r}(t) = \langle \cos(2t) + t, \sin(2t), t^2\rangle$$

at time $t = 0$.



b) (4 points) The parametrization $\vec{v}(t) = \vec{r}\,'(t)$ defines a new curve. It is located on a surface. Which of the following surface is it?

	Check one
cylinder	A
cone	B
plane	C
ellipsoid	D
paraboloid	E

Solution:
a) Start differentiating:
$\vec{r}\,'(t) = \langle -2\sin(2t) + 1, 2\cos(2t), 2t)\rangle$. velocity
$\vec{r}\,''(t) = \langle -4\cos(2t), -4\sin(2t), 2)\rangle$. acceleration
$\vec{r}\,'''(t) = \langle 8\sin(2t), -8\cos(2t)0)\rangle$. jerk
$\vec{r}\,''''(t) = \langle 16\cos(2t), 16\sin(2t)0)\rangle$. snap
$\vec{r}\,'''''(t) = \langle -32\sin(2t), 32\cos(2t)0)\rangle$. crackle
$\vec{r}\,''''''(t) = \langle -64\cos(2t), -32\sin(2t)0)\rangle$. pop
$\vec{r}\,'''''''(t) = \langle 128\sin(2t), -64\cos(2t)0)\rangle$. Harvard
$\vec{r}\,'''''''(0) = \langle 0, -128, 0\rangle$.
b) $\vec{r}\,'(t) = \langle -2\sin(2t) + 1, 2\cos(2t), 2t)\rangle$.
The surface is located on a cylinder as $(x-1)^2 + y^2 = 4$. The answer is $\boxed{\text{A}}$.

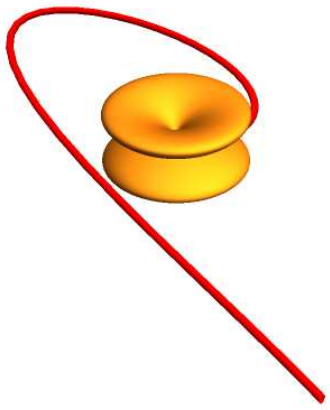
Problem 6) (10 points)

A kid plays with a **Yo-Yo**. It is accelerated periodically with $\vec{r}\,''(t) = \langle \sin(t), 0, \cos(t) - 10\rangle$. Find the position of the Yo-Yo at time $t = 2\pi$ if the initial position is

$$\vec{r}(0) = \langle 5, 5, 0\rangle$$

and the initial velocity is

$$\vec{r}\,'(0) = \langle 1, 1, 1\rangle \ .$$



Solution:
a) Just integrate and fix the constant $\vec{r}\,'(t) = \langle -\cos(t), 0, \sin(t) - 10t\rangle + \langle 2, 1, 1\rangle$.
Now integrate again: $\vec{r}(t) = \langle -sin(t) + 2t + 5, t + 5, \cos(t) - 5t^2 + t + 1\rangle$. Evaluated at $t = 2\pi$ this is $\vec{r}(2\pi) = \langle 4\pi + 5, 2\pi 5, -20\pi^2 + 2\pi\rangle$.

Problem 7) (10 points)

a) (5 points) Compute the arc length of the curve

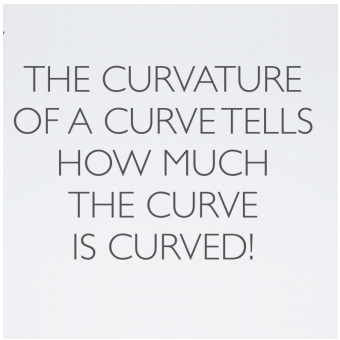
$$\vec{r}(t) = \langle t, t^2, 2t^3/3 \rangle$$

if $0 \leq t \leq 4$.

b) (5 points) What is the curvature

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

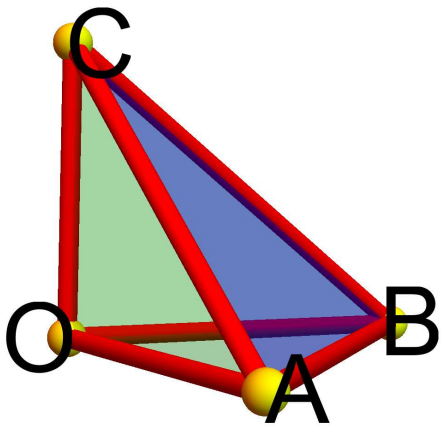
of this curve at $t = 0$?



Solution:
a) $\vec{r}\,'(t) = \langle 1, 2t, 2t^2 \rangle$. $|\vec{r}\,'(t)| = 2t^2 + 1$. Integration $\int_0^4 2t^2 + 1 \, dt$ gives $140/3$.
b) The curvature is $|\langle 1, 0, 0 \rangle \times \langle 0, 2, 0 \rangle| / |\langle 1, 0, 0 \rangle|^3$. This is $\boxed{2}$.

Problem 8) (10 points)

The triangle ABC is obtained by slicing a corner ABC off from a cube. One obtains a so called **trirectangular tetrahedron**.



a) (5 points) Find the square of the area of the triangle ABC, where $A = (3, 0, 0), B = (0, 6, 0), C = (0, 0, 8)$.

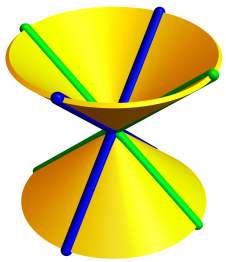
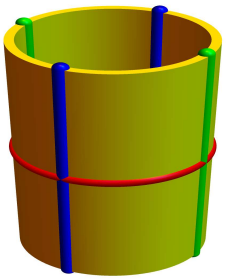
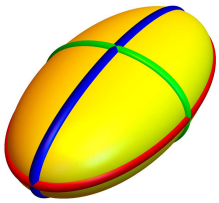
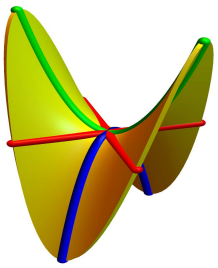
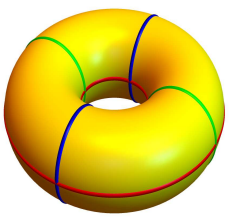
b) (5 points) Compute also the sum of the squares of the areas of the triangles OAB, OBC and OCA, where $O = (0, 0, 0)$ is the origin. The sum should get the same value you got in a).

P.S. the same computation can be repeated for arbitrary points $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$. It proves a not so well known theorem telling that the sum of the squares of the side wall areas is the square of the face area. It is a 3 dimensional version of Pythagoras and also goes by the name **de Gua theorem** or **Faulhaber theorem**.

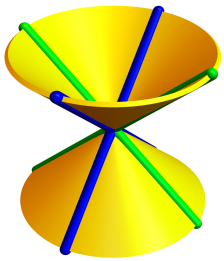
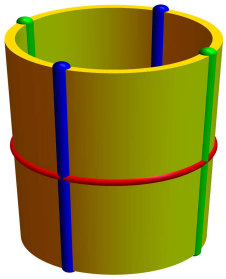
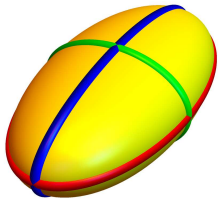
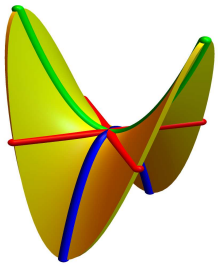
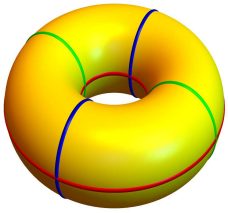
Solution:
a) The area is half the area of a parallelogram. Its square is therefore $|AB \times AC|^2/4$ which is $|\langle 3, -6, 0\rangle \times \langle 3, 0, -8\rangle|^2/5$. This is $\boxed{801}$.
b) The area of a side triangle is half the area of a square. The sum of the areas squared is $18^2/4 + 48^2/4 + 24^2/4 = 801$.
P.S. The full computation is the same $\langle a, -b, 0\rangle \times \langle a, 0, -c\rangle = \langle bc, ac, ab\rangle$ which has square length $(bc)^2 + (ac)^2 + (ab)^2$. The triangle has 1/4'th of this. The area of the triangle is therefore $(bc)^2/4 + (ac)^2/4 + (ab)^2/4$ and each summand is the square area of a side triangle.

Problem 9) (10 points)

The 3D printing venture "Math-Candy" (math-candy.com) asks you to do some product development. In each of the 5 following parametrizations, two entries are still missing, each entry being worth one candy (1 point).

a)		The surface $\boxed{x^2+y^2=z^2}$ is parametrized by $\vec{r}(\theta,z)=\langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z\rangle$
b)		The surface $\boxed{x^2+y^2=1}$ is parametrized by $\vec{r}(\theta,z)=\langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z\rangle$
c)		The surface $\boxed{2(x-1)^2+(y-5)^2+4z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=\langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, \cos(\phi)/2\rangle$
d)		The surface $\boxed{x^2-y^2=z}$ is parametrized by $\vec{r}(x,y)=\langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, x^2-y^2\rangle$
e)		The surface $\boxed{(\sqrt{x^2+y^2}-2)^2+z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=\langle (2+\cos(\phi))\cos(\theta), \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\rangle$

Solution:

a)		The surface $\boxed{x^2 + y^2 = z^2}$ is parametrized by $$\vec{r}(\theta, z) = \langle z\cos(\theta), z\sin(\theta), z\rangle$$
b)		The surface $\boxed{x^2 + y^2 = 1}$ is parametrized by $$\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z\rangle$$
c)		The surface $\boxed{2(x-1)^2 + (y-5)^2 + 4z^2 = 1}$ is parametrized by $$\vec{r}(\theta, \phi) = \langle 1 + \sin(\theta)\cos(\theta)/\sqrt{2}, 5 + \sin(\theta)\sin(\phi), \cos(\phi)/2\rangle$$
d)		The surface $\boxed{x^2 - y^2 = z}$ is parametrized by $$\vec{r}(x, y) = \langle x, y, x^2 - y^2\rangle$$
e)		The surface $\boxed{(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1}$ is parametrized by $$\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi)\rangle$$

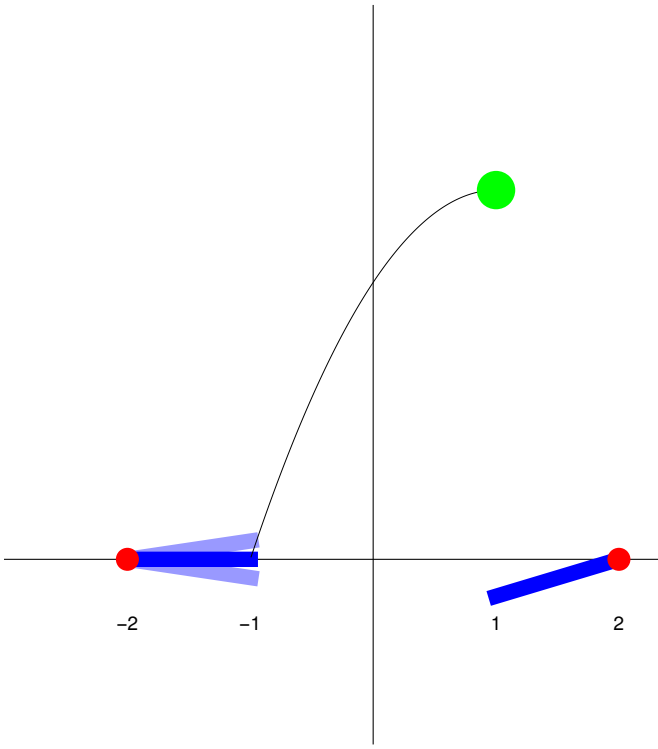
Problem 10) (10 points)

A **pinball machine** is tilted in such a way that a ball in the xy-plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}\,'(0) = \langle 1/2, 5 \rangle$.

a) (4 points) Find the velocity $\vec{r}\,'(t)$.

b) (4 points) What trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ does the ball follow?

c) (2 points) As the ball hits the line $y = 0$, is it reachable by the player? In other words, does it hit $y = 0$ within the interval $x \in [1, 2]$?



Solution:
a) We know the acceleration $\vec{r}\,''(t) = \langle 0, -2 \rangle$. Integrating once gives the velocity

$$\vec{r}\,'(t) = \langle 0, -2t \rangle + \langle 1/2, 5 \rangle \ .$$

Integrating again gives

$$\vec{r}(t)\langle x(t), y(t) \rangle = \langle -1 + t/2, 5t - t^2 \rangle \ .$$

b) We have $y(t) = 0$ for $t = 0$ and $t = 5$. At this time, the ball is at $(3/2, 0)$. This is right in the middle of the right flipper. The player will hit the ball.

9/27/2017 FIRST HOURLY PRACTICE 2 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The vector $\langle 2, 3, 6\rangle$ has a length which is an integer.

Solution:
Yes, its length is equal to 7.

2) [T] [F] The surface $x^2 + y^2 + z^2 - 2x = 3$ is a sphere.

Solution:
Complete the square

3) [T] [F] If $\vec{v} \cdot \vec{w}$ is negative, then the angle between $\vec{v}$ and $\vec{w}$ is acute (= smaller than $\pi/2$).

Solution:
True

4) [T] [F] The level curves $f(x, y) = 1$ and $f(x, y) = 0$ do not intersect for $f(x, y) = (xy + \cos(x))^6$.

Solution:
The function would have to be schizophrenic (multi-valued) and have the value 1 and 0 at the same point. This can not happen for a perfectly smooth function like the one given.

5) [T] [F] For any nonzero $\vec{a}$, the equation $\vec{a} \times \vec{x} = \vec{b}$ always has a solution $\vec{x}$.

Solution:
We need $\vec{b}$ to be perpendicular.

6) [T] [F] For two non-parallel $\vec{a}, \vec{b}$, the equation $(\langle x, y, z\rangle \times \vec{a}) \cdot \vec{b} = 1$ defines a plane.

Solution:
This was not an easy problem. Look at the volume of the parallel epiped spanned by $\vec{a}, \vec{b}$ and $\vec{x}$. This volume is fixed if the vector $\vec{x}$ is on the plane of points which have fixed distance to the plane spanned by $\vec{a}$ and $\vec{b}$.

7) [T] [F] The curvature of $\vec{r}(t) = \langle t^3, 1 - t^3, t^3 \rangle$ is 0 if $t = 1$.

Solution:
It is a line

8) [T] [F] If $\vec{N}$ is the normal vector and $\vec{T}$ the unit tangent vector to a curve $\vec{r}(t)$ then the vector projection of $\vec{N}(t)$ onto $\vec{T}(t)$ is zero.

Solution:
They are perpendicular.

9) [T] [F] There exist non-parallel vectors $\vec{v}, \vec{w}$ such that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

Solution:
Yes, in that case, one of the vectors has to be zero.

10) [T] [F] The point given in spherical coordinates as $\rho = 3, \phi = \pi/2, \theta = \pi$ is on the x-axes.

Solution:
It is the north pole.

11) [T] [F] The parametrized curve $\vec{r}(t) = \langle 5\cos(3t), 3\sin(3t), 0 \rangle$ is an ellipse.

Solution:
Indeed, and it is contained in the xz-plane.

12) [T] [F] If the vector projection of $\vec{v}$ onto $\vec{w}$ is $\vec{w}$ then $\vec{v} = \vec{w}$.

Solution:
We can take $\vec{v}$ plus an orthogonal vector.

13) [T] [F] Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \times \vec{v}) \times \vec{w}| \leq |\vec{u}||\vec{v}||\vec{w}|$.

Solution:
Use the identity for the length of the cross product

14) [T] [F] The surface $y^2 + z = x^2$ is a hyperbolic paraboloid.

Solution:
Look at the traces

15) [T] [F] The curvature of a curve $\vec{r}(t)$ at time $t = 0$ is the same as the curvature of $\vec{r}(\sin(t))$ at time $t = 0$.

Solution:
This is a re-parametrization

16) [T] [F] The arc length of the curve $\langle \sin(t), 0, \cos(t) \rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

Solution:
It is a circle

17) [T] [F] The curve $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t) + \sin(t) \rangle$ is on the intersection of $x^2 + y^2 = 1$ and $x + y - z = 0$.

Solution:
Just plug in

18) [T] [F] Using $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle$, the identity $(\vec{i} \times \vec{j}) \times \vec{j} = \vec{i} \times (\vec{j} \times \vec{j})$ holds.

Solution:
Indeed, associativity fails.

19) [T] [F] Using the same notation, the identity $(\vec{i} \cdot \vec{j})\vec{j} = \vec{i}(\vec{j} \cdot \vec{j})$ holds.

Solution:
Indeed, associativity fails.

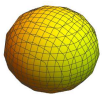
20) [T] [F] $\langle \cos t, \sin t, t \rangle$, $0 \leq t \leq 2$ and $\langle \cos(t^3), \sin(t^3), t^3 \rangle$, $0 \leq t \leq 2$ have the same arc length.

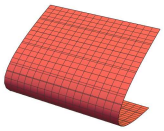
Solution:
While it is true that arc length is invariant under reparametrization, we have in this case a different curve because the end points do not agree. If the time interval would have been $[0, 1]$ (like in some practice exam), then the arc length would agree.

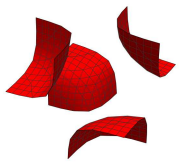
[ ] Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contours $g(x, y, z) = 1$. Enter O, if there is no match.

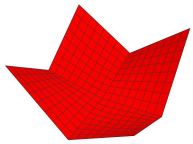

I

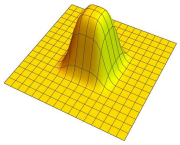

II

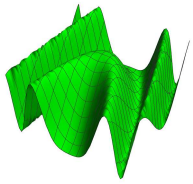

III

Function $g(x, y, z) = 1$	Enter O,I,II or III
xyz	
$x^2 + y^2 + z^2$	
$z^2 - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

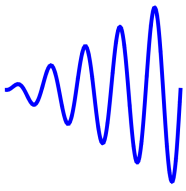

I

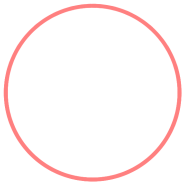

II

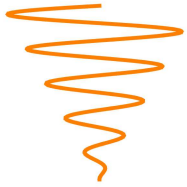

III

Function $f(x, y) =$	Enter O,I,II or III
$\cos(x^2 + y)$	
$\|x + y\| + \|xy\|$	
$\exp(-x^4 - y^4)$	
x^3	

c) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

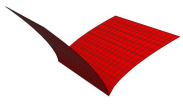

I

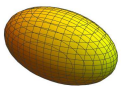

II

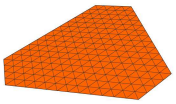

III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = \langle t, t\sin(5t)\rangle$	
$\vec{r}(t) = \langle t\sin(5t), t\rangle$	
$\vec{r}(t) = \langle \sin(5t), \cos(5t)\rangle$	
$\vec{r}(t) = \langle \cos(5t), \cos(5t)\rangle$	

d) (2 points) Match functions g with level surface $g(x, y, z) = 1$. Enter O, if there is no match.

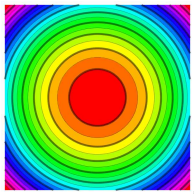

I

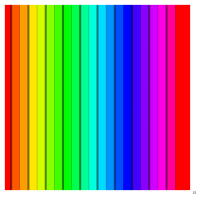

II

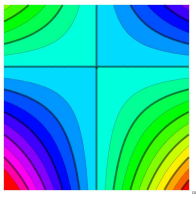

III

Function $g(x, y, z) = 1$	Enter O, I,II or III
$x^2 - y^2 + z^2 = 1$	
$x - y - z = 1$	
$y^3 = z^2$	
$x^2/4 + y^2 + z^2/2 = 1$	

e) (2 points) Match the contour maps to a function $f(x, y)$. Enter O if no match.


I


II

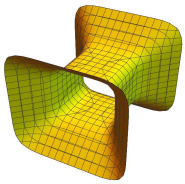

III

$f(x, y) =$	Enter O,I,II or III
$x^2 - y^4$	
$xy - x$	
x	
y	
$x^2 + y^2$	

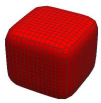
Solution:
a) III,I,II,0
b) III,I,II,0
c) I,III,II,0
d) 0,III,I,II
e) 0,III,II,0,I

Problem 3) (10 points)	(Only answers are needed)

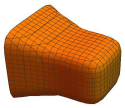
a) (4 points) The following contour surfaces were deformed by setting $X = x^3, Y = y^3, Z = z^3$. Can you label the original quadrics from which it was deformed?



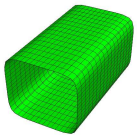
I



II



III



IV

Surface	I-IV	name A-D
$X^2 + Y^2 + Z^2 = 1$		
$X + Y^2 + Z^2 = 1$		
$X^2 - Y^2 + Z^2 = 1$		
$X^2 + Z^2 = 1$		

Fill in:
A) Ellipsoid'esque
B) Paraboloid'esque
C) Hyperboloid'esque
D) Cylinder'esque

b) Help Larry the physicist in the movie "A serious man", to compute some quantities:

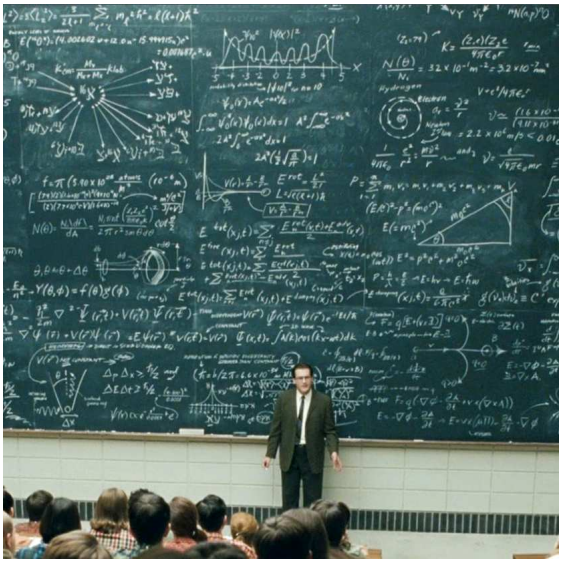
$\vec{v} = \langle 1, 2, 3 \rangle$ represent the velocity
$\vec{w} = \langle 0, 1, 1 \rangle$ represent angular velocity
$\vec{B} = \langle 1, 0, 1 \rangle$ represent magnetic field
$\vec{r} = \langle 0, 0, 1 \rangle$ represent position. Compute:

(i) (2 points) Coriolis force $\vec{v} \times \vec{w}$.

(ii) (2 points) Lorentz force $\vec{v} \times \vec{B}$.

(iii) (1 point) Kinetic energy $(\vec{v} \cdot \vec{v})/2$.

(iv) (1 point) Magnetic energy $\vec{B} \cdot \vec{B}/2$.



Larry: "I mean - even I don't understand the dead cat. The math is how it really works."

(i)

ii)

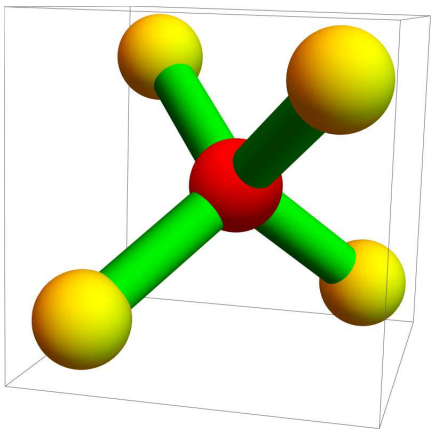
(iii)

iv)

Solution:
a) II A
III,B
I,C
IV,D
b) $< -1, -1, 1 >$
$< 2, 2, -2 >$
7
1

Problem 4) (10 points)

Methane CH_4 is the number two greenhouse gas emitted by human activity in the US. The four hydrogen atoms of **methane** are located at the vertices $P = (2,2,2), Q = (2,0,0), R = (0,2,0), S = (0,0,2)$ and form a regular tetrahedron, while C is the central carbon atom located at $(1,1,1)$.



a) (2 points) Find one bond distance $|CP|$ and the distance $|PQ|$

b) (4 points) Find the cosine of the bond angle between $\overrightarrow{PC}$ and $\overrightarrow{PQ}$.

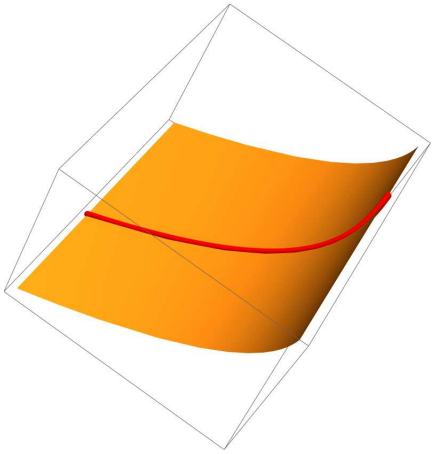
c) (4 points) What is volume of the parallelepiped spanned by $\overrightarrow{PC}, \overrightarrow{QC}, \overrightarrow{RC}$?

Solution:
a) $|CP| = \sqrt{3}$
$|PQ| = 2\sqrt{2}\sqrt{8}$
b) We use the basic cos-formula for getting the angle. $\cos(\theta) = 4/(\sqrt{3}\sqrt{8}) = 2/\sqrt{12} = \sqrt{6}/3 = \sqrt{2}/\sqrt{3}$.
c) We compue the triple scalar product between the three vectors. The answer is 4. It makes sense since it is just half the volume of the cube of side length 8.

Problem 5) (10 points)

Consider the curve

$$\vec{r}(t) = \langle 2e^t, t, e^{2t} \rangle \ .$$



a) (3 points) Compute the speed $|\vec{r}\,'(0)|$.

b) (5 points) Find the arc length from $t = -2$ to $t = 1$.

c) (2 points) There exists a constant a such that the curve lies on the cylindrical paraboloid $x^2 = az$. Which a does apply?

Solution:
a) The velocity is $\vec{r}\,'(t) = \langle 2e^t, 1, 2e^{2t}\rangle$. At $t = 0$, this is $\vec{r}\,'(0) = \langle 2, 1, 2\rangle$. The speed is $\sqrt{4+4+1} = 3$.
b) We have to integrate

$$\int_{-2}^{1} \sqrt{4e^{2t} + 1 + 4e^{4t}}\, dt$$

The term inside the square root can be written as $(2e^{2t} + 1)^2$. The integration is now elementary and becomes $e^2 - e^{-4} + 3$
c) Since $x = 2e^t$ and $z = e^{2t}$ we see $x^2 = 4e^{2t} = az$ for $a = 4$.

Problem 6) (10 points)

The highest **bungee jump** ever recorded was done from the 233 meter high Macau Tower. Assume the rope pulls back with a force $2t$ so that the acceleration is

$$\vec{r}\,''(t) = \langle 0, 0, 2t - 10\rangle \ .$$



Assume the initial velocity is $\langle 1, 0, 0\rangle$ and that the daredevil jumps from $\vec{r}(0) = \langle 0, 0, 233\rangle$:

a) (5 points) Find $\vec{r}\,'(t)$ and determine t_0 for which the third component $z'(t_0) = 0$. This is the time of the lowest point.

b) (5 points) Find $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\vec{r}(t_0)$. Did the jumper hit the ground $z = 0$?

Solution:
a) $\vec{r}' = \langle 0, 0, t^2 - 10t\rangle + \langle C_1, C_2, C_3\rangle$.
Fixing the initial velocity shows $\vec{r}' = \langle 1, 0, t^2 - 10t\rangle$. The roots of $t^2 - 10t$ are $t = 0$ and $t = 10$. The first one is the start of the jump, the second one is at the lowest point.
b) Integrate $\vec{r}\,'(t)$ again and fix the constants. We get $\vec{r} = \langle t, 0, t^3/3 - 5t^2 + 233\rangle$
At time $t = 10$, we are at $\vec{r}(10) = \langle 10, 0, 199/3\rangle$. The jumper survives.
The situation modeled here is pretty close to what you see in the video. The jumpers don't get too close to the ground. The t part of the acceleration assumes that the rope satisfies the Hook law telling that the force pulling back gets bigger if is is getting longer. After that moment, the differential equation does not model the situation any more well.

Problem 7) (10 points)

The **logarithmic spiral** is parametrized by $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), 0\rangle$.



a) (5 points) Find the angle between $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at time $t = 0$.

b) (5 points) Compute the curvature at $t = 0$.

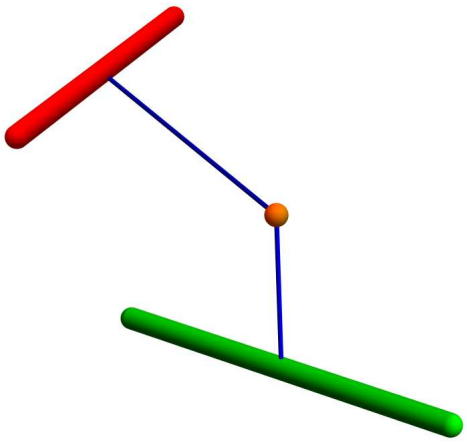
$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \ .$$

One miracle about the spiral is that arc length from 0 to t multiplied with curvature at t is constant. Jacob Bernoulli called it the curve the "Spira mirabilis" which means "miraculous spiral".

Solution:
a) $\vec{r}\,'(t) = \langle e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), 0\rangle$ $\vec{r}\,''(t) = \langle 2e^t \sin(t), 2e^t \cos(t), 0\rangle$
$\vec{r}\,'(0) = \langle 1, 1, 0\rangle$
$\vec{r}\,''(0) = \langle 0, 2, 0\rangle$.
$\theta = \arccos(1/\sqrt{2}) = \pi/4$
b) We compute $\langle 1, 1, 0\rangle \times \langle 0, 2, 0\rangle = \langle 0, 0, 2\rangle$. Therefore, $\kappa = |\langle 1, 1, 0\rangle \times \langle 0, 2, 0\rangle|/2 = \sqrt{2}/2 = 1/\sqrt{2}$

Problem 8) (10 points)

Given a line $\vec{r}(t) = \langle 1 + t, t, t\rangle$ and a line $\vec{s}(t) = \langle 1 - t, 1 + t, 1 - t\rangle$.



a) (6 points) Find the sum of the distances of the point $(0, 0, 0)$ to the two lines.

b) (4 points) Find the distance between the two lines.

Solution:
This is a routine distance problem, but three problems.
a) The first distance is $|\langle 1,0,0\rangle \times \langle 1,1,1\rangle|/\sqrt{3} = \sqrt{2}/\sqrt{3}$. The second distance is $|\langle 1,1,1\rangle \times \langle -1,1,-1\rangle|/\sqrt{3} = \sqrt{8}/\sqrt{3}\ \sqrt{6}$. In both cases, we have computed the area of a parallelogram and divided by the base length.
b) Also here, we think about geometry and get the volume of the parallel epiped spanned by $\langle 1,1,1\rangle, \langle -1,1,-1\rangle$ and $\langle 0,1,1\rangle$ (which is 2) divided by the area of the parallelogram spanned by $\langle 1,1,1\rangle, \langle -1,1,-1\rangle$ which is the length of their cross product (which is $\sqrt{4+4} = \sqrt{8}$. The answer is $2/\sqrt{8} = 1/\sqrt{2}$

Problem 9) (10 points)

Parametrize the following surfaces in space. As usual r, θ, z are cylindrical and ρ, θ, ϕ are spherical coordinate variables. You do not need to give bounds on the parameters.

a) (2 points) Parametrize $y = \cos(3x) - \sin(3z)$ as

$\vec{r}(x,z) =$

b) (2 points) Parametrize $\rho = 2 + \cos(8\theta + 5\phi)$ as

$\vec{r}(\theta,\phi) =$

c) (2 points) Parametrize $r^2 - z^2 = 1$ as

$\vec{r}(\theta,z) =$

d) (2 points) Parametrize $x = 0$ as

$\vec{r}(y,z) =$

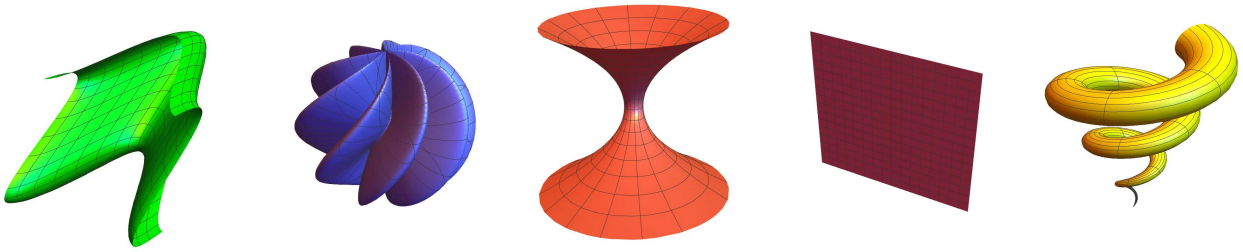
e) Decide whether none, one, or both of the grid curves u = 1, v = 1 is a circle, if

$$\vec{r}(u,v) = \langle (3u + u\cos(v))\cos(2u), (3u + u\cos(v))\sin(2u), (3u + u\sin(v))\rangle$$

(1 point) Is the curve $u = 1$ a circle? Yes or No

(1 point) Is the curve $v = 1$ a circle? Yes or No

Illustrations:



a) b) c) d) e)

Solution:
a) $\langle x, \cos(3x) - \sin(3z), z\rangle$.
b) $\langle (2+\cos(8\theta+5\phi))\sin(\phi)\cos(\theta), (2+\cos(8\theta+5\phi))\sin(\phi)\sin(\theta), (2+\cos(8\theta+5\phi))\cos(\phi)\rangle$.
c) $\sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z\rangle$.
d) $\langle 0, y, z\rangle$ e) First "Yes" then "No". For seeing that $u = 1$ is a circle, you can write down $\vec{r}(1, v)$ and see that $x^2 + y^2 + z^2 = 1$ and $x\sin(1) - y\cos(1) = 0$ which means that the curve is on the intersection of a sphere with a plane. The curve $v = 1$ is clearly a spiral lying on the cone $x^2 + y^2 = z^2$. (To see that the first one was a circle was probably the hardest point to earn in this exam!)

Problem 10) (10 points)

We enjoy the fall sun outside and sit in a local restaurant for a refreshment. In each of the ordered items, give a **surface parametrization** of the form

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle \ .$$

As indicated, we can use also other variables. Your task is to fill in the three parametrization functions in each case, using the variables provided.

a) (2 points) Parametrize the **lemonade glass** $x^2 + y^2 = 1$.

$\vec{r}(\theta, z) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$.

b) (2 points) Parametrize the **sorbet glass** $x^2+y^2 = z^2$.

$\vec{r}(\theta, z) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$.

c) (2 points) Parametrize the **lemon** surface $x^2 + y^2 = \sin(z)$.

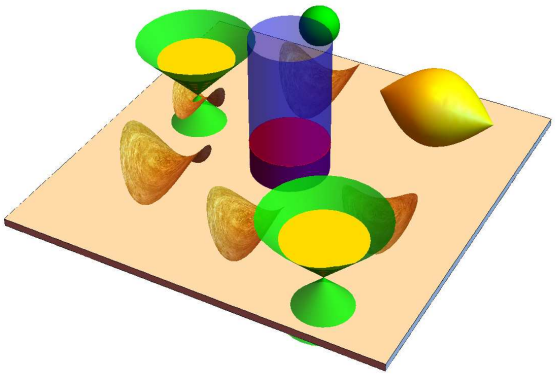
$\vec{r}(\theta, z) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$.

d) (2 points) Parametrize one of the **chips** $z = x^2 - y^2$.

$\vec{r}(x, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$.

e) (2 points) Parametrize the **lime** $x^2+y^2+(z-3)^2 = 1$.

$\vec{r}(\theta, \phi) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$.



Solution:
a) $\langle \cos(\theta), \sin(\theta), z \rangle$.
b) $\langle z\cos(\theta), z\sin(\theta), z \rangle$.
c) $\langle \sqrt{\sin(z)}\cos(\theta), \sqrt{\sin(z)}\sin(\theta), z \rangle$.
d) $\langle x, y, x^2 - y^2 \rangle$.
e) $\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), 3 + \cos(\phi) \rangle$.

9/27/2017 FIRST HOURLY PRACTICE 3 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 or problem 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		11
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The surface $-x^2 + y^2 + z^2 = -1$ is a one-sheeted hyperboloid.

Solution:
Look at the traces

2) [T] [F] The equation $y = 3x + 2$ in space defines a plane.

Solution:
It looks like a line because only two variables are involved, but remember that we are in space.

3) [T] [F] Whenever $|\vec{r}\,'(t)| = 1$ then $|\vec{T}\,'(t)| = 1$.

Solution:
We know that $|\vec{T}'|$ is curvature. This can be arbitrary even if the speed is 1 at all times

4) [T] [F] The length of the vector projection $\text{Proj}_{\vec{v}}(\vec{w})$ is smaller than or equal to the length of $\vec{w}$.

Solution:
One can see this geometrically, or by expanding the dot product in the formula.

5) [T] [F] The velocity vector of $\vec{r}(t) = \langle t, t, t\rangle$ at time $t = 2$ is the same as the velocity vector at $t = 1$.

Solution:
It is $\langle 1, 1, 1\rangle$ in both cases.

6) [T] [F] If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$ then $\vec{v}, \vec{w}$ are parallel.

Solution:

7) [T] [F] The vector $\langle -2, 1, 0\rangle$ is perpendicular to the line $\langle 1+t, 2t, 3t\rangle$.

Solution:
Take the dot product with the velocity vector.

8) [T] [F] The point given in spherical coordinates as $\rho = 3, \phi = 0, \theta = \pi$ is the same point as the point $\rho = 3, \phi = 0, \theta = 0$.

Solution:
It is the north pole.

9) [T] [F] The parametrized curve $\vec{r}(t) = \langle 0, 3\cos(t), 5\sin(t)\rangle$ is an ellipse.

Solution:
Indeed, and it is contained in the xz-plane.

10) [T] [F] The curvature of the line $\vec{r}(t) = \langle t, t, t\rangle$ is $\sqrt{3}$ everywhere.

Solution:
No, it is 0.

11) [T] [F] If $|\vec{v} \times \vec{w}| = \vec{v} \cdot \vec{w}$ then either $\vec{v}$ is parallel to $\vec{w}$ or perpendicular to $\vec{w}$.

Solution:
They can be unit vectors of angle 45 degrees for example

12) [T] [F] If the dot product between two unit vectors $\vec{v}, \vec{w}$ is -1, then $\vec{v} = -\vec{w}$.

Solution:
Yes, $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\alpha) = \cos(\alpha) = -1$ shows that $\alpha = \pi$.

13) [T] [F] Writing $\vec{k} = \langle 0, 0, 1\rangle$, we have $|(\vec{k} \times \vec{v}) \times \vec{w}| \leq |\vec{v}||\vec{w}|$ for all vectors $\vec{v}, \vec{w}$.

Solution:
Use the identity for the length of the cross product

14) [T] [F] The curvature of a curve $\vec{r}(t)$ is given by $\kappa(t) = |\vec{T}\,'(t)|/|\vec{r}\,'(t)|$. If $|\vec{r}\,'(t)| = 1$ for all times, then $\kappa(t) = |\vec{r}\,''(t)|$.

Solution:
Yes, under the assumption we have $\vec{T}(t) = \vec{r}\,'(t)$ and $\vec{T}' = \vec{r}\,''$ so that by definition of curvature we have $\kappa(t) = |\vec{r}\,''(t)|$.

15) [T] [F] The arc length of the curve $\langle \sin(t/2), 0, \cos(t/2)\rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

Solution:
We only trace half the circle.

16) [T] [F] If L, K are skew lines in space, there is a unique plane which is equidistant from L, K.

Solution:
Draw two planes through the lines which are spanned by vectors in L and K. These planes are parallel. There exists exactly one plane between. But there is a caveat. You can find many planes which intersect both lines and have distance zero to both. We graded both answers yes as we actually planned to assume the distance to be nonzero and the lines disjoint.

17) [T] [F] The curve $\vec{r}(t) = \langle t, t^2, 1 - t\rangle$ is the intersection curve of a plane $x + z = 1$ and $y = x^2$

Solution:
Just plug in

18) [T] [F] The lines $\vec{r}_1(t) = \langle 5 + t, 3 - t, 2 - t\rangle$ and $\vec{r}_2(t) = \langle 6 - t, 2 + t, 1 - 2t\rangle$ intersect at $(6, 2, 1)$ perpendicularly.

Solution:
First check that the point appears in both curves. Now compute the velocity vector.

19) [T] [F] The vector $\langle 3/13, 12/13, 4/13\rangle$ is a unit vector.

Solution:
Yes, its length is equal to 1.

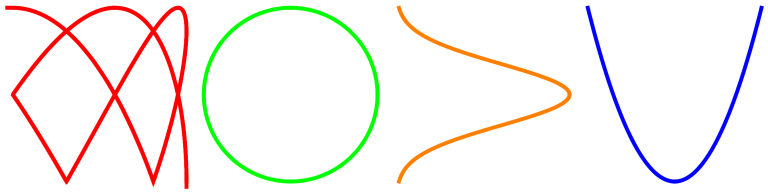
20) [T] [F] $\vec{v} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Solution:
A counter example is $u = i, v = j$.

[ ] Total

Problem 2) (10 points) No justifications are needed in this problem.

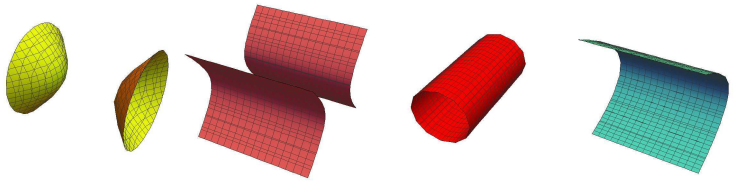
a) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match. In each of the problems a) - e), every of the entries O, I, II, III, IV appears exactly once.



I II III IV

$\vec{r}(t) =$	Enter O-IV
$\langle \exp(-t^2), t \rangle$	
$\langle \cos(t), \sin(t) \rangle$	
$\langle \lvert\cos(3t)\rvert, \lvert\sin(5t)\rvert \rangle$	
$\langle 2t, 3t \rangle$	
$\langle t, t^2 + 1 \rangle$	

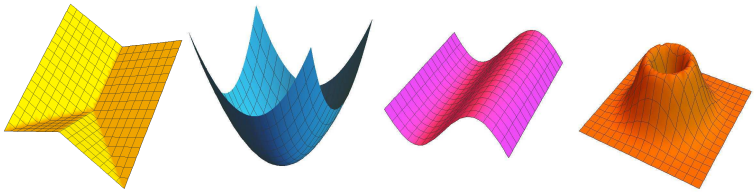
b) (2 points) Match the contour surfaces. Enter O, if there is no match.



I II III IV

$g(x, y, z) = 1$	Enter O-IV
$x + 2y = 1$	
$x^2 - y^2 - z^2 = 1$	
$z^2 + 2y = 1$	
$y^2 - z^2 = 1$	
$x^2 + z^2 = 1$	

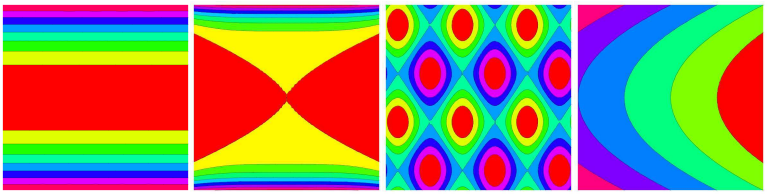
c) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.



I II III IV

$f(x, y) =$	Enter O-IV
$\exp(-x^2 - y^2)(x^2 + y^2)$	
$\sin(x)$	
$\exp(-x^2 - y^2)\sin(x^2)$	
$\lvert\lvert x\rvert + y\rvert$	
$x^2 + y^2$	

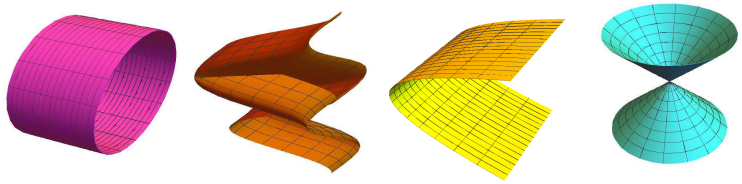
d) (2 points) Match functions $g(x, y)$ with contour maps. Enter O, if no match.



I II III IV

$g(x, y) =$	Enter O-IV
$\sin(3x) + \sin(2y)$	
$y^2 + x^2$	
$y^2 - 2x$	
y^2	
$y^6 - x^4$	

e) (2 points) Match the surface parametrization. Enter O, where is no match.



I II III IV

$\vec{r}(u, v) =$	Enter O,I-IV
$\langle u^2, v, u \rangle$	
$\langle u\cos(v), u\sin(v), u \rangle$	
$\langle \cos(u), \sin(v), u + v \rangle$	
$\langle v, \cos(u), \sin(u) \rangle$	
$\langle v, u, v \rangle$	

Solution:
a) III,II,I,O,IV
b) O,I,IV,II,III
c) IV,III,O,I,II
d) III,O,IV,I,II
e) III,IV,II,I,O

Problem 3) (10 points) No justifications are needed

In this problem $\vec{v}, \vec{w}$ are arbitrary vectors in space, $\vec{r}(t)$ is an arbitrary space curve.
The vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ are assumed to be nonzero where $\vec{N}$ is the normal vector and $\vec{B}$ the bi-normal vector.
All these vectors $\vec{r}, \vec{T}, \vec{B}, \vec{N}$ and its derivatives are evaluated at the fixed time $t = 0$.

first vector	second vector	always parallel	always perpendicular	depends
$\vec{r}\,'$	$\vec{r}\,''$			
$\vec{B}$	$\vec{N}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{v}$			
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			

Solution:

first vector	second vector	always parallel	always perpendicular	depends
$\vec{r}\,'$	$\vec{r}\,''$			X
$\vec{B}$	$\vec{N}$		X	
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$	X		
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			X
$\vec{v} \times \vec{w}$	$\vec{v}$		X	
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			X
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$	X		
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	X		
$\vec{T}$	$\vec{r}\,'$	X		
$\vec{T}$	$\vec{T}\,'$		X	

Problem 4) (10 points)

A parallelepiped has vertices at $A = (0,0,0), B = (1,1,1), C = (2,3,4)$ and $A' = (3,4,8)$ and contains the sides AB, AC and AA'.

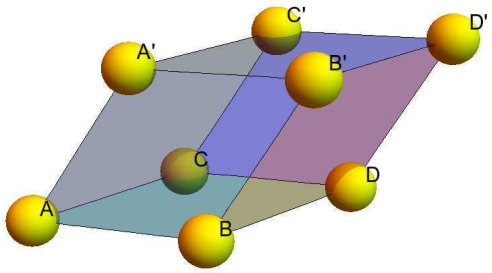
a) (2 points) Find a fourth point D so that A, B, C, D is a parallelogram.

b) (2 points) What is the area of that parallelogram $ABCD$?

c) (2 points) What is the volume of the parallelepiped?

d) (2 points) Find the height of the parallelepiped with floor $ABCD$ and roof A', B', C', D'.

e) (2 points) Find the distance between the face diagonals AD and $B'C'$.



Solution:

a) $\vec{A} + \vec{AB} + \vec{AC} = \langle 0,0,0\rangle + \langle 1,1,1\rangle + \langle 2,3,4\rangle = \boxed{\langle 3,4,5\rangle}$

b) $\vec{AB} \times \vec{AC} = \langle 1,1,1\rangle \times \langle 2,3,4\rangle = \langle 1,-2,1\rangle$. Its length is $\boxed{\sqrt{6}}$ which is the area.

c) $\vec{AA'} \cdot (\vec{AB} \times \vec{AC}) = \langle 3,4,8\rangle \cdot \langle 1,-2,1\rangle = 3$.

d) The height is Volume/area $= \boxed{3/\sqrt{6}}$.

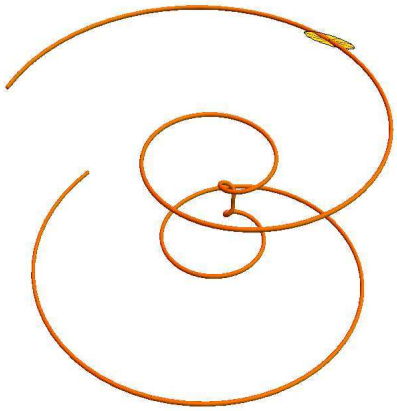
e) The distance is equal to the height and therefore also $\boxed{\sqrt{6}/3}$.

Problem 5) (10 points)

Autumn is here. A leaf tumbles down along the curve

$$\vec{r}(t) = \langle t^2 \cos(t), t^2 \sin(t), 16 - 2t \rangle$$

in space.



a) (3 points) What is the speed of the leaf at $t = \pi$?

b) (7 points) Find the arc length of the curve traced in the time interval $-8 \leq t \leq 8$.

Solution:
a) $\vec{r}\,'(t) = \langle 2t\cos(t) - t^2 \sin 9t), 2t\sin(t) + t^2\cos(t), -2\rangle$. The length $|\vec{r}\,'(t)| = \sqrt{(2+t^2)^2}$ simplifies to $2 + t^2$. At $t = \pi$ we have the speed $\boxed{2 + \pi^4}$.
b) $\int_{-8}^{8} 2 + t^2 = 2t + t^3/3|_{-8}^{8} = \boxed{1120/3}$.

Problem 6) (10 points)

On September 21, 2014, SpaceX launched a Dragon capsule with tons of supplies and experiments including a 3D printer to the space station. Assume the rocket experiences an acceleration



$$\vec{r}\,''(t) = \langle 2t, 0, 3t^2 - 5t^4 \rangle$$

starts at Cape Canaveral Air force station $\vec{r}(0) = \langle 2, 3, 0\rangle$ with zero velocity $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$.

a) (5 points) Where is the capsule at time $t = 1$?

b) (5 points) What is the curvature of the path at $t = 1$? You can use the formula $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$.

Solution:
a) Integrate twice and fix the initial condition

$$\vec{r}\,''(t) = \langle 2t, 0, 3t^2 - 5t^4 \rangle$$

$$\vec{r}\,'(t) = \langle t^2, 0, t^3 - t^5 \rangle$$

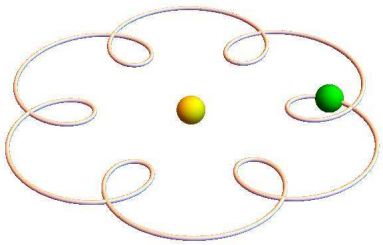
$$\vec{r}\,(t) = \langle t^3/3, 0, t^4/4 - t^6/6 \rangle + \langle 2, 3, 0 \rangle$$

At $t = 1$, this is $\langle 2 + 1/3, 3, 1/4 - 1/6 \rangle = \langle 7/3, 2, 1/12 \rangle$.
b) Compute the velocity and acceleration at $t = 1$ to get $\vec{r}\,'(1) = \langle 1, 0, 0 \rangle$ and $\vec{r}\,''(1) = \langle 2, 0, 2 \rangle$ which have the cross product $\langle 0, 2, 0 \rangle$. The curvature is $\kappa(1) = |\vec{r}\,'(1) \times \vec{r}\,''(1)|/|\vec{r}\,'(1)|^3 = 2$.

Problem 7) (10 points)

Before Kepler and Newton clarified planetary motion, there was the **Ptolemaic universe** which was based on the idea that planets move on epicycles like

$$\vec{r}(t) = \langle 3\cos(t) + \cos(7t), \sin(t) + \sin(7t), 3 \rangle \ .$$



a) (2 points) What is the velocity $\vec{v} = \vec{r}\,'(t)$ at $t = \pi$?

b) (2 points) What is the velocity $\vec{w} = \vec{r}\,'(t)$ at $t = \pi/2$?

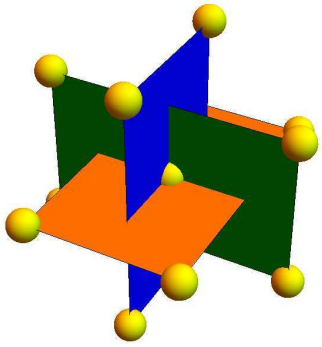
c) (2 points) Yes or no? Is $\vec{v} \times \vec{w}$ parallel to the binormal vector $\vec{B}(t)$ for all times t?

d) (4 points) Parametrize the line tangent to the curve at the point $A = \vec{r}(\pi)$.

Solution:
a) $\langle 0, -8, 0 \rangle$.
b) $\langle 4, 0, 0 \rangle$.
c) Yes, the binormal vector is perpendicular to the velocity vector and the acceleration. In this case, both the velocity and acceleration are in the xy plane.
d) $\vec{r}(\pi) = \langle -4, 0, 3 \rangle$. Now the parametrization is $\langle -4, 0, 3 \rangle + t\langle 0, -8, 0 \rangle = \langle -4, -8t, 3 \rangle$.

Problem 8) (10 points)

In this problem, the symbol φ is used to represent the golden ratio $\varphi = (\sqrt{5}+1)/2 \sim 1.618$ which satisfies the equation $\varphi^2 = \varphi + 1$.
The centers of four unit spheres are placed in the xy-plane at $A = (1, \varphi, 0), C = (-1, \varphi, 0), B = (1, -\varphi, 0)$ and $D = (-1, -\varphi, 0)$. 8 further points are located in the same way in the yz and xz plane so that we get 12 points which form the vertices of an **icosahedron** and surround a unit sphere centered at $(0, 0, 0)$.



a) (3 points) Consider the distances between the points A and B. Verify that the unit spheres centered at A and B do not intersect. Likewise, verify that the unit spheres centered at A and C do just intersect in a point.
b) (2 points) Using the concept of an icosahedron, explain why all 12 spheres either pairwise do not intersect or intersect in a point.

c) (3 points) The centers of all spheres have equal distance d from $(0, 0, 0)$. What is d in terms of φ?

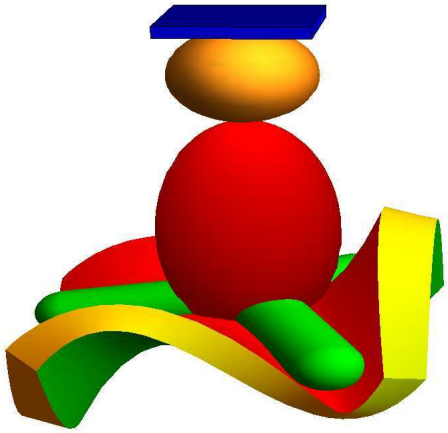
d) (2 points) Why does the central unit sphere intersect all other unit spheres?

Solution:
a) $d(A, B) = 2\varphi$ is larger than 2. We also have $d(A, C) = 2$. They just touch.
b) Symmetry. All other distances are the same. Either 2φ or 2.
c) $\sqrt{1+\varphi^2} = \sqrt{2+\varphi}$ is the distance of the center to the origin so that $\sqrt{1+\varphi^2} - 1$ is the distance of the center to the sphere.
d) The distance between the centers is smaller than 2 so that any of the 12 spheres intersects with the central sphere

Isaac Newton and **James Gregory** argued whether 13 unit spheres can be placed around a central unit sphere just "kissing the central sphere". They knew that 12 work. Newton believed 13 is impossible, but it was only proven in 1954 that the **kissing number** is 12. Here we have seen how to place 12 spheres: by pushing the 12 spheres constructed here a bit so that they just touch the central sphere, you showed that they have positive distance from each other and solve the 12 sphere kissing problem. It is known since 2003 that the kissing number in 4 dimensions is 24 but nobody has any clue what the kissing number in 5 dimensions is! It is only known that the answer is between 40 and 44.

Problem 9) (10 points)

As a souvenir for this exam, we build a Monkey riding a "Monkey saddle" and 3D print it. No explanations are necessary.



a) (2 points) Parametrize the hat $z = 5$.

b) (2 points) Parametrize the saddle $z = yx^2 - x^3$.

c) (2 points) Parametrize the torso $x^2 + y^2 + \frac{(z-1)^4}{4} = 1$.

d) (2 points) Parametrize the head $4x^2+y^2+(z-4)^2 = 1$.

e) (2 points) Parametrize the monkey tail $x^2 + z^2 = \frac{1}{4}$.

Solution:
In each case we have a function $\vec{r}$ of two variables.
a) $\langle x, y, 5\rangle$.
b) $\langle x, y, yx^2 - x^3\rangle$.
c) $\langle \cos(\theta)\sqrt{1 - \frac{(z-1)^4}{4}}, \sin(\theta)\sqrt{1 - \frac{(z-1)^4}{4}}, z\rangle$.
d) $\langle \cos(\theta)\sin(\phi)/2, \sin(\theta)\sin(\phi), 4 + \cos(\phi)\rangle$.
e) $\langle \cos(\theta)/2, y, \sin(\theta)/2\rangle$.

Problem 10) (10 points)



This June 2014, the Swiss extreme sports women **Géraldine Fasnacht** jumped with a wing-suit from the Matterhorn (a mountain in Switzerland). When flying with a wingsuit, there is the gravitational force, the force from the wind and a force from the wing. Assume

$$\vec{r}\,''(t) = \langle 1, t, \exp(-t) - 10\rangle$$

and $\vec{r}(0) = \langle 0, 0, 4500\rangle$ and $\vec{r}\,'(0) = \langle 0, 2, 0\rangle$. Find the path $\vec{r}(t)$.

Solution:
Integrate twice and compare initial conditions: $\vec{r}(t) = \langle t^2/2, t^3/6+2t, e^{-t}-5t^2+t+4499\rangle$.

9/27/2017 FIRST HOURLY PRACTICE 4 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The vector $\langle 0, 6/10, 8/10\rangle$ is a direction = unit vector.

Solution:
Yes, its length is equal to 1.

2) [T] [F] Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v} \times \vec{w} = \vec{0}$.

Solution:
yes

3) [T] [F] For any vectors $\vec{u}$ and $\vec{v}$, we must have $\vec{v} \cdot \text{Proj}_{\vec{u}}\vec{v} = \vec{u} \cdot \text{Proj}_{\vec{v}}\vec{u}$.

Solution:
The left hand side depends on the length of v, the right hand side not.

4) [T] [F] The plane parametrized by $\vec{r}(t, s) = \langle t, s, 1\rangle$ is the same as $z = 1$.

Solution:
Indeed

5) [T] [F] The surface $x^2 + y^2 - 2y - z^2 = 0$ is a cone.

Solution:
After completing the square there is a constant.

6) [T] [F] The curvature of the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ at any t is less than 1.

Solution:
The helix is less curved than the unit circle and the unit circle has curvature 1. The actual result 1/2.

7) [T] [F] The volume of a parallelepiped generated by the vectors $\vec{u}, \vec{v}, \vec{w}$ is equal to $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

Solution:
Yes, this is a basic fact

8) [T] [F] If a curve in space is parametrized by $\vec{r}(t)$ with $0 \leq t \leq 1$, then the same curve in the opposite direction can be parametrized by $\vec{r}(1-t)$ with $0 \leq t \leq 1$.

Solution:
The second parametrization reverses time but the curve is the same.

9) [T] [F] The two-sheeted hyperboloid $x^2+y^2-z^2 = -1$ separates space into regions. The points $(3, 4, 6)$ and $(5, 12, -14)$ lie in the same region.

Solution:
That was a tough cookie. One could think that because $g(x, y, z) = x^2 + y^2 - z^2$ has the same sign for both points that the regions are connected, but the regions are separated by a region in which the sign is different. Intuitively, one point is above the upper bowl, the second below the lower one.

10) [T] [F] Given two vectors $\vec{u}$ and $\vec{v}$ which are perpendicular. Then $\text{Proj}_{\vec{u}}(\text{Proj}_{\vec{v}}\vec{w}) = \vec{0}$ for any vector $\vec{w}$.

Solution:

11) [T] [F] The velocity vector $\vec{r}\,'(t)$ is always perpendicular to the curve.

Solution:
It is parallel to the curve

12) [T] [F] If a point P with cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) has the property that $r = \rho$, then it must be on the xy plane.

Solution:
Yes, $x^2 + y^2 = x^2 + y^2 + z^2$ implies that $z = 0$

13) [T] [F] The curvature of a circle of radius 3 is $1/3$.

Solution:
yes $1/3$.

14) [T] [F] The triple scalar product satisfies $\vec{u} \cdot (\vec{v} \times \vec{w}) \leq |\vec{u}||\vec{v}||\vec{w}|$.

Solution:
Use the formulas for the lengths.

15) [T] [F] If the dot product between two vectors is positive, then the two vectors form an acute angle.

Solution:
Yes, if $\cos(\alpha) > 0$, then this means that $\alpha < \pi/2$.

16) [T] [F] The surface given in cylindrical coordinates as $z^2 + r^2 = 1$ is a sphere.

Solution:
yes, it means $x^2 + y^2 + z^2 = 1$.

17) [T] [F] The arc length of the curve $\langle \sin(t), \cos(t) \rangle$ from $t = 0$ to $t = 1$ is equal to 1.

Solution:
Yes, the speed is equal to 1 so that the integral is 1.

18) [T] [F] The curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ hits the plane $z = 0$ at a right angle.

Solution:
The velocity vector is $-1, 0, 1$. This is not perpendicular to the plane.

19) [T] [F] The parametrized curve $\langle 0, 7\cos(1+t), 3\sin(1+t)\rangle$ is an ellipse.

Solution:
Indeed, it is part of the yz-plane.

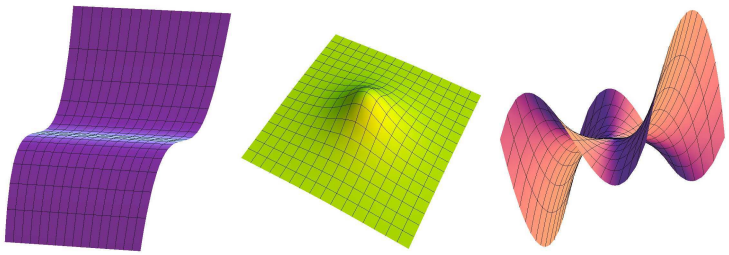
20) [T] [F] $\vec{u} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Solution:
Take $u = i, v = j$.

[ ] Total

Problem 2) (10 points) No justifications are needed here.

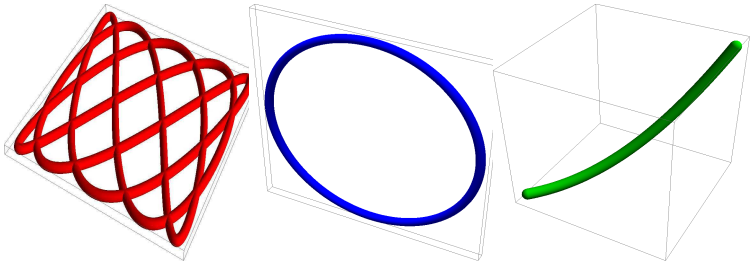
a) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.



I II III

Function $f(x,y) =$	Enter O,I,II or III
$x^3 - xy^2$	
y^3	
$1/(1+x^2+y^2)$	
$x^4 + y^4$	

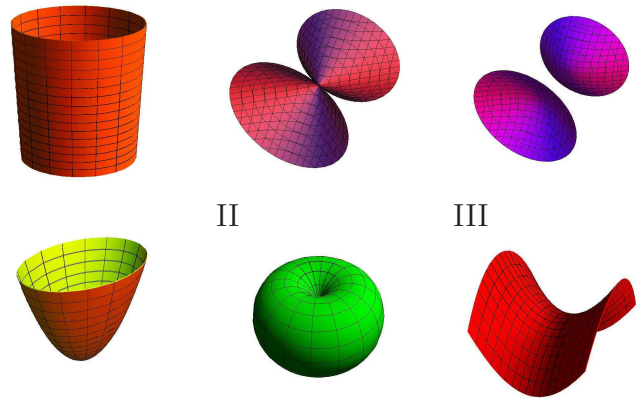
b) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\vec{r}(t) = \langle \cos(3t), \sin(5t), 0 \rangle$	
$\vec{r}(t) = \langle t, t, t^2 \rangle$	
$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$	
$\vec{r}(t) = \langle \sin(t), \sin(t), \sin(t) \rangle$	

c) (4 points) Match the surfaces to the pictures. There is an exact match here.

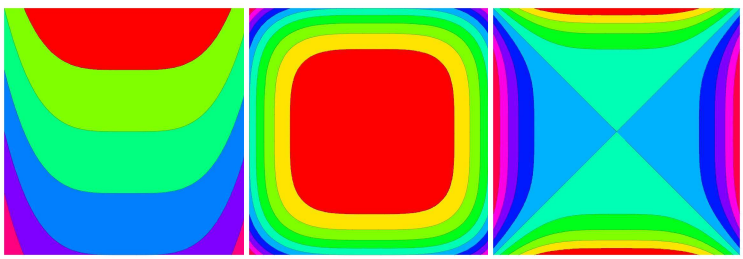


I II III

IV V VI

Description	I,II,III,IV,V,VI
$\langle 2u\cos(v), 4u\sin(v), u^2 \rangle$	
$\langle u^3, v^3, u^6 - v^6 \rangle$	
$\rho = \sin(\phi)$	
$r = 1$	
$x^2 - y^2 + z^2 = -1$	
$x^2 = y^2 - z^2$	

d) (2 points) Match the contour maps for $f(x,y)$. Enter O if no match.



I II III

function $f(x,y) =$	O,I,II,III
$f(x,y) = x^4 + y^4$	
$f(x,y) = x^4 - y^4$	
$f(x,y) = x - y$	
$f(x,y) = x^4 - y$	

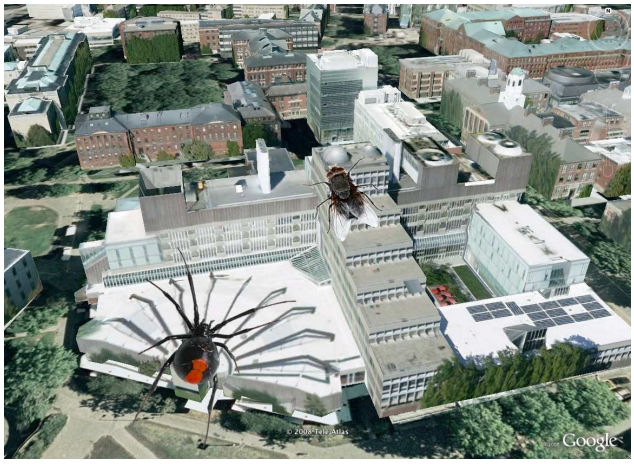
Solution:
III,I,II,0
I,III,II,0
IV,VI,V,I,III,II
II,III,O,I

Problem 3) (10 points)

The front roof line of the "spider" on the Harvard lecture halls forms a line

$$\vec{r}(t) = \langle 1 + t, 2 + t, 1 \rangle \ .$$

On top of the telescope sits a fly at the point $P = (0, 0, 10)$. Find the distance of P to the line.



Solution:
The line contains the point $Q = (1, 2, 1)$ and the vector $\vec{v} = \langle 1, 1, 0 \rangle$. We need $\vec{PQ} = \langle 1, 2, -9 \rangle$ and $\vec{v}$. We have $\vec{v} \times \vec{PQ} = \langle -9, 9, 1 \rangle$ which has length $\sqrt{163}$. The distance is $\boxed{\sqrt{163}/\sqrt{2}}$.

Problem 4) (10 points)

The kinect sensor can be used to scan objects. An infrared laser is used to measure distances in the horizontal plane.



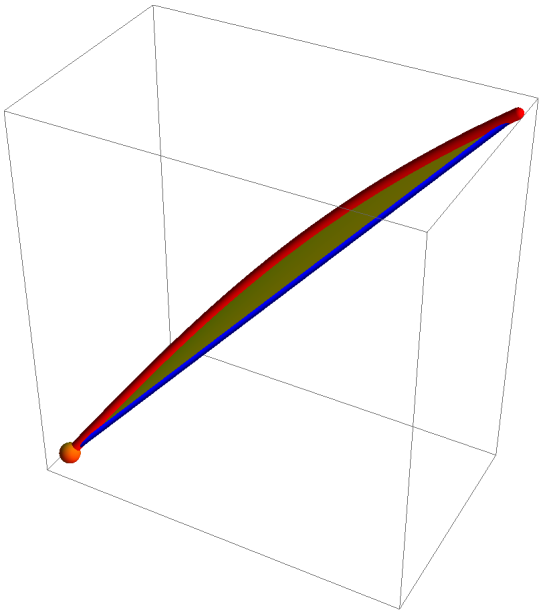
a) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 5 from the sensor $(0, -1)$.

b) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 4 from the sensor $(0, 1)$.

c) (6 points) Assume we know that P has distance 5 from $(0, -1)$ and distance 4 from $(0, 1)$. Where is this point (x, y) if we assume that it has a positive x-coordinate?

Solution:
a) $x^2 + (y + 1)^2 = 25$.
b) $x^2 + (y - 1)^2 = 16$.
c) Take both equations and subtract to get$(y+1)^2 - (y-1)^2 = 9$.This gies $y = 9/4$. From one of the equations we get $x = \sqrt{231}/4$.

Problem 5) (10 points)



a) (6 points) Given $\vec{r}(t) = \langle t + t^3/3, \arctan(t), \sqrt{2}t \rangle$. Find the arc length from $t = 0$ to $t = 1$.

b) (4 points) Compute the vector integral

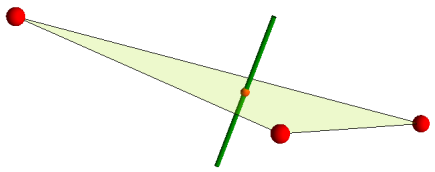
$$\int_0^1 \vec{r}\,'(t)\, dt$$

by integrating coordinate by coordinate. Verify that the length of this vector agrees with the arc length of the straight line connecting $\vec{r}(0)$ with $\vec{r}(1)$.

Solution:
a) $\vec{r}\,'(t) = \langle 1+t^2, 1/(1+t^2), \sqrt{2}\rangle$ so that $|\vec{r}\,'(t)| = 1 + t^2 + (1+t^2)^{-1}$. Integrating this from 0 to 1 gives $\boxed{4/3 + \pi/4}$.
b) If we integrate $\vec{r}\,'(t) = \langle 1+t^2, 1/(1+t^2), \sqrt{2}\rangle = \vec{r}(t)$, we get $\langle t + t^3/3, \arctan(t), \sqrt{2}t\rangle$ and evaluating this at $t = 0$ and $t = 1$. The length of this vector is just the distance of the two points.
Remark. This problem had the purpose to make you aware what the difference is between the arc length $\int_a^b |\vec{r}\,'(t)|\, dt$ and $|\int_a^b \vec{r}\,'(t)|$, which is the distance between the two end points. Of course the later is always smaller or equal than the former.

Problem 6) (10 points)

Given four points $A = (1,2,1), B = (1,0,1), C = (0,1,1), D = (1,1,2)$.



a) (4 points) Find an equation $ax + by + cz = d$ for the plane which contains A, B, C.
b) (3 points) Parametrize the line L which passes through D perpendicular to the plane ABC.
c) (3 points) Where does L hit the plane through A, B, C?

Solution:
a) We take the cross product of $\vec{AB} = \langle 0, -2, 0\rangle$, $\vec{AC} = \langle -1, -1, 0\rangle$ to get $\vec{AB} \times \vec{AC} = \langle 0, 0, -2\rangle$. The equation of the plane is $-2z = d$ where d is a constant. We can get d by pluggin in a point. We have $-2z = -2$ or $\boxed{z = 1}$.
b) $\vec{r}(t) = \langle 1, 1, 2\rangle + t\langle 0, 0, -2\rangle = \langle 1, 1, 2 - 2t\rangle$.
c) In order to get the intersection with the plane, we have to see where $2 - 2t = 1$, this gives $t = 1/2$. The point $\boxed{\vec{r}(1) = \langle 1, 1, 1\rangle}$ is the intersection point.

Problem 7) (10 points)

British stuntman **Gary Connery** made aviation history last year by becoming the first skydiver to land without parachute. He landed in 18000 boxes. Assume he started with an initial velocity $\langle 0, 100, 0\rangle$ from the initial point $\langle 0, 0, 800\rangle$. He was exposed to an acceleration $\vec{r}\,''(t) = \langle 0, 0, -10 + t\rangle$. Where is his location at time t=6?



Solution:
Start with the acceleration, then integrate to get the velocity, then integrate again to get the position. In each step we add a constant so that the setup is correct for $t = 0$, where the initial velocity and position are given.

$$\vec{r}\,''(t) = \langle 0, 0, -10 + t\rangle \ .$$

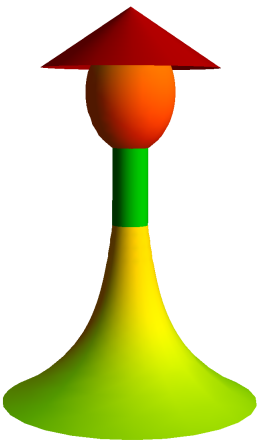
$$\vec{r}\,'(t) = \langle 0, 100, -10t + t^2/2\rangle \ .$$

$$\vec{r}\,(t) = \langle 0, 100t, 800 - 5t^2 + t^3/6\rangle \ .$$

Now plug in $t = 6$ to get $\boxed{\langle 0, 600, 656\rangle}$.

Problem 8) (10 points)

We parametrize the queen in a fancy chess set. It consists of 5 surfaces. Parametrize them. You do not have to give bounds for the parameters. In each case, just give an answer of the form $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ without further explanations.



a) (2 points) "hat" Cone $x^2 + y^2 = (1 - z)^2$.

b) (2 points) "head" Sphere $x^2 + y^2 + (z + 1/2)^2 = 1$.

c) (2 points) "neck" Cylinder $x^2 + y^2 = 1/4$.

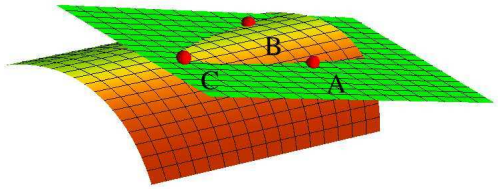
d) (2 points) "robe" Hyperboloid $x^2 + y^2 - (z + 4)^2 = 1$.

e) (2 points) "floor" Plane $z = -8$

Solution:
a) Since the relation between z and r is $r = 1-z$, we have $\langle (1-v)\cos(u), (1-v)\sin(u), v\rangle$.
b) We just have to shift the last coordinate a bit from the standard parametrization of the sphere: $\langle \sin(v)\cos(u), \sin(v)\sin(u), \cos(v) - 1/2\rangle$.
c) The cylinder has radius $1/2$ so that $\langle \cos(u)/2, \sin(u)/2, v\rangle$.
d) This is a surface of revolution. We just have to see that the equation tells us that $r^2 = 1 + (z+4)^2$, so that $\langle \sqrt{1+(v+4)^2}\cos(u), \sqrt{1+(v+4)^2}\sin(u), v\rangle$.
e) This is best done as a graph: $\langle u, v, -8\rangle$.

Problem 9) (10 points)

We are given a surface parametrized as $\vec{r}(u,v) = \langle u+v, u^2, v\rangle$.



a) (2 points) Locate the points $A = \vec{r}(1,2), B = \vec{r}(-1,2)$ and $C = \vec{r}(0,0)$.

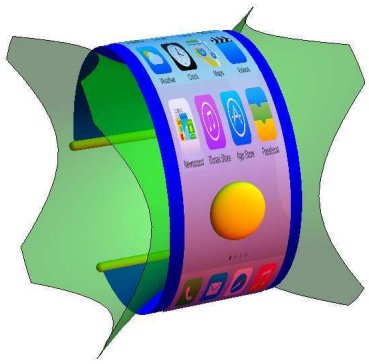
b) (4 points) Parametrize the plane through A, B, C.

c) (4 points) Find the area of the triangle with vertices A, B, C.

Solution:
a) $A = (3,1,2), B = (1,1,2), C = (0,0,0)$.
b) $\vec{r}(u,v) = \langle 3,1,2\rangle + t\langle 2,0,0\rangle + s\langle 3,1,2\rangle$.
c) $\langle 2,0,0\rangle \times \langle 3,1,2\rangle = \langle 0,-4,2\rangle$ which has length $2\sqrt{5}$. The answer is $\boxed{\sqrt{5}}$.

Problem 10) (10 points)

The reason for the name AppleWatch is that we can still hope for an **iWatch** which does not need pairing with a phone and which is waterproof. Simplicity is the rule: it consists of a band and "home button". As for fancy packaging, we strap it around a one sheeted hyperboloid. For each of the following surfaces, find a parametrization of the form



$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \ .$$

a) (3 points) "Band" cylinder

$$x^2 + y^2 = 9 \ .$$

b) (4 points) "Button" ellipsoid

$$10(x-3)^2 + y^2 + z^2 = 1/4 \ .$$

c) (3 points) "Package" 1-sheeted hyperboloid

$$x^2 + y^2 - z^2 - 1/2 = 0 \ .$$

Solution:

a) $\vec{r}(u,v) = \langle 3\cos(u), 3\sin(u), v \rangle$.

b) $\vec{r}(u,v) = \langle 3 + \frac{1}{\sqrt{40}}\cos(u)\sin(v), \frac{1}{2}\sin(u)\sin(v), \frac{1}{2}\cos(v) \rangle$.

c) $\vec{r}(u,v) = \langle \sqrt{v^2 + 1/2}\cos(u), \sqrt{v^2 + 1/2}\sin(u), v \rangle$.

9/27/2017 FIRST HOURLY PRACTICE 5 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] For

Solution:
Take $\vec{w} = -\vec{v}$, then the length of the sum is zero but the sum of the lengths is twice the length of $\vec{v}$.

2) [T] [F] There are unit vectors $\vec{v}$ and $\vec{w}$ in space for which $|\vec{v} \times \vec{w}| = 2$.

Solution:
Use the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$ to see that the length of the cross product is maximally 1.

3) [T] [F] The vector $\langle 4, 5, 0 \rangle$ is perpendicular to the plane $-5x + 4y + z = 2$.

Solution:
We know that the normal vector to $ax + by + cz = d$ is $\langle a, b, c \rangle$.

4) [T] [F] The distance between the cylinders $x^2 + z^2 = 1$ and $x^2 + (z-3)^2 = 1$ is 3.

Solution:
The two cylinders are parallel.

5) [T] [F] The vector projection of $\langle 2, 3, 1 \rangle$ onto $\langle 1, 1, 1 \rangle$ is parallel to $\langle 1, 1, 1 \rangle$.

Solution:
Yes by definition of the projection.

6) [T] [F] The equation $\rho \sin(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, we have $y = 2$.

7) [T] [F] There is a planar curve for which the arc length is 2π and the curvature is constant 1.

Solution:
The unit circle

8) [T] [F] If we know the intersection of a graph $z = f(x, y)$ with the coordinate planes $x = 0, y = 0$ and $z = 0$, the function f is determined uniquely.

Solution:
$f(x, y) = xy$ and $f(x, y) = x^2y^2$ have the same traces but the functions are different.

9) [T] [F] The surface given in cylindrical coordinates as $r = z^2$ is a paraboloid.

Solution:
The surface $z = r^2$ is a paraboloid. This surface is a "pointy shape".

10) [T] [F] If the curvature of a space curve is constant 2 everywhere along the curve then the curve is a circle.

Solution:
A spiral is a counter example.

11) [T] [F] If $\vec{u}, \vec{v}$, and $\vec{w}$ are unit vectors then the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$, and $\vec{w}$ is largest when the parallelepiped is a cube.

Solution:
Look at the formulas for the dot and cross product. The volume is $|u||v||w| \cos(\alpha) \sin(\beta)$ which is maximal if $\alpha = 0, \beta = \pi/2$.

12) [T] [F] If a point is moving along a straight line parametrized by $\vec{r}(t)$ then the velocity $\vec{r}\,'(t)$ vector and acceleration vector $\vec{r}\,''(t)$ must be parallel.

Solution:
The curvature is zero. Look at the curvature formula

13) [T] [F] The parametrization $\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v\rangle$ with $0 \leq u < 2\pi$ and $v \in \mathbf{R}$ is a cylinder.

Solution:
It is cone

14) [T] [F] If two lines in space are not parallel, then they must intersect.

Solution:
They can be skew.

15) [T] [F] If two planes do not intersect, then their normal vectors are parallel.

Solution:
Yes, otherwise, the cross product between the two normal vectors and a common point defines the intersection line.

16) [T] [F] $(\vec{i} \times \vec{j})$ and $(\vec{i} \times (\vec{i} \times (\vec{i} \times \vec{j})))$ are parallel.

Solution:
The latter is a negative scalar multiple of the former.

17) [T] [F] The surface parametrized by $\vec{r}(u,v) = \langle \sin(u)\sin(v), \sin(u)\cos(v), \cos(u)\rangle$ with $0 \leq v < 2\pi, 0 \leq u \leq \pi$ is a sphere.

Solution:
It looks as if something is false here, but it is the standard sphere. Just that u, v are switched.

18) [T] [F] The unit tangent vector $\vec{T}$ to a curve at a given point is independent of the parametrization up to a factor of -1.

Solution:
The curve at the point defines two unit tangent vectors. This is the only ambiguity.

19) [T] [F] $z^2 = r^2(\cos^2(\theta) - \sin^2(\theta)) + 1$ is a one-sheeted hyperboloid.

Solution:
Convert into cartesian coordinates to get $z^2 = x^2 - y^2 + 1$.

20) [T] [F] If $\vec{a} \cdot \vec{b} > 0$ and $\vec{b} \cdot \vec{c} > 0$, then $\vec{a} \cdot \vec{c} > 0$.

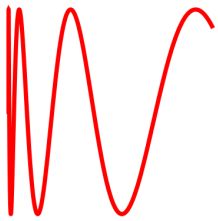
Solution:
Two acute angles can sum up to an obtuse angle.

[ ] Total

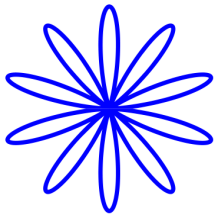
Problem 2) (10 points)

No explanations needed. I,II,III,O appear all once in each box.

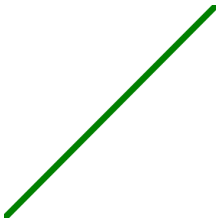
a) (2 points) Match curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I



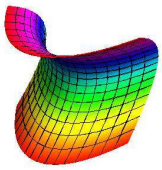
II



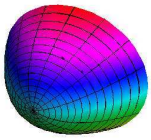
III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\sin^2(5t)\langle\cos(t), \sin(t)\rangle$	
$\langle t^3, \sin(7t)\rangle$	
$\langle t^5, 1 + t^5\rangle$	
$\langle\sin(t), \cos(t)\rangle$	

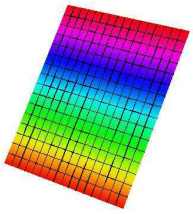
b) (2 points) Match the parametrization. Enter O, where no match.



I



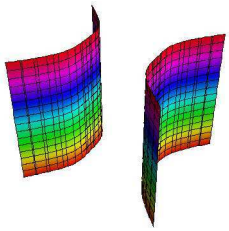
II



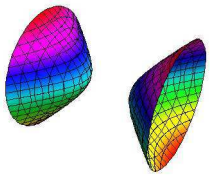
III

$\vec{r}(s,t)$	O,I,II,III
$\langle 1 - t, 1 + s, 2 + s\rangle$	
$\langle s, t^2 - s^2, t\rangle$	
$\langle t\cos(s), t\sin(s), s\rangle$	
$\langle s\cos(t), s^2, s\sin(t)\rangle$	

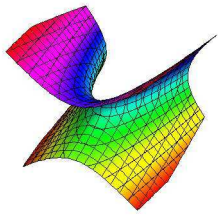
c) (2 points) The pictures show contour surfaces. Enter O, where no match.



I



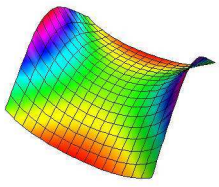
II



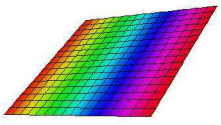
III

$g(x,y,z) =$	O, I,II,III
$x^2 - y^2 + z^2 = -1$	
$x^2 - y^2 = 1$	
$x^4 + z = 1$	
$x^2 + y - z^2 = 1$	

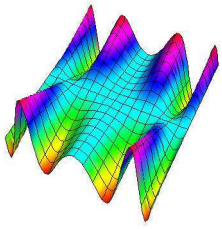
d) (2 points) Match the graphs $z = f(x,y)$ with the functions. Enter O, where no match.



I



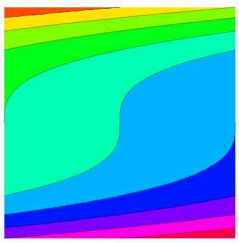
II



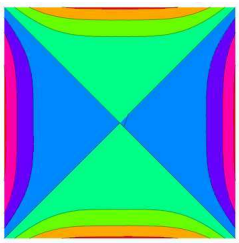
III

function $f(x,y) =$	O,I,II,III
$2x$	
$e^{-2x^2-2y^2}$	
$e^{x^2-y^2}$	
$y\sin(x^2)$	

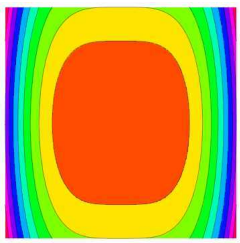
e) (2 points) Match the family of level curves with $f(x,y)$. Enter O, where no match.



I



II



III

Function $f(x,y) =$	O,I,II,III
$x^4 + y^2$	
$x^4 - y^4$	
$x - y$	
$x - y^3$	

Solution:
II,I,III,O
III,I,O,II
II,I,0,III
II,0,I,III
III,II,0,I

Problem 3) (10 points)

No explanations needed. In 3a), in each row check only one box.

a) (4 points) The intersection of a plane with a cone $S : x^2 + y^2 - z^2 = 0$ is called a **conic section**. What curve do we get?

Intersect S with	hyperbola	parabola	circle	line
$z = 1$ gives a				
$z = x$ gives a				
$z = x + 1$ gives a				
$x = 1$ gives a				

b) (3 points) By intersecting the upper half sphere $x^2 + y^2 + z^2 = 5, z > 0$ with the hyperboloid $x^2 + y^2 - z^2 = -3$ we get a curve. Which one? Check exactly one box.

$\vec{r}(t) = \langle \cos(t), \sin(t), 2 \rangle$	
$\vec{r}(t) = \langle 0, 0, t \rangle$	
$\vec{r}(t) = \langle \cos(t), \sin(t), 2t \rangle$	

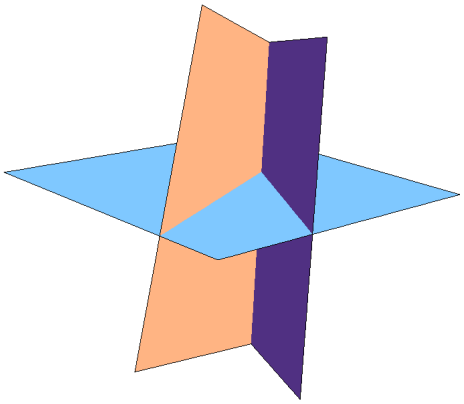
c) (3 points) Which of the following surface parametrizations gives a one sheeted hyperboloid? Check exactly one box.

$\vec{r}(t, s) = \langle s, t, s^2 - t^2 \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 + s^2} \cos(t), \sqrt{1 + s^2} \sin(t), s \rangle$	
$\vec{r}(t, s) = \langle \sqrt{1 - s^2} \cos(t), \sqrt{1 - s^2} \sin(t), s \rangle$	

Solution:
a) Circle, line, parabola, hyperbola. This constructions are important and give the etymology for the name "conic sections". We cut a cone with a plane and can get all the basic quadratic curves.
b) first box
c) second box.

Problem 4) (10 points)

We are given two planes $x+y+z=1$ and $x-y-z=2$. Find a third plane which contains the point $(1,0,0)$ and which is perpendicular to both.



Solution:
A normal vector to the third plane can be obtained by taking the cross product of the two normal vectors of the first two planes.

$$\langle 1,1,1\rangle \times \langle 1,-1,-1\rangle = \langle 0,2,-2\rangle \ .$$

The third plane has the equation $2y-2z=d$. In order that the point $(1,0,0)$ is there, the constant $d=0$. The equation is $2y-2z=0$ or $\boxed{y=z}$
A completely different solution was also valid: we can directly write down a parametrization of the plane

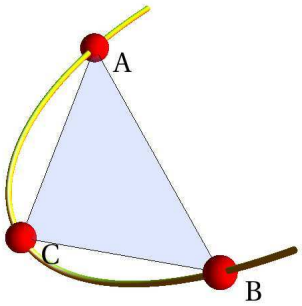
$$\vec{r}(s,t)\langle 1,0,0\rangle + s\langle 1,1,1\rangle + t\langle 1,-1,-1\rangle \ .$$

Problem 5) (10 points)

We are given a curve $\vec{r}(t) = \langle 1+t, t^2, t^3 \rangle$.

a) (5 points) Find the area of the triangle with vertices $A = \vec{r}(-1), B = \vec{r}(1)$ and $C = \vec{r}(0)$.

b) (5 points) Find an equation $ax + by + cz = d$ for the plane through A, B, C.

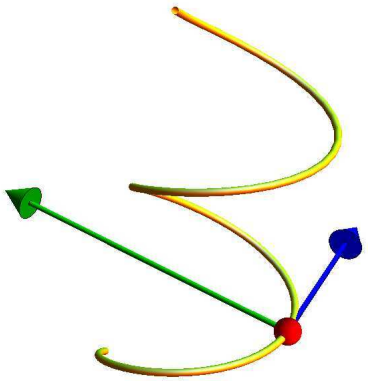


Solution:
a) $\vec{r}(0) = \langle 1, 0, 0 \rangle$ $\vec{r}(1) = \langle 2, 1, 1 \rangle$ $\vec{r}(-1) = \langle 0, 1, -1 \rangle$. We have $\vec{v} = \vec{r}(1) - \vec{r}(0)$ and $\vec{w} = \vec{r}(-1) - \vec{r}(0)$ and $\vec{n} = \vec{v} \times \vec{w} = \langle 1, 1, 1 \rangle \times \langle -1, 1, -1 \rangle = \langle -2, 0, 2 \rangle$. The area of the triangle is one half of the area of the parallelogram which is $\sqrt{4+4}/2 = 2\sqrt{2}/2 = \sqrt{2}$.
b) $-2x + 2z = d$. We get the constant by plugging in the point $(1, 0, 0)$. The solution is $-2x + 2z = -2$ or $\boxed{x - z = 1}$.

Problem 6) (10 points)

a) (3 points) Find the unit tangent vector $\vec{T}(t)$ of the curve $\vec{r}(t) = \langle t^2, \cos(t^2\pi), \sin(t^2\pi) \rangle$ at $t = 1$.
b) (3 points) What is the acceleration vector $\vec{r}\,''(t)$ at $t = 1$?
c) (4 points) Find the curvature at the time $t = 1$. You may use the formula

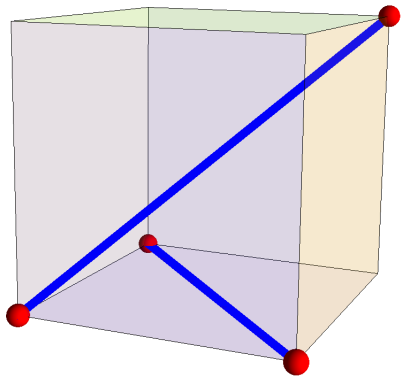
$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \ .$$



Solution:
a) $\vec{r}\,'(t) = \langle 2t, -2t\pi\sin(t^2\pi), 2t\pi\cos(t^2\pi)\rangle$ is $\vec{r}\,'(1) = \langle 2, 0, -2\pi\rangle$. We have $\boxed{\vec{T}(1) = \langle 1, 0, -\pi\rangle/\sqrt{1+\pi^2}}$.
b) We have $\vec{r}\,''(t) = \langle 2, -2\pi\sin(t^2\pi) - 4t^2\pi^2\cos(t^2\pi), 2\pi\cos(t^2\pi) - 4t^2\pi^2\sin(t^2\pi)\rangle$ and $\vec{r}\,''(1) = \boxed{\langle 2, 4\pi^2, -2\pi\rangle}$.
c) $|\langle 1, 0, \pi\rangle \times \langle 2, 4\pi^2, -2\pi\rangle|/(2^{3/2}(1+\pi^2)^{3/2})$ simplifies to $\boxed{\pi^2/(1+\pi^2)}$.
Bu the way, the figure to this problem as well as in the previous figure has been produced in Mathematica. The computer algebra system would by default not make such shiny curves. We computed the T,N,B frame using the formulas and use it to create a "tube" around the curve. The circles which make that tube use the N,B directions. This is an example of a situation, where the TNB frame is useful!

Problem 7) (10 points)

What's the closest that the long diagonal of the unit cube connecting the corners $(0,0,0)$ to $(1,1,1)$, comes to the diagonal of a face connecting the corners $(1,0,0)$ and $(0,1,0)$?



Solution:
This is a distance problem. $\vec{n} = \langle 1,1,1\rangle \times \langle 1,-1,0\rangle = \langle 1,1,-2\rangle$ and

$$d = |\langle 1,0,0\rangle \cdot \langle 1,1,-2\rangle|/|\langle 1,1,-2\rangle| = 1/\sqrt{6}\,.$$

The distance is $\boxed{1/\sqrt{6}}$.

Problem 8) (10 points)

In a parallel universe of ours, the inhabitants live under a "Newton's law" of gravity in which the "jerk" $\vec{r}\,'''(t)$ rather than the acceleration is constant. Suppose that $\vec{r}\,'''(t) = \langle 0, 0, -10\rangle$ for all t.



a) (3 points) Find $\vec{r}\,''(t)$ if you know $\vec{r}\,''(0) = \langle 0, 0, 0\rangle$.
b) (3 points) Now find $\vec{r}\,'(t)$ if we know also $\vec{r}\,'(0) = \langle 1, 0, 0\rangle$.
c) (4 points) Finally find $\vec{r}(t)$ if we know additionally $\vec{r}\,(0) = \langle 0, 0, 10\rangle$.

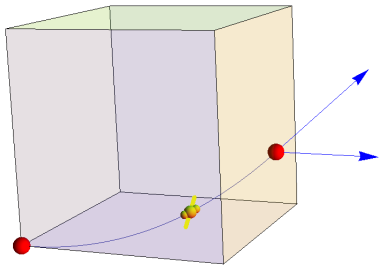
Solution:
Integrate

$$\vec{r}\,''(t) = \langle 0, 0, -10t\rangle$$
$$\vec{r}\,'(t) = \langle 0, 0, -5t^2\rangle + \langle 1, 0, 0\rangle$$
$$\vec{r}\,(t) = \langle 0, 0, -5t^3/3\rangle + \langle t, 0, 0\rangle + \langle 0, 0, 10\rangle = \langle t, 0, 10 - 5t^3/3\rangle\ .$$

Problem 9) (10 points)

a) (2 points) A fly is trapped inside a unit cubicle made of planar glass panes. It flies, starting at $t = 0$ at the origin $(0, 0, 0)$ along the curve

$$\vec{r}(t) = \langle t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\rangle\ .$$

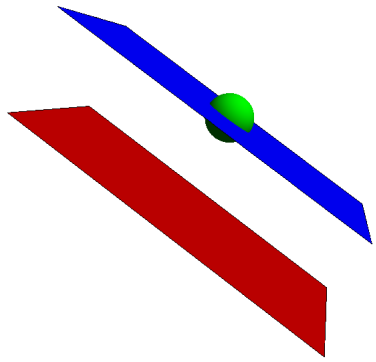


At what time does it bump into the glass wall $x = 1$?
b) (4 points) Find the impact angle (= the angle between the normal vector of the plane and the velocity vector).
c) (4 points) How long is the path it has followed from $t = 0$ to the impact point?

Solution:
a) Compare the first coordinates to get $\boxed{t = 1}$.
b) The velocity vector is $\langle 1, 2t/\sqrt{2}, t^2\rangle$. At time $t = 1$, it is $\boxed{\vec{v} = \langle 1, \sqrt{2}, 1\rangle}$. The normal vector to the plane is $\vec{n} = \langle 1, 0, 0\rangle$. The formula for the angle is $\vec{v} \cdot \vec{n}/(|\vec{n}||\vec{v}|) = 1/(1 \cdot 2)$. The angle satisfies $\cos(\theta) = 1/2$ which is $\boxed{\theta = \pi/3}$.
c) $\int_0^1 (1 + t^2)\ dt = 1 + 1/3 = \boxed{4/3}$.

Problem 10) (10 points)

When two uncharged metallic parallel plates are put close together, there is an attractive force between them which can be explained by quantum field theory only. In May 14, 2013, an article suggested to use this Casimir effect for microchip designs. (Source Nature: http://www.nature.com/ncomms/journal/v4/n5/full/nc



a) (3 points) Locate a point P on the plane $x + 2y + 2z = 4$.

b) (7 points) Find the distance d between the plane $x + 2y + 2z = 1$ and plane $x + 2y + 2z = 4$.

Solution:
a) There are many possibilities, $P = (4, 0, 0), (0, 2, 0), (0, 0, 2), (0, 1, 1)$ were popular choices.
b) Compute the distance of P to the plane. The second plane has a point $Q = (1, 0, 0)$. Now compute $d = |\vec{PQ} \cdot \vec{n}|/|\vec{n}| = 1$, where $\vec{n} = \langle 1, 2, 2 \rangle$. It is also possible to compute the distance as $|e - d|/|\vec{n}| = 3/3 = 1$ where $e = 4$ is the constant in the first plane and $d = 1$ is the constant in the second plane.

9/27/2017 FIRST HOURLY PRACTICE 6 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] If t
$a =$

Solution:
We can have $A = 2a, B = 2b, C = 2c, D = 2d$ for example.

2) [T] [F] The point $(x, y, z) = (1, 1, \sqrt{2})$ has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$.

Solution:
Use the transformation formula.

3) [T] [F] Every point on the parametric curve $\vec{r}(t) = \langle t, t^2, -t \rangle$ lies on the surface $xz + y = 0$.

Solution:
Check with $x = t, y = t^2, z = -t$.

4) [T] [F] The two surfaces $f(x, y, z) = 3$ and $f(x, y, z) = 5$ of the function $f(x, y, z) = 2x^2 + y^3 + z^4$ do not intersect at any point in space.

Solution:
The function is continuous so that level surfaces to different values can not intersect. The function would take two values at such a point.

5) [T] [F] $\vec{u} \times \vec{i}$ and $\vec{u} \times \vec{j}$ are perpendicular for all vectors $\vec{u}$.

Solution:
Take $\vec{u} = \vec{i} + \vec{j}$.

6) [T] [F] If $\vec{u}$ and $\vec{v}$ are parallel (remember that this means $\vec{u} = \lambda \vec{v}$ for some real λ) then $\vec{u} \cdot \vec{v} \geq |\vec{u} \times \vec{v}|$.

Solution:
We can have $\vec{u} = -\vec{v}$ in which case the left hand side is negative if $\vec{v}$ has positive length.

7) [T] [F] If a surface has the property that all intersections with the planes $y =$ *constant* are straight lines, then the surface is a plane.

Solution:
Take the function $y = x^2$ for example. Its graph is not a plane but $y =$ *constant* are lines.

8) [T] [F] For any non-zero vectors $\vec{u}$ and $\vec{w}$, we must have $\text{Proj}_{\vec{u}}\vec{w} = -\text{Proj}_{\vec{w}}\vec{u}$.

Solution:
The projection onto $\vec{u}$ is parallel to $\vec{u}$ and the projection onto $\vec{v}$ is parallel to $\vec{v}$.

9) [T] [F] In the parametric surface $\vec{r}(s,t) = \langle\, \sqrt{1+e^t}\cos(s)\,,\, \sqrt{1+e^t}\sin(s)\,,\, t\,\rangle$ the grid curves with constant s are ellipses.

Solution:
Take $s = 0$ to get the curve $\langle\, \sqrt{1+e^t}, 0, t\,\rangle$ which is the graph of a function of one variable in the xz plane.

10) [T] [F] There is a vector $\vec{v}$ with the property that $\vec{v} \times \langle 1,1,1\rangle = \langle 0,0,1\rangle$.

Solution:
Whatever the vector is, the right hand side would be perpendicular to $\langle 1,1,1\rangle$.

11) [T] [F] We can assign a value $f(0,0)$ such that the function $f(x,y) = (x^3+y^3)/(x^2+y^2)$ is continuous at $(0,0)$.

Solution:
Use polar coordinates to get $f = r^3(\cos^3(\theta)+\sin^3(\theta))/r^2 = r(\cos^3(\theta)+\sin^3(\theta))$.

12) [T] [F] The curvature of a curves $\vec{r}(t) = \langle t, t^2, t^3\rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6\rangle$ are the same at $t = 1$.

Solution:
Curvature is independent of the parametrization.

13) [T] [F] The curve given in spherical coordinates as $\phi = \pi/2, \rho = \pi/2$ is a circle.

Solution:
The result is zero

14) [T] [F] Two nonparallel planes with normal vectors $\vec{n}, \vec{m}$ intersect in a line parallel to $\vec{n} \times \vec{m}$.

Solution:
Make a picture. This is a way to compute the intersection line.

15) [T] [F] If $f(x, y) = x^3/3 - y^2$, then the graph of the function $f(x, y)$ is called an elliptic paraboloid.

Solution:
It is a hyperbolic paraboloid

16) [T] [F] The equation $\rho \cos(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, we have $x = \rho \cos(\theta) \sin(\phi)$.

17) [T] [F] The vector $\langle 3, -2 \rangle$ in the two dimensional plane is perpendicular to the line $3x - 2y = 7$.

Solution:
It is the gradient $\langle 1, 2 \rangle$.

18) [T] [F] The volume of the parallelepiped spanned by the vectors $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle$ and $\langle 1, 1, 1 \rangle$ is 2.

Solution:
Compute the triple scalar product which is 2.

19) [T] [F] If $\vec{r}(t)$ is a curve and $|\vec{r}\,'(t)| > 0$ and $|\vec{T}'| > 0$, we have $\vec{T}(t) \cdot (\vec{N}(t) \times \vec{B}(t)) = 1$.

Solution:
The three vectors are defined and are all perpendicular to each other and have length 1. They span a cube of volume 1.

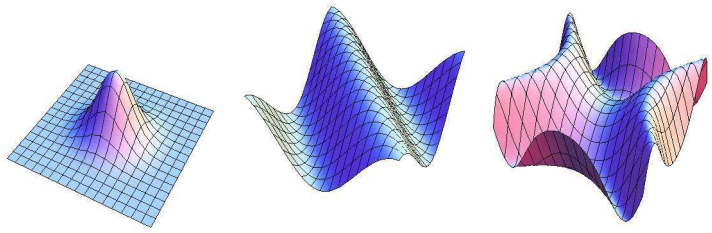
20) [T] [F] The arc lengths of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same for $0 \leq t \leq 1$.

Solution:
This is an important property of arc length.

[ ] Total

Problem 2) (10 points)

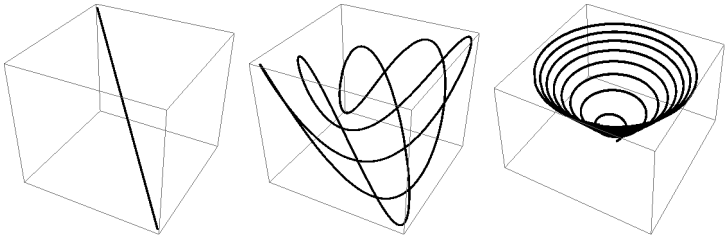
a) (2 points) Match the graphs $z = f(x, y)$ with the functions. Enter O, if there is no match. In each of the problems a) - d), each entry O,I,II,III appears exactly once.



I II III

Function $f(x, y) =$	O,I,II or III
$e^{-x^2-y^2}$	
$\cos(x + y)$	
$\sin(x^2 - y^2)$	
$x^4 + y^4$	

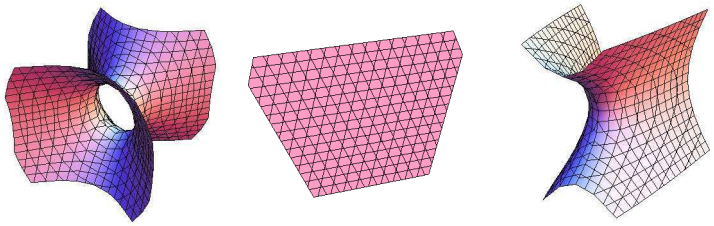
b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\langle 1 + t, 1 - t, t\rangle$	
$\langle t\cos(t^2), t\sin(t^2), t\rangle$	
$\langle t, t, \sin(t^3)\rangle$	
$\langle \cos(3t), \sin(2t), \sin(5t)\rangle$	

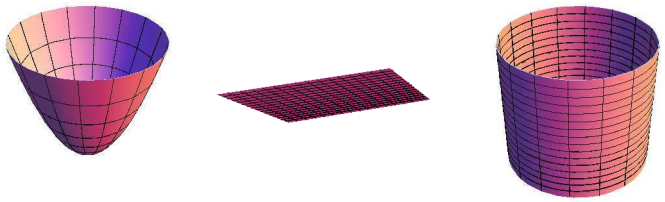
c) (2 points) Match the functions g with the level surface $g(x, y, z) = 1$. Enter O, where no match.



I II III

$g(x, y, z) =$	O, I,II,III
$(x - 1)^2 - y^2 + z^2 = 1$	
$(x - 1)^2 + y + z^2 = 1$	
$(x - 1) + y + z = 1$	
$(x - 1)^2 - y - z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.



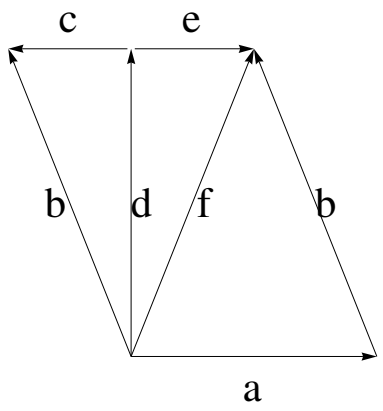
I II III

Parametriztion $\vec{r}(s, t) =$	O,I,II,III
$\langle s\cos(t), s\sin(t), s^2\rangle$	
$\langle t - 1, s, s + t\rangle$	
$\langle \cos(t), \sin(t), s\rangle$	
$\langle s\cos(t), s\sin(t), s^2\sin(t)\rangle$	

Solution:
a) I,II,III,O
b) I,III,O,II
c) I,O,II,III
d) I,II,III,O

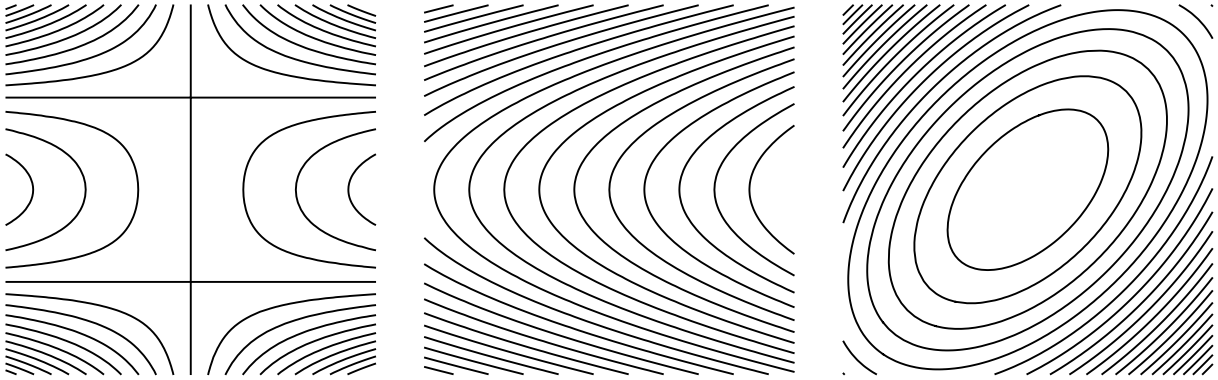
Problem 3) (10 points)

a) (7 points) Each of the vectors $a, b, c, d, e, f, 0$ (written without arrows for clarity) will appear in the blanks exactly once. As the picture indicates, you know $d \cdot e = d \cdot c = 0$.



the vector	is equal to
$\text{Proj}_d f$	
$f - d$	
$-2c$	
$d - c$	
$-e$	
$\text{Proj}_d e$	
$d + c$	

b) (3 points) Match the contour maps with the functions



I II III

Function $f(x,y) =$	Enter O,I,II or III
$y - x$	
$(y^2 - 1)x$	
$y^2 + x^2 - xy$	
$y^2 - x$	

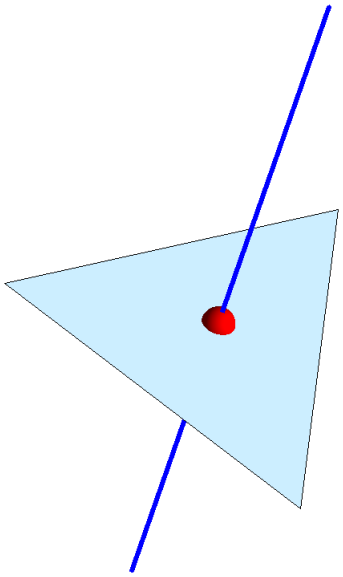
Solution:
Its obviously a "deaf cob":

the vector	is equal to
$\text{Proj}_{\vec{d}}\vec{f}$	$\vec{d}$
$\vec{f} - \vec{d}$	$\vec{e}$
$-2\vec{c}$	$\vec{a}$
$\vec{d} - \vec{c}$	$\vec{f}$
$-\vec{e}$	$\vec{c}$
$\text{Proj}_{\vec{d}}\vec{e}$	$\vec{o}$
$\vec{d} + \vec{c}$	$\vec{b}$

b) O,I,III,II

Problem 4) (10 points)

a) (4 points) The **center** of the triangle $A = (3,2,1), B = (1,1,1), C = (2,0,4)$ is the point $P = (A+B+C)/3 = (2,1,2)$. Find the line L perpendicular to the plane which contains A, B, C and which goes through P.



b) (3 points) Find the equation of the plane through A, B, C.

c) (3 points) Find the area of the triangle ABC.

Solution:
The vectors $\vec{BA} = \langle 2,1,0\rangle, \vec{BC} = \langle 1,-1,3\rangle$ are in the plane. Their cross product $\vec{n} = \langle 3,-6,-3\rangle$ gives the direction normal to the plane as well as the direction of the line.
a) The equation of the line is $\vec{OP} + t\vec{n} = \langle 2,1,2\rangle + t\langle 3,-6,-3\rangle$.
b) Since the plane contains the point $(1,1,1)$, the equation of the plane is $3x-6y-3z = -6$.
c) The area of the triangle is half the area of the parallelogram which is the length of the cross product divided by 2. This is $3\sqrt{6}/2$.

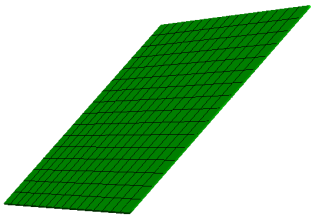
Problem 5) (10 points)

Complete the parametrizations:

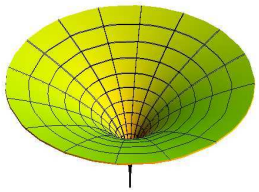
a) (3 points) $\vec{r}(u,v) = \langle 2 + 3\cos(u)\sin(v), 3 + \sin(u)\sin(v), \boxed{}\rangle$ parametrizes the ellipsoid $(x-2)^2/9 + (y-3)^2 + (z-5)^2/16 = 1$.



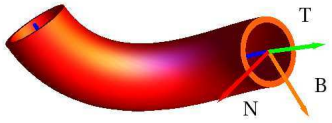
b) (2 points) $\vec{r}(u,v) = \langle u, v, \boxed{} \rangle$ parametrizes the plane $x + y + z = 1$.



c) (3 points) $\vec{r}(u,v) = \langle v^3 \cos(u), \boxed{}, v \rangle$ parametrizes the surface of revolution $x^2 + y^2 = z^6$.



d) (2 points) $\vec{r}(u,v) = \vec{r}(v) + \cos(u)\vec{N}(v) + \sin(u)\boxed{}$ parametrizes a tube around a curve $\vec{r}(v)$ which has unit tangent vector $\vec{T}(v)$, normal vector $\vec{N}(v)$ and binormal vector $\vec{B}(v)$.



Solution:

a) This is an ellipsoid centered at $(2,3,5)$ which is deformed. The last entry is $\boxed{5 + 4\cos(v)}$.

b) This is a plane. Solve for $z = 1 - x - y$ and use u, v to get $\boxed{1 - u - v}$.

c) This is a surface of revolution and we have $\boxed{v^3 \sin(u)}$.

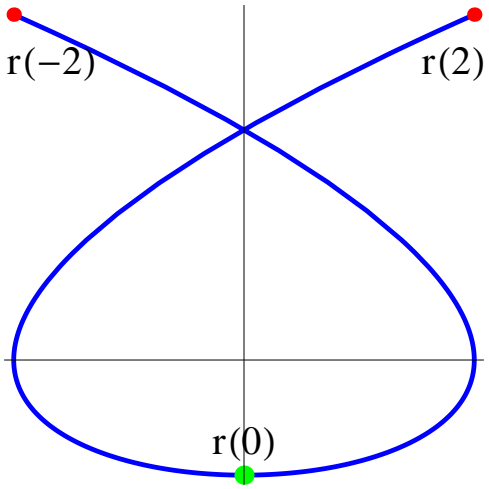
d) Since the tube circular and the grid curves with constant v are circles, we can use N, B to draw the circle. This is almost completed. W only have to fill in $\boxed{\vec{B}(v)}$. This example is actually one of the main motivations to use the TNB frame. It allows to draw beautiful tubes around a given curve.

Problem 6) (10 points)

We look at the parametrized curve

$$\vec{r}(t) = \langle \frac{t^3}{3} - t, t^2 - 1, 0 \rangle$$

whose image you see in the picture showing it in the xy plane for $-2 \leq t \leq 2$.



a) (3 points) Find the velocity $\vec{r}\,'(t)$, the acceleration $\vec{r}\,''(t)$ and speed $|\vec{r}\,'(t)|$.

b) (2 points) Evaluate this at $t = 0$ to get $\vec{r}\,'(0), \vec{r}\,''(0)$ and $|\vec{r}\,'(0)|$.

c) (2 points) Find the curvature $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$ at $(0, -1, 0)$.

d) (3 points) Find the arc length of the curve $\vec{r}(t)$ from $-2 \leq t \leq 2$.

Solution:
a) $\vec{r}\,'(t) = \langle t^2 - 1, 2t, 0 \rangle$. $\vec{r}\,''(t) = \langle 2t, 2, 0 \rangle$ and $|\langle t^2 - 1, 2t, 0 \rangle| = \sqrt{t^4 + 2t + 1} = \sqrt{(t^2+1)^2} = t^2 + 1$.
b) $\vec{r}\,'(0) = \langle -1, 0, 0 \rangle, \vec{r}\,''(0) = \langle 0, 2, 0 \rangle$ and $|\vec{r}\,'(0) = 1$.
c) Since the speed is 1, the curvature is

$$|\langle -1, 0, 0 \rangle \times \langle 0, 2, 0 \rangle| = |\langle 0, 0, -2 \rangle| = 2$$

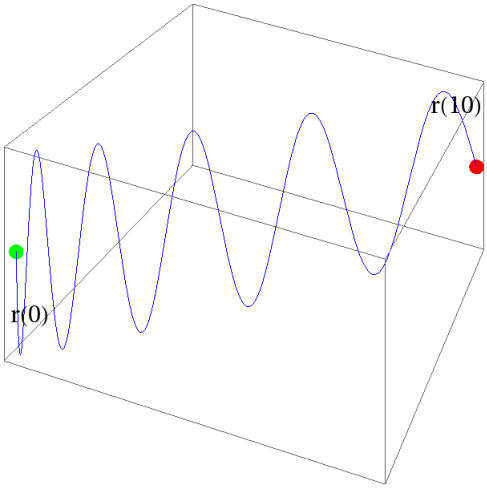
d) The arc length is $\int_{-2}^{2} |\vec{r}\,'(t)|\, dt = \int_{-2}^{2} t^2 + 1\, dt = t^3/3 + t|_{-2}^{2} = 28/3$.

Problem 7) (10 points)

a) (4 points) We know $\vec{r}\,''(t) = \langle 1, 2, \pi^2 \sin(\pi t)\rangle$ and the initial velocity $\vec{r}\,'(0) = \langle 1, 0, -\pi\rangle$. Find $\vec{r}\,'(t)$.

b) (3 points) Assume we know also $\vec{r}(0) = \langle 0, 0, 10\rangle$. Find $\vec{r}(10)$.

c) (3 points) What is the projection of $\vec{r}\,'(10)$ onto $\langle 1, 1, 0\rangle$?



Solution:
a) Integrate to get

$$\vec{r}\,'(t) = \langle t, 2t, -\pi\cos(\pi t)\rangle + \langle c_1, c_2, c_3\rangle \ .$$

Comparing the initial velocity gives the constants and so

$$\vec{r}\,'(t) = \langle 1 + t, 2t, -(\pi\cos(\pi t))\rangle \ .$$

b) Integrate again and compare coefficients to get

$$\vec{r}(t) = t + t^2/2, t^2, 10 - \sin(\pi t) \ .$$

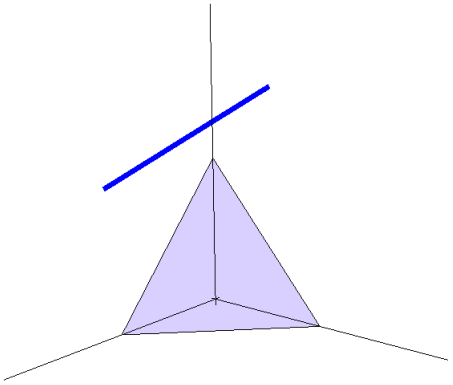
we have $r(10) = \langle 60, 100, 10\rangle$.
c) We have $\vec{r}\,'(10) = \langle 10, 20, -\pi\rangle$. The vector projection is $\langle 1, 1, 0\rangle = \vec{r}\,'(10) \cdot \langle 1, 1, 0\rangle/2 = 15\langle 1, 1, 0\rangle$

Problem 8) (10 points)

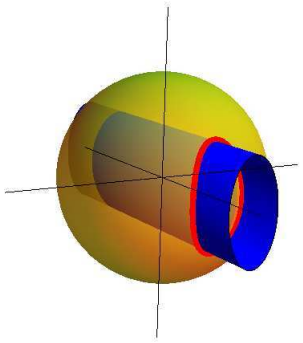
a) (5 points) Find the distance between the plane $x + y + z = 1$ and the line

$$x - 1 = \frac{(y-1)}{-2} = z - 1$$

which is parallel to the plane.
(You do not have to check that it is parallel).



b) (5 points) The intersection of the cylinder $4x^2+z^2 = 1$ with the sphere centered at $(0, 0, 0)$ with radius $\rho = \sqrt{2}$ cuts out two curves. Parametrize the curve which contains the point $(0, 1, 1)$.



Solution:
a) Choose a point on the plane $P = (1, 0, 0)$ and a point $Q = (1, 1, 1)$ on the line and compute the distance from P to the plane We have $\vec{PQ} = \langle 0, 1, 1\rangle$ and

$$d = \frac{|\langle 0, 1, 1\rangle \cdot \langle 1, 1, 1\rangle|}{|\langle 1, 1, 1\rangle|} = 2/\sqrt{3} \, .$$

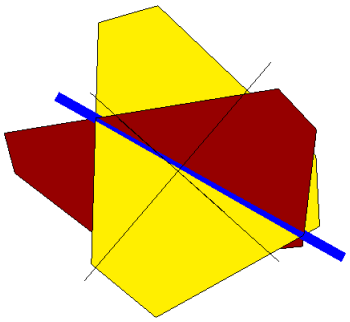
b) First parametrize the ellipse in the xz-plane with $\vec{r}(t) = \langle \cos(t)/2, ..., \sin(t)\rangle$ where we do not know y yet. The sphere has the equation $x^2 + y^2 + z^2 = 2$ and we can solve for y and get the parametrization

$$\vec{r}(t) = \langle \cos(t)/2, \sqrt{2 - \cos^2(t)/4 - \sin^2(t)}, \sin(t)\rangle \, .$$

Problem 9) (10 points)

a) (5 points) Find a parametrization of the intersection line L of the two planes

$$\begin{aligned} 2x - 2y + z &= 1 \, , \\ x + y + z &= 1 \, . \end{aligned}$$



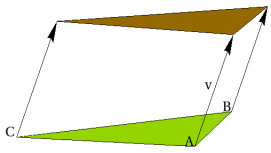
b) (5 points) Find a parametrization for the line M parallel to the line L computed in a) which passes through $(1, 2, 3)$.

Solution:
a) A point in the intersection of the plane is $P = (0, 0, 1)$. The cross product between the normal vectors is $\langle -3, -1, 4\rangle$. The parametrization of the line is $\vec{r}(t) = \langle 0, 0, 1\rangle + t\langle -3, -1, 4\rangle$.
b) Now translate the line through $(1, 2, 3$ to get $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle -3, -1, 4\rangle$.

Problem 10) (10 points)

a) (5 points) What is the area of the triangle through the points $A = (1, 1, 1)$ and $B = (0, 1, 0)$ and $C = (1, 2, 4)$.

b) (5 points) Find the volume of the prism which has the triangle T as base as well as a by $\vec{v} = \langle 0, 1, 1\rangle$ translated triangle as roof.



Solution:
a) It is half the length of

$$\vec{AB} \times \vec{AC} = \langle -1, 0, -1\rangle \times 0, 1, 3\rangle = \langle 1, 3, -1\rangle$$

which has length $\sqrt{11}$ so that the answer is $\boxed{\sqrt{11}/2}$.
b) The volum is half of the volume of the parallel epiped and therefore half of the absolute value of the triple scalar product

$$\vec{v} \cdot (\vec{AB} \times \vec{AC}) = \langle 0, 1, 1\rangle \cdot \langle 1, 3, -1\langle = 2$$

so that the answer is $\boxed{1}$.

9/27/2017 FIRST HOURLY PRACTICE 7 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

Solution:
It is the square root of $4^2 + 2^2 + 4^2$ which is 6.

2) [T] [F] Any three distinct points A, B, C in space determine a unique plane which passes through these points.

Solution:
The three points can be on a line.

3) [T] [F] For any two non-intersecting lines L, K, there are two parallel planes Σ, Δ whose distance $d(\Sigma, \Delta)$ is equal to the distance $d(L, K)$ such that L is in Σ and K is in Δ.

Solution:
The two planes are spanned by the vectors in the lines.

4) [T] [F] If $z - f(x, y) = g(x, y, z)$ then the graph of $f(x, y)$ is a level surface $g(x, y, z) = c$ of $g(x, y, z)$.

Solution:
It is the level surface to $c = 0$.

5) [T] [F] The graph of the function $f(x, y) = x^2 + y$ is called an elliptic paraboloid.

Solution:
It is a parabolic cylinder

6) [T] [F] The equation $\rho \sin(\phi) \sin(\theta) = 1$ in spherical coordinates defines a plane.

Solution:
Use spherical coordinates. We have $y = 1$.

7) [T] [F] The vector $\langle 1, 2, 3\rangle$ is parallel to the plane $2x + 4y + 6z = 4$.

Solution:
The vector $\langle 2, 4, 6\rangle$ is the normal vector.

8) [T] [F] The cross product between $\langle 2, 3, 1\rangle$ and $\langle 1, 1, 1\rangle$ is 6.

Solution:
It is the dot product which is 6. The cross product is a vector.

9) [T] [F] The curve $\vec{r}(t) = \langle \cos(t), t^2, \sin(t)\rangle, 1 \leq t \leq 9$ and the curve $\vec{r}(t) = \langle \cos(t^2), t^4, \sin(t^2)\rangle, 1 \leq t \leq 3$ have the same length.

Solution:
This is a change of parametrization

10) [T] [F] The point $(1, -1, 1)$ has the spherical coordinates of the form $(\rho, \theta, \phi) = (\sqrt{3}, \pi/4, \pi/4)$.

Solution:
Apply the transformation formulas. We have $\theta = -\pi/4$.

11) [T] [F] The distance between two parallel lines in space is the distance of any point on one line to the other line.

Solution:
Note that this is only true for parallel lines.

12) [T] [F] For two nonzero vectors $\vec{v}$ and $\vec{w}$, the identity $\text{Proj}_{\vec{w}}(\vec{v} \times \vec{w}) = \vec{0}$ holds.

Solution:
the vector $(\vec{v} \times \vec{w})$ is projected onto a vector perpendicular to it.

13) [T] [F] The vector projection of $\langle 2, 3, 4 \rangle$ onto $\langle 1, 0, 0 \rangle$ is $\langle 2, 0, 0 \rangle$.

Solution:
Apply the formula. Because the vector on which we project has length 1, the result is the dot product times this vector.

14) [T] [F] The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ between three vectors $\vec{u}, \vec{v}, \vec{w}$ is zero if and only if two or more of the 3 vectors are parallel.

Solution:
They can be nonparallel but in the same plane.

15) [T] [F] There are two vectors $\vec{v}$ and $\vec{w}$ so that the dot product $\vec{v} \cdot \vec{w}$ is equal to the length of the cross product $|\vec{v} \times \vec{w}|$.

Solution:
Take two vectors which make an angle of 45 degrees. Then $\sin(\theta) = \cos(\theta)$.

16) [T] [F] The distance between two spheres of radius 2 whose centers have distance 8 is 4.

Solution:
The connection between the centers is also the connection between the nearest points on the sphere.

17) [T] [F] If two vectors $\vec{v}$ and $\vec{w}$ are both parallel and perpendicular, then at least one of the vectors must be the zero vector.

Solution:
If $\vec{v} = \lambda \vec{w}$, then $\vec{v} \cdot \vec{w} = |\vec{v}|\vec{w}| = 0$ implies that one must be empty.

18) [T] [F] The curvature $\kappa(\vec{r}(t))$ is always smaller than or equal to the length $|\vec{r}\,''(t)|$ of the acceleration vector $\vec{r}\,''(t)$.

Solution:
If you drive along a circle very slowly the acceleration is small but the curvature is the same.

19) [T] [F] The curve $\vec{r}(t) = \langle \cos(t^2)\sin(t^2), \sin(t^2)\sin(t^2), \cos(t^2) \rangle$ is located on a sphere.

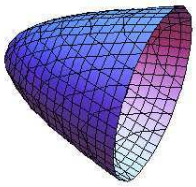
Solution:
Check $x^2 + y^2 + z^2 = 1$.

20) [T] [F] The surface $x^2 + y^2 + z^2 = 2z$ is a sphere.

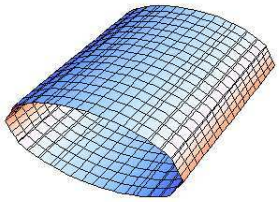
Solution:
Complete the square to see that it is indeed a sphere centered at $(0, 0, 1)$ with radius 1.

Problem 2) (10 points)

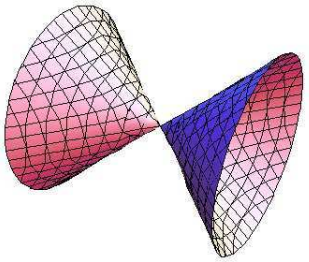
a) (6 points) Match the surfaces the equations $g(x, y, z) = 0$.



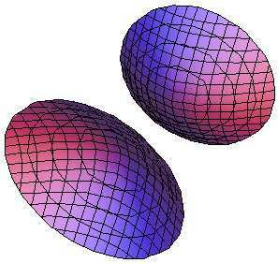
I



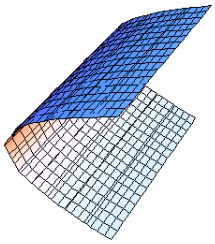
II



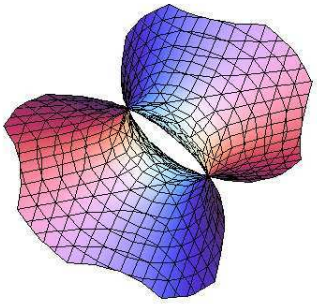
III



IV



V



VI

Function $g(x, y, z) = 0$	Enter I,II,III,IV,V,VI
$y^2 + z^2 - x$	
$y^2/4 + z^2/4 - 1$	
$x^2 - y^2 - z^2 - 1$	

Function $g(x, y, z) =$	Enter I,II,III,IV,V,VI
$x^2 - y^2 - z^2 + 1$	
$x - z^2$	
$y^2 - z^2 + x^2$	

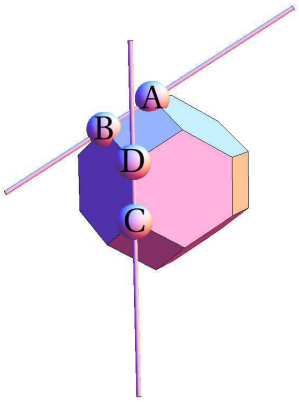
b) (4 points) Match the surfaces given in cylindrical and spherical coordinates with the surfaces given in Cartesian coordinates:

	surface
A	$r = 1$
B	$\sin(\theta) = 0$
C	$\cos(2\phi) = 0$
D	$\rho = 1$

Enter A-D	surface
	$x^2 + y^2 = 1$
	$x^2 + y^2 = z^2$
	$x^2 + y^2 + z^2 = 1$
	$y = 0$

Solution:
a) left: I II IV right: VI,V,III
b) ACDB

Problem 3) (10 points)



A **truncated octahedron** has an edge connecting the vertices $A = (-1, 3, 0), B = (-1, 1, -1)$ and an edge connecting the vertices $C = (-3, -1, 0), D = (-3, 1, 0)$.

a) (5 points) Find the distance of C to the line through A, B.

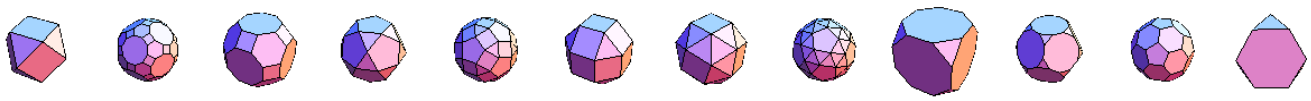
b) (5 points) Find the distance between the line L through A, B and the line K through C, D.

Solution:
a) These are typical distance formula problems: the first is a distance between a point and a line

$$d = \frac{|\vec{AC} \times \vec{AB}|}{|\vec{AB}|} = 6/\sqrt{5} \; .$$

b) The second problem asks for the distance between a line and a point:

$$d = \frac{|\vec{AC} \cdot (\vec{AB} \times \vec{CD})|}{|\vec{AB} \times \vec{CD}|}| = |-4|/2 = 2 \; .$$



Here are the remaining 12 **Archimedean solids**. These are polyhedra bound by different types of regular polygons but for which each vertex of the polyhedron looks the same. There are 13 such semiregular polyhedra. Archimedes studied them first in 287BC. Kepler was the first to describe the complete set of 13 in his work **"Harmonices Mundi"** of 1619.

Problem 4) (10 points)

a) (3 points) Give a parametrization $\vec{r}(\theta, z) = \langle x(\theta, z), y(\theta, z), z(\theta, z)\rangle$ of the surface which is in cylindrical coordinates given by

$$r = z^4 \ .$$

b) (2 points) Find a parametrization $\vec{r}(u, v)$ of the graph $z = \sin(xy)$.

c) (2 points) Find a parametrization $\vec{r}(u, v)$ of the yz-plane $x = 0$.

d) (3 points) Give a parametrization $\vec{r}(\phi, \theta)$ of the surface which is in spherical coordinates given by

$$\rho = 2 + \cos(13\phi) \ .$$

Solution:
a) This is a typical surface of revolution

$$\vec{r}(\theta, z) = \langle z^4 \cos(\theta), z^4 \sin(\theta), z\rangle \ .$$

b) This is a typical graph of a function of two variables:

$$\vec{r}(u, v) = \langle u, v, \sin(uv)\rangle$$

c) This is a typical plane

$$\vec{r}(u, v) = \langle 0, u, v\rangle$$

d) This is a typical modification of a sphere. It is called a bumpy sphere:

$$\vec{\rho}(\phi, \theta) = \langle (2 + \cos(13\phi)\sin(\phi))\cos(\theta), (2 + \cos(13\phi))\sin(\phi)\sin(\theta), (2 + \cos(13\phi))\cos(\phi)\rangle$$

Problem 5) (10 points)

a) (7 points) Find the arc length of the curve

$$\vec{r}(t) = \langle \cos(t^2/2), \sin(t^2/2), (1/3)(1 - t^2)^{3/2}\rangle$$

from $0 \leq t \leq 1$.

b) (3 points) Decide whether the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous.

Solution:
a) We have

$$\vec{r}\,'(t) = \langle -t\sin(t^2/2), t\cos(t^2/2), -t(1-t^2)^{1/2}\rangle$$

so that the speed is

$$|\vec{r}(t)| = t\sqrt{2-t^2}$$

and

$$\int_0^1 |\vec{r}(t)|\, dt = \int_0^1 t\sqrt{2-t^2}\, dt = -(1/3)(2-t^2)^{3/2}|_0^1 = (2\sqrt{2}-1)/3$$

b) Yes, the function is continuous. It is obviously continuous everywhere except the origin. To investigate the origin, us polar coordinates. We have

$$f(r,\theta) = r\cos(\theta)\sin^2(\theta)\ .$$

Since $\cos(\theta)\sin^2(\theta)$ stays bounded as $r \to 0$, the value of f approaches 0 as we approach the origin.

Problem 6) (10 points)



Wall-e explores a planet, where a strong solar wind produces a time-dependent magnetic field and where the combined force of gravity and magnetic lift produces a time dependent vertical acceleration

$$\vec{r}\,''(t) = \langle 0, 0, 10\cos(t)\rangle\ .$$

a) (6 points) Wall-e knows that he is at time $t = 0$ at $\vec{r}(0) = \langle 1, 2, 3\rangle$ with velocity $\langle 0, 1, 2\rangle$. Where is he at time $t = \pi$?

b) (4 points) What speed does he have at time $t = \pi$?

Solution:
a) Integrate up the equation

$$\vec{r}\,''(t) = \langle 0, 0, 10\cos(t)\rangle$$

to get

$$\vec{r}\,'(t) = \langle 0, 0, 10\sin(t)\rangle + \langle 0, 1, 2\rangle$$

The constant part to the right was obtained by comparing with $\vec{r}\,'(0)$.
Now integrate again:

$$\vec{r}\,(t) = \langle 0, 0, -10\cos(t)\rangle + \langle 0, t, 2t\rangle + \langle 1, 2, 3\rangle\ ,$$

where the constant part to the right was obtained by comparing with $\vec{r}(0)$. At time $t = \pi$, he is at

$$\vec{r}(\pi) = \langle 0, 0, 10\rangle + \langle 0, \pi, 2\pi\rangle + \langle 1, 2, 13\rangle = \langle 1, 2+\pi, 2\pi+23\rangle\ .$$

b) We obtain the speed, the length of the velocity vector $\vec{r}\,'(t)$ at $t = \pi$ as follows:

$$\vec{r}\,'(\pi) = |\langle 0, 1, 2\rangle| = \sqrt{5}\ .$$

Problem 7) (10 points)



Potter plays Quidditch. At time $t = 0$ he is at $P = (1, 3, 5)$. At time $t = 1$ he is at $Q = (0, 1, 3)$. Harry is spell-bound and can not change direction, nor change speed and crashes into a tilted side wall of the stadium crushing his knee (*).

a) (3 points) If Potter flies on a straight line through PQ, find a parametrization for that line.

b) (4 points) Where and when does he hit the tilted side wall $x + y + z = 1$ of the stadium?

c) (3 points) What is the angle between Harry's velocity vector and the upwards pointing normal vector of the side wall?

(*) Don't worry, Madam Pomfrey will fix it.

Solution:
a) $\vec{r}(t) = \langle 1, 3, 5\rangle + t\langle -1, -2, -2\rangle = \langle 1 - t, 3 - 2t, 5 - 2t\rangle$.
b) We look for the time t, where $x(t)+y(t)+z(t) = 1$. This is $(1-t)+(3-2t)+(5-2t) = 9 - 5t = 1$. which gives $t = 8/5$. Now $\vec{r}(8/5) = \langle -3, -1, 9\rangle/5$.
c) The velocity vector is $\langle -1, -2, -2\rangle$. The upwards pointing normal vector to the side wall is the vector $\langle 1, 1, 1\rangle$. The dot product is -5 and the cos of the angle is $-5/(3\sqrt{3})$. The angle is $\arccos(-5/(3\sqrt{3}))$.

Problem 8) (10 points)

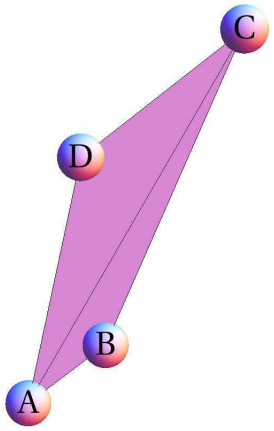
No justifications are needed in this problem. All vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ can be assumed to be nonzero. The vector $\vec{N} = \vec{T}'/|\vec{T}'|$ is the normal vector and $\vec{B}$ is the binormal vector. Recall that two vectors are perpendicular, if and only if their dot product is zero and that two vectors are parallel if and only if their cross product is the zero vector.

first vector	second vector	always parallel	always perpendicular	neither
$\vec{T}'(0)$	$\vec{r}\,''(0)$			
$\vec{w}$	$(\vec{v} \times \vec{w}) \times \vec{v}$			
$\vec{v}$	$\vec{v}/\lvert\vec{v}\rvert$			
$\vec{v} = \langle a, b, c\rangle$	normal to $ax + by + cz = 4$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{B}$	$\vec{N}$			

Solution:

first vector	second vector	always parallel	always perpendicular	neither
$\vec{T}'(0)$	$\vec{r}''(0)$			*
$\vec{w}$	$(\vec{v} \times \vec{w}) \times \vec{v}$			*
$\vec{v}$	$\vec{v}/\lvert\vec{v}\rvert$	*		
$\vec{v} = \langle a, b, c \rangle$	normal to $ax + by + cz = 4$	*		
$\vec{T}$	$\vec{r}\,'$	*		
$\vec{T}$	$\vec{T}\,'$		*	
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$	*		
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			*
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	*		
$\vec{B}$	$\vec{N}$		*	

Problem 9) (10 points)



The four points $A = (0, 0, 5), B = (1, 1, 6), C = (2, 4, 11), D = (0, 2, 9)$ are in a plane.

a) (5 points) Find the equation $ax + by + cz = d$ for this plane.

b) (5 points) The quadrilateral $ABCD$ is the union of two triangles ABC and ACD. Find the area of the quadrilateral.

Solution:
a) To get the equation of the plane, we find the normal vector

$$\vec{n} = \vec{AB} \times \vec{AC} = \langle 2, -4, 2 \rangle \ .$$

The equation is $2x - 4y + 2z = d$, where d can be obtained by plugging in one point. It is 10. The plane is $x - 2y + z = 5$.
b) The first triangle area is half the length of the vector $\vec{AB} \times \vec{AC}$ which is $\sqrt{6}$. The second triangle area is half the length of the vector $\vec{AC} \times \vec{AD}$ which is half the length of $\langle 4, 8, 4 \rangle$ and which is $2\sqrt{6}$. The quadrilateral therefore has the area $3\sqrt{6} = \sqrt{54}$.

Problem 10) (10 points)

In this problem we find some parametrizations of surfaces which is of the form

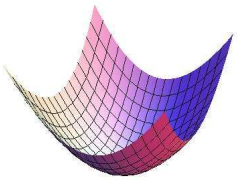
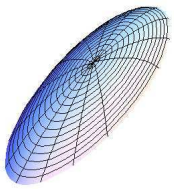
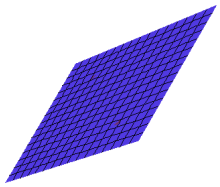
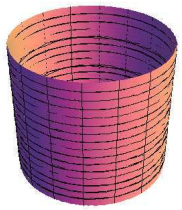
$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \ .$$

a) (2 points) Parametrize the **paraboloid** $z = x^2 - y^2$.

b) (3 points) Parametrize (the entire!) **ellipsoid** $(x-1)^2 + \frac{(y-2)^2}{4} + z^2 = 1$.

c) (2 points) Parametrize the **plane** $x + y + z = 3$.

d) (3 points) Parametrize the **cylinder** $x^2 + z^2 = 1$.

Solution:
a) $\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$. This is a graph.
b) $\vec{r}(u, v) = \langle 1 + \cos(u)\sin(v), 2 + 2\sin(u)\sin(v), \cos(v) \rangle$. In this problem, a graph parametrization like $\langle u, v, f(u, v) \rangle$ would not give the entire ellipsoid.
c) $\vec{r}(u, v) = \langle 3, 0, 0 \rangle + u\langle 3, -3, 0 \rangle + v\langle 3, 0, -3 \rangle$. There are of course many possibilities here. An other simple solution is $\langle u, v, 3 - u - v \rangle$.
d) $\vec{r}(u, v) = \langle \cos(u), v, \sin(u) \rangle$. In this problem, the order was often incorrect and the standard cylinder $\langle \cos(u), \sin(u), v \rangle$ along the z-axes taken.

9/27/2017 FIRST HOURLY PRACTICE 8 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

Solution:
Complete the square: add 1 on both sides, to get $x^2+y^2+(z+1)^2=1$ which is a sphere.

2) [T] [F] The length of the vector $\langle 1,2,2\rangle$ is an integer.

Solution:
Indeed, $1+4+4=9$ is a perfect square.

3) [T] [F] The vector $\langle 3,4\rangle$ appears as a velocity vector of the curve $\vec{r}(t) = \langle \cos(5t), \sin(5t)\rangle$. Namely, there is a t such that $\vec{r}\,'(t) = \langle 3,4\rangle$.

Solution:
The velocity vector of that curve has length 5 and takes any possible direction, so also the direction of the vector $\langle 3,4\rangle$.

4) [T] [F] If $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the unit normal vector, and $\vec{B}$ is the binormal vector, then $\vec{B}\times\vec{N}=\vec{T}$.

Solution:
The sign is wrong.

5) [T] [F] The curvature of a larger circle $r=2$ is greater than the curvature of a smaller circle $r=1/2$.

Solution:
False. The curvature of a circle of radius r is 1/r.

6) [T] [F] The surface $x^2-y^2-z^2-1=0$ is a one sheeted hyperboloid.

Solution:
It is a two sheeted one. Make a completion of a square

7) [T] [F] The function $f(x, y) = y^2 - x^2$ has a graph that is an elliptic paraboloid.

Solution:
It is a hyperbolic paraboloid.

8) [T] [F] Let $\vec{r}(t)$ be a parametrization of a curve. If $\vec{r}(t)$ is always parallel to the tangent vector $\vec{r}\,'(t)$, then the curve is part of a line through the origin.

Solution:
$\vec{T}\,'$ is zero at all times, so since $\vec{r}\,'(t)$ is parallel to the position vector $\vec{r}(t)$, the curve must lie on a line through the origin.

9) [T] [F] If $\text{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\vec{u}$ is the zero vector.

Solution:
Take $\vec{u} = \vec{i}$.

10) [T] [F] If $\text{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\text{Proj}_{\vec{k}}(\vec{u})$ is the zero vector.

Solution:
Assume $\mathbf{Proj}_{\vec{k}}(\vec{u})$ is a nonzero vector in the direction of $\vec{k}$. This means that $\vec{k}$ is perpendicular to $\vec{u}$, so $\text{Proj}_{\vec{k}}(\vec{u}) = \vec{0}$ by the definition of the projection.

11) [T] [F] If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

Solution:
The two vectors can be parallel.

12) [T] [F] There are two vectors $\vec{a}$ and $\vec{b}$ such that the scalar projection of $\vec{a}$ onto $\vec{b}$ is 100 times the magnitude of $\vec{b}$.

Solution:
Take $\vec{b} = 100\vec{a}$.

13) [T] [F] The curve $\vec{r}(t) = \langle \cos(t), e^t + 10, t^2 \rangle, 2 \leq t \leq 6$ and the curve $\vec{r}(t) = \langle \cos(2t), e^{2t}, 4t^2 \rangle, 1 \leq t \leq 3$ have the same length.

Solution:
Make a change of variables.

14) [T] [F] The equation $\rho \sin(\phi) - 2\sin(\theta) = 0$ in spherical coordinates defines a two sheeted hyperboloid.

Solution:
The equation means $r = 2\sin(\theta)$ or $x^2 + y^2 = 2y$.

15) [T] [F] If triple scalar product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is larger than $|\vec{u} \times \vec{v}|$ then $|\vec{w}| > 1$.

Solution:
Think of the triple scalar product as the volume. If that is larger than the base area, the height has to be bigger than than 1.

16) [T] [F] The distance between the x-axis and the line $x = y = 1$ is $\sqrt{2}$.

Solution:
The distance is 1. The distance between $x = y = 1$ and the z-axis would be $\sqrt{2}$.

17) [T] [F] The vector $\langle -1, 2, 3 \rangle$ is perpendicular to the plane $x - 2y - 3z = 9$.

Solution:
It is. Because $\langle 1, -2, -3 \rangle$ is perpendicular to the plane.

18) [T] [F] The curve $\vec{r}(t) = t^3\langle 1, 2, 3\rangle$ is a line.

Solution:
Even so t appears not in a linear way, the curve is still a line.

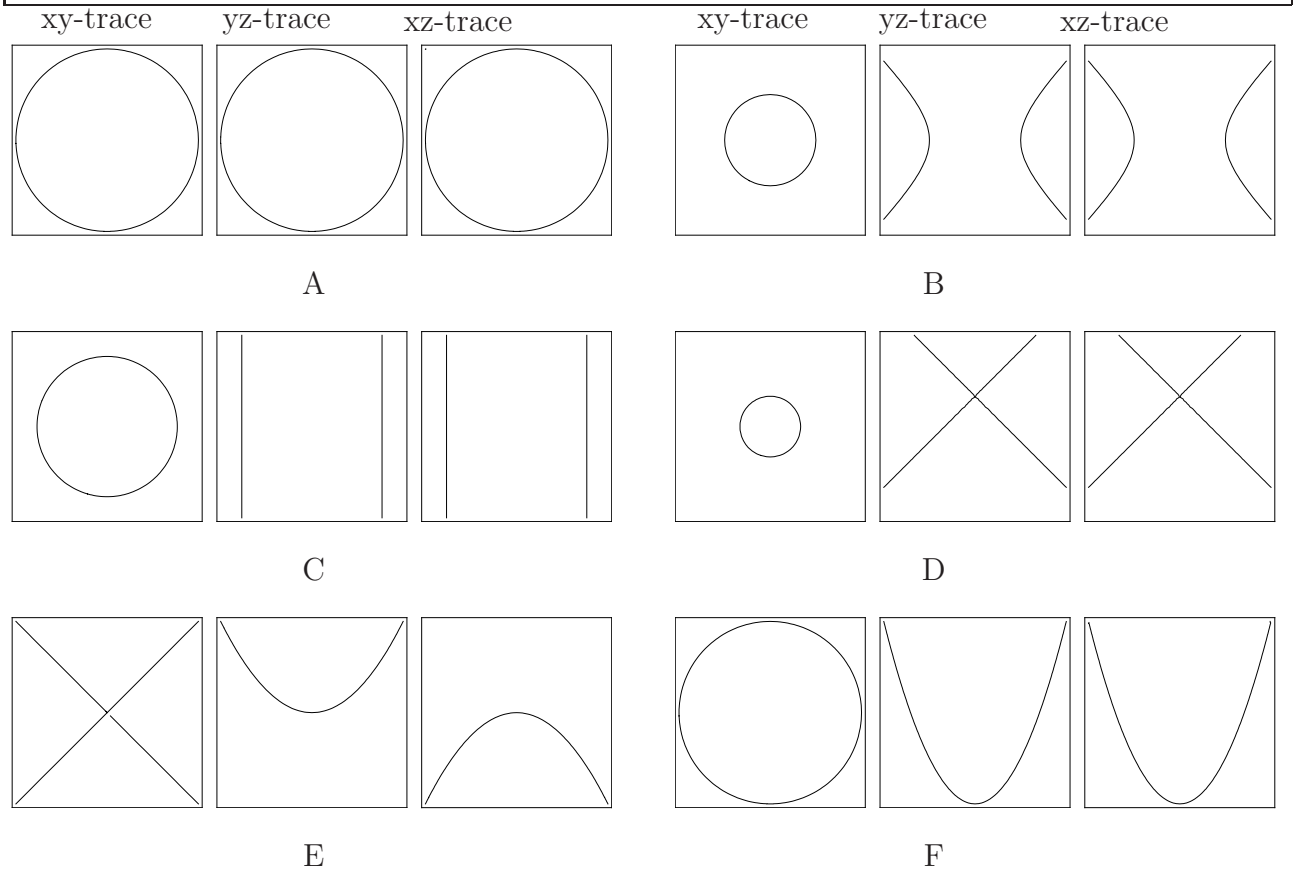
19) [T] [F] The point $(1, 1, -\sqrt{3})$ is in spherical coordinates given by $(\rho, \theta, \phi) = (\sqrt{5}, \pi/4, 2\pi/3)$.

Solution:
Also the z coordinate is wrong.

20) [T] [F] If the cross product satisfies $(\vec{v} \times \vec{w}) \times \vec{v} = \vec{0}$ then $\vec{v}$ and $\vec{w}$ are orthogonal.

Solution:
One can have $\vec{v} = \vec{w}$ in which case the two vectors are not orthogonal and still the product in question is zero.

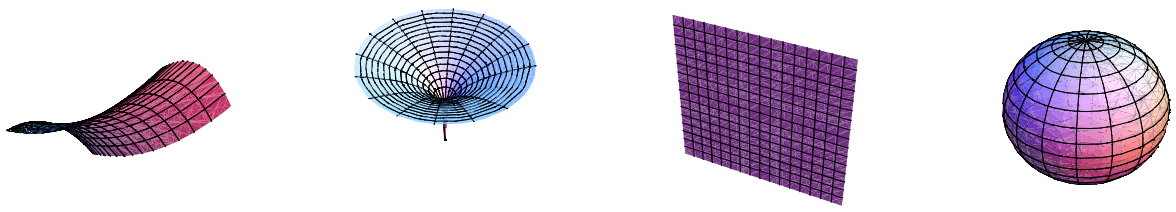
Problem 2a) (6 points)



The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2 + y^2 - (z - 1/3)^2 = 0$
	$x^2 + y^2 + z^2 - 1 = 0$
	$x^2 - y^2 - z = 0$
	$x^2 + y^2 - 1 = 0$
	$x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + y^2 - z = 1$

Problem 2b) (4 points)



I II III IV

Match the parametric surfaces with their parameterization. No justifications are needed.

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u, v) = \langle u^2, v^2, u^4 - v^4 \rangle$
	$\vec{r}(u, v) = \langle \cos(u)\sin(v), 1 + \sin(u)\sin(v), \cos(v) \rangle$
	$\vec{r}(u, v) = \langle v\cos(u), v\sin(u), v^{1/4} \rangle$
	$\vec{r}(u, v) = \langle u, 3, v \rangle$

Solution:
a) DAECBF.
b) I,IV,II,III

Problem 3) (10 points)

Find the distance of the point $P = (3, 4, 5)$ to the line

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} .$$

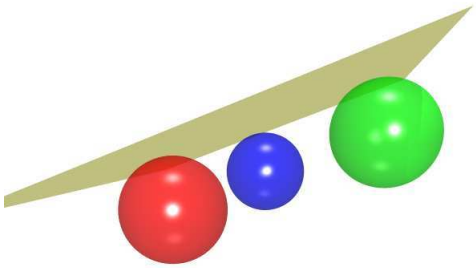
Solution:
First parametrize the line as $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle 4, 56\rangle = \langle 1, 2, 3\rangle + t\vec{v}$. We also need the vector $\vec{QP} = \langle 2, 2, 2\rangle$ connecting a point $Q = (1, 2, 3)$ on the line with the point $P = (3, 4, 5)$. Use the distance formula

$$d = |\langle 4, 5, 6\rangle \times \langle 2, 2, 2\rangle|/|\langle 4, 5, 6\rangle| = |\langle -2, 4, -2\rangle|/|\langle 4, 5, 6\rangle|$$

$\boxed{= \sqrt{24/77}}$.

Problem 4) (10 points)

Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches each of three spheres from the same side.



Solution:
The normal vector to the plane is $\vec{n} = \langle 3, 3, 0\rangle \times \langle 0, 1, 2\rangle = \langle 6, -6, 3\rangle$. The plane touching the three spheres has the equation $6x - 6y + 3z = d$, where d is a constant still to be determined. To find this constant, we have to find a point P on the plane. We do that by going from the point A by 1 unit in the direction of the normal vector. The point $P = A + \vec{n}/|\vec{n}| = (1, 2, 0) + \langle 6/9, -6/9, 3/9\rangle = (5, 4, 1)/3$ is on the plane. Plug in this point into the equation gives $d = 3$. The equation of the plane is $\boxed{6x - 6y + 3z - 3 = 0}$.

Problem 5) (10 points)

Find the arc length of the curve

$$\vec{r}(t) = \langle t^3/3, t^4/2, 2t^5/5\rangle$$

from $0 \leq t \leq 1$.

Solution:
We have $\vec{r}\,'(t) = \langle t^2, 2t^3, 2t^4\rangle$ and $|\vec{r}\,'(t)| = \sqrt{t^4 + 4t^6 + 4t^8} = t^2 + 2t^4$. The arc length is

$$\int_0^1 t^2 + 2t^4 \, dt = 1/3 + 2/5 = 11/15 \, .$$

The arc length is $\boxed{11/15}$.

Problem 6) (10 points)

An apple at position $(0, 0, 20)$ rests 20 meters above Newton's head, the tip of whose nose is at $(1, 0, 0)$. The apple falls with constant acceleration $\vec{r}\,''(t) = \langle a, 0, -10\rangle$ (where $\langle 0, 0, -10\rangle$ is caused by gravity and $\langle a, 0, 0\rangle$ by the wind) precisely onto the nose of Newton. Find the wind force $\langle a, 0, 0\rangle$ which achieves this. Give a parametrization for the path along which the apple falls.



Solution:
From

$$\vec{r}\,''(t) = \langle a, 0, -10\rangle$$

we get by integration

$$\vec{r}\,'(t) = \langle at, 0, -10t\rangle + \langle 0, 0, 0\rangle$$

and

$$\vec{r}\,(t) = \langle at^2/2, 0, -5t^2\rangle + \langle 0, 0, 20\rangle \ = \langle at^2/2, 0, -5t^2 + 20\rangle \ .$$

Now, in order that we reach the nose $(1, 0, 0)$, we have to get the time t such the apple is at the ground $5t^2 = 20$ gives $t = 2$. In order that $at^2/2 = 1$ we have $a = 1/2$. The wind force is $\langle 1/2, 0, 0\rangle$. The path is

$$\vec{r}\,(t) = \langle t^2/4, 0, 20 - 5t^2\rangle \ .$$

Problem 7) (10 points)

a) (5 points) A red maple leaf falls to the ground $z = 0$. It falls along the curve $\vec{r}(t) = \langle 3\sqrt{3}\cos(t), 3\sqrt{3}\sin(t), 5-t-4t^2\rangle$. At which angle does it hit the xy-plane?

b) (5 points) Find the tangent line to the curve at the impact point.



Solution:
a) The leaf hits the ground at the point $\vec{r}(1) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0\rangle$ at time $t = 1$. We compute the velocity vector at that time

$$\vec{r}\,'(1) = \langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9\rangle \ .$$

In order to compute the impact angle, we compute the angle with the normal vector $\langle 0, 0, 1\rangle$ which is $\cos(\alpha) = -9/(6\sqrt{3}) = -\sqrt{3}/2$ so that $\alpha = 5\pi/6$. The angle between the plane and the velocity vector is $5\pi/6 - \pi/2 = \pi/3$. The result is $\boxed{\pi/3}$.
b) the tangent line has the parametrization $\vec{r}(t) = \vec{r}(1) + t\vec{r}\,'(1)$ which is

$$\vec{r}(t) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0\rangle + t\langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9\rangle \ .$$

Problem 8) (10 points)

a) (5 points) The surface

$$\vec{r}(t,s) = \langle 1+t+s, 1-t-2s, 1+t-s \rangle$$

with $0 \leq t \leq 1, 0 \leq s \leq 1$ is a parallelogram in space. Find the area of this parallelogram.

b) (5 points) Another surface is given in spherical coordinates by $\rho = 2\sin(\phi)\cos(\theta)$. Write down the equation of this surface in rectangular coordinates as well as in cylindrical coordinates.

Solution:
a) The parallelogram is spanned by $\langle 1, -1, 1 \rangle$ and $\langle 1, -2, -1 \rangle$. The area is the length of the cross product $\langle 3, 2, -1 \rangle$ which is $\boxed{\sqrt{14}}$.
b) To get the equation in rectangular coordinates, multiply both sides of the equation with ρ, this gives

$$x^2 + y^2 + z^2 = \rho^2 = 2\rho\sin(\phi)\cos(\theta) = 2x \ .$$

Complete the square to get $\boxed{(x-1)^2 + y^2 + z^2 = 1}$.
To get from rectangular to cylindrical coordinates, just replace $x^2 + y^2$ with r^2 and x with $r\cos(\theta)$ and leave z as it is. In cylindrical coordinates the surface is given by the equation $\boxed{r^2 - 2r\cos(\theta) + z^2 = 0}$.

Problem 9) (10 points)

a) (5 points) Parametrize the curve obtained by intersecting the surface $z - x^2 + y^3 = 0$ with the cylindrical surface $x^2/4 + 9y^2 = 1$.

b) (5 points) Find the unit tangent vector $\vec{T}$ and the normal vector $\vec{N}(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$ to the curve

$$\vec{r}(t) = \langle 3, t^2, t \rangle$$

at the point $(3, 0, 0)$. What is the binormal vector $\vec{B} = \vec{T} \times \vec{N}$?

Solution:
a) First parametrize the first two coordinates: $\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), ...\rangle$, then fill in the third coordinate $z = x^2 - y^3$ to get

$$\vec{r}(t) = \langle 2\cos(t), 1/3\sin(t), 4\cos^2(t) - \sin^3(t)/27\rangle$$

b) $\vec{T}(t) = \langle 0, 2t, 1\rangle/\sqrt{1+4t^2}$. At $t = 0$ we have $\boxed{\vec{T} = \langle 0, 0, 1\rangle}$. Now $\vec{T}\,'(t) = \langle 0, 2, 0\rangle/\sqrt{1+4t^2} - 4t\langle 0, 2t, 1\rangle/(1+4t^2)^{3/2}$. Which is at $t = 0$ equal to $\langle 0, 2, 0\rangle$. Normalized, we get $\boxed{\vec{N} = \langle 0, 1, 0\rangle}$. The third vector is just the cross product of the first two $\boxed{\vec{B} = \langle -1, 0, 0\rangle}$.

Problem 10) (10 points)

a) (4 points) Give a parametrization of the hyperboloid $x^2 + y^2 = z^2 + 1$.

b) (3 points) Give a parametrization of the plane $x + y = 1$.

c) (3 points) Give a parametrization of the ellipsoid $x^2 + y^2 + z^2/4 = 1$.

Solution:
a) Since the hyperboloid is in cylindrical coordinates given as $r^2 = z^2 + 1$, we have

$$\vec{r}(z, \theta) = \langle \sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z\rangle \ .$$

b) Since the points $(1, 0, 0), (0, 1, 0), (1, 0, 1)$ are in the plane, we have two vectors $\vec{v} = \langle -1, 1, 0\rangle, \vec{w} = \langle 0, 0, 1\rangle$ parallel to the plane and can parametrize

$$\vec{r}(s, t) = \langle 1, 0, 0\rangle + t\langle -1, 1, 0\rangle + s\langle 0, 0, 1\rangle \ .$$

c) The ellipsoid is a deformation of the sphere. We just have to make it higher:

$$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), 2\cos(\phi)\rangle \ .$$

9/27/2017 FIRST HOURLY PRACTICE 9 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications needed

1) [T] [F] The length of the sum of two vectors is always the sum of the length of the vectors.

Solution:
There is a triangle inequality in general. But equality only holds for parallel vectors pointing in the same direction

2) [T] [F] For any three vectors, $\vec{v} \times (\vec{w} + \vec{u}) = \vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.

Solution:
The cross product is distributive but not commutative.

3) [T] [F] The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

Solution:
$x^2 + y^2 - z^2 = 0$ is a cone. Completion of the square adds an other constant and the surface is a one sheeted hyperboloid.

4) [T] [F] The surface $\vec{r}(u, v) = \langle \cos(u^2)\sin(v^2), \sin(u^2)\sin(v^2), \cos(v^2) \rangle$ with $0 \leq u < \sqrt{2\pi}, 0 \leq v \leq \sqrt{\pi}$ is a sphere.

Solution:
Just write $u^2 = \theta, v^2 = \phi$.

5) [T] [F] If P, Q, R are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing P, Q, R.

Solution:
The vectors $\vec{PQ}$ and $\vec{RQ}$ are both in the plane. The cross product is perpendicular to the plane.

6) [T] [F] The line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ hits the plane $2x + 3y + 4z = 9$ at a right angle.

Solution:
The vector $\langle 2, 3, 4 \rangle$ is in the line and perpendicular to the plane.

7) [T] [F] The function $f(x,y) = \sin(xy)/y$ is continuous everywhere.

Solution:
The problem is the y axes. For fixed x, the limit $y \to 0$ exists and is x. The function can be defined on the y-axes as $f(x,0) = x$ and with this, it becomes continuous.

8) [T] [F] For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

Solution:
The cross product is anti commutative.

9) [T] [F] If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.

Solution:
Assume $\vec{v}$ is not $\vec{0}$, then take $\vec{w}$ as a vector which is perpendicular to $\vec{v}$.

10) [T] [F] If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

Solution:
The vector in question is perpendicular to $\vec{u}$ and perpendicular to $\vec{u} \times \vec{v}$. Also $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{u} \times \vec{v}$.

11) [T] [F] Every vector contained in the plane $x + y + z = 1$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

Solution:
It is perpendicular to the vector $\langle 1, 1, 1 \rangle$, not parallel.

12) [T] [F] The sphere can in cylindrical coordinates described as $r^2 = 1 - z^2$.

Solution:
Just substitute $r = \sqrt{x^2 + y^2}$.

13) [T] [F] The curvature of the curve $2\vec{r}(4t)$ at $t = 0$ is twice the curvature of the curve $\vec{r}(t)$ at $t = 0$.

Solution:
The curvature of the first curve is $1/2$ of the curvature of the second curve.

14) [T] [F] The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.

Solution:
The surface is an elliptical cone.

15) [T] [F] If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.

Solution:
The two vectors can be parallel and nonzero.

16) [T] [F] Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

Solution:
It is parallel to $\langle 2, 3, 4 \rangle$

17) [T] [F] Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

Solution:
You can use the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$. If this is zero, then either one of the vectors is the zero vector or $\sin(\alpha) = 0$. In all cases, this can be considered parallel.

18) [T] [F] The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is always in the same plane together with $\vec{v}$ and $\vec{w}$.

Solution:
Let $\vec{n} = (\vec{v} \times \vec{w})$ be the vector perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$. Then $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \vec{n}$ is perpendicular to $\vec{n}$. It is therefore parallel to the plane.

19) [T] [F] The line $\vec{r}(t) = \langle 1+2t, 1+2t, 1-4t \rangle$ hits the plane $x+y+z=9$ at a right angle.

Solution:
In order to be perpendicular, the velocity vector would have to be parallel to $\langle 1, 1, 1 \rangle$.

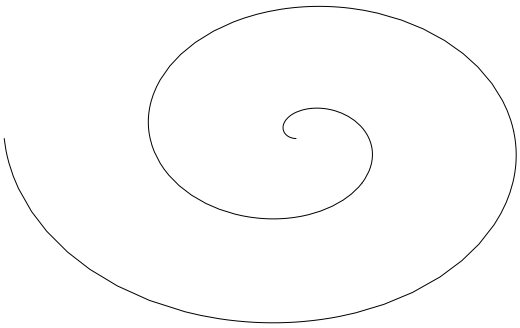
20) [T] [F] The intersection of the ellipsoid $x^2/3+y^2/4+z^2/3=1$ with the plane $y=1$ is a circle.

Solution:
Just set $y=1$ in that equation.

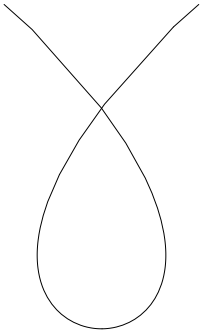
Problem 2a) (3 points)

Match the curves with their parametric definitions.

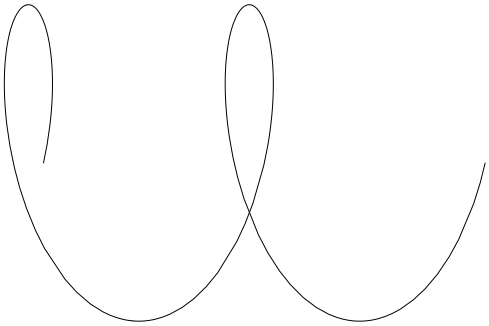
I



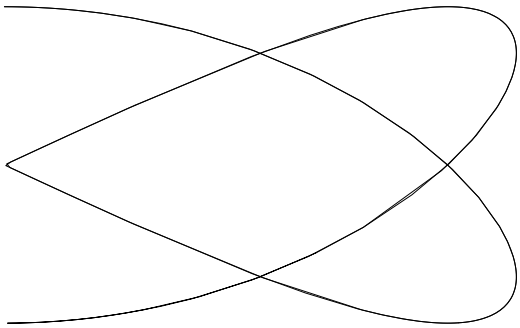
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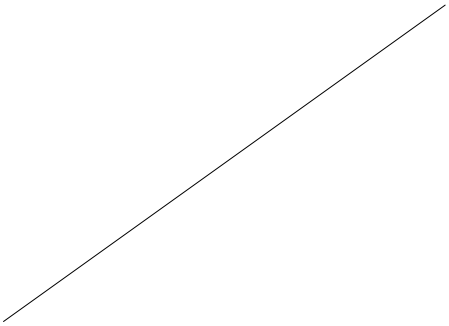
III



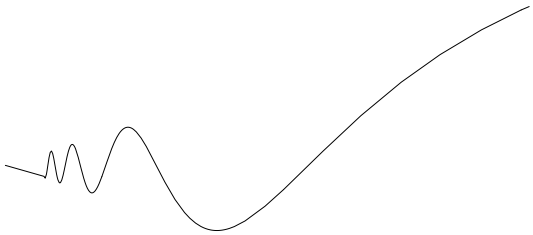
IV



V



VI



Enter I,II,III,IV,V or VI here	Parametric equation for the curve
	$\vec{r}(t) = \langle t, \sin(1/t)t \rangle$
	$\vec{r}(t) = \langle t^3 - t, t^2 \rangle$
	$\vec{r}(t) = \langle t + \cos(2t), \sin(2t) \rangle$
	$\vec{r}(t) = \langle \lvert \sin(2t) \rvert, \cos(3t) \rangle$
	$\vec{r}(t) = \langle 1 + t, 5 + 3t \rangle$
	$\vec{r}(t) = \langle -t\cos(t), 2t\sin(t) \rangle$

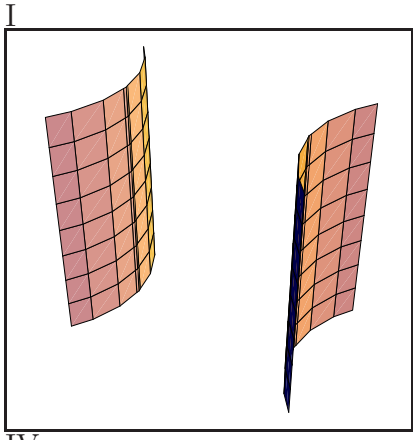
Solution:

Enter I,II,III,IV,V or VI here	Parametric equation for the curve
VI	$\vec{r}(t) = \langle t, \sin(1/t)t \rangle$
II	$\vec{r}(t) = \langle t^3 - t, t^2 \rangle$
III	$\vec{r}(t) = \langle t + \cos(2t), \sin(2t) \rangle$
IV	$\vec{r}(t) = \langle \lvert \sin(2t) \rvert, \cos(3t) \rangle$
V	$\vec{r}(t) = \langle 1 + t, 5 + 3t \rangle$
I	$\vec{r}(t) = \langle -t\cos(t), 2t\sin(t) \rangle$

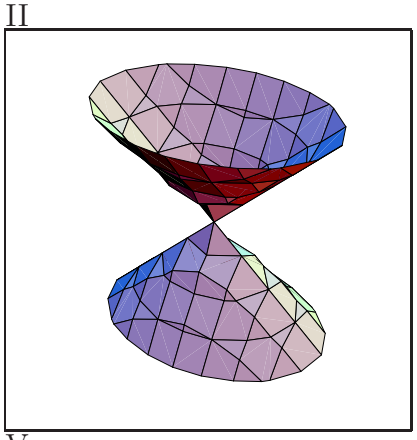
Problem 2b) (3 points)

Match the equations with the surfaces.

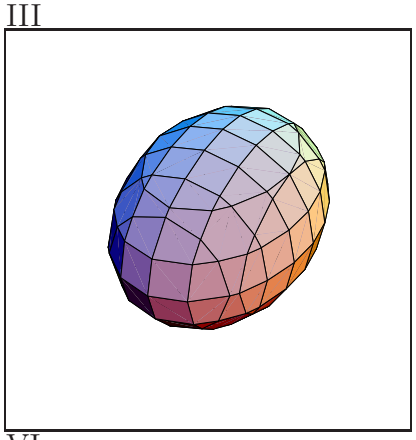
I



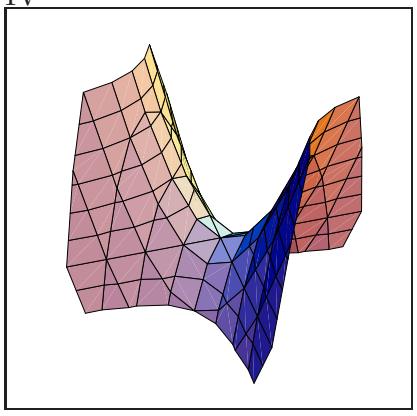
II



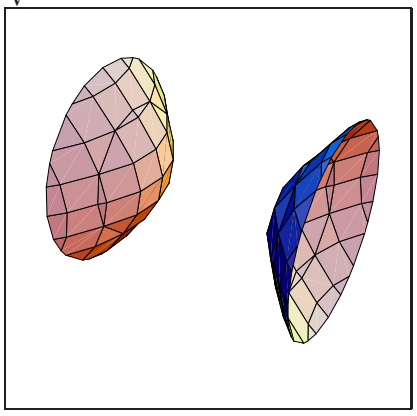
III



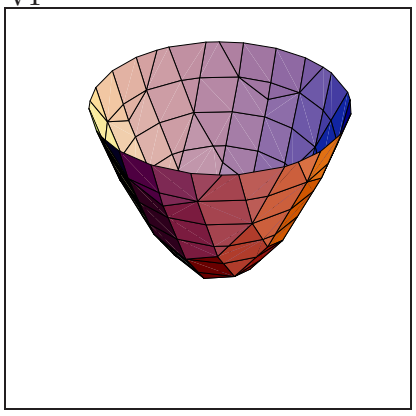
IV



V



VI



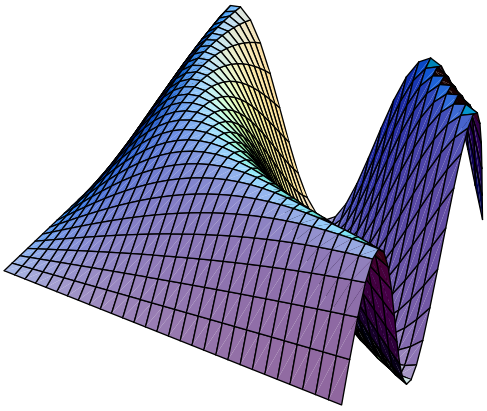
Enter I,II,III,IV,V,VI here	Equation
	$x^2 - y^2 - z^2 = 1$
	$x^2 + 2y^2 = z^2$
	$2x^2 + y^2 + 2z^2 = 1$
	$x^2 - y^2 = 5$
	$x^2 - y^2 - z = 1$
	$x^2 + y^2 - z = 1$

Solution:

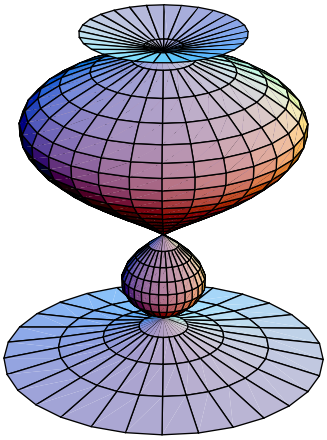
Enter I,II,III,IV,V,VI here	Equation
V	$x^2 - y^2 - z^2 = 1$
II	$x^2 + 2y^2 = z^2$
III	$2x^2 + y^2 + 2z^2 = 1$
I	$x^2 - y^2 = 5$
IV	$x^2 - y^2 - z = 1$
VI	$x^2 + y^2 - z = 1$

Problem 2c) (4 points)

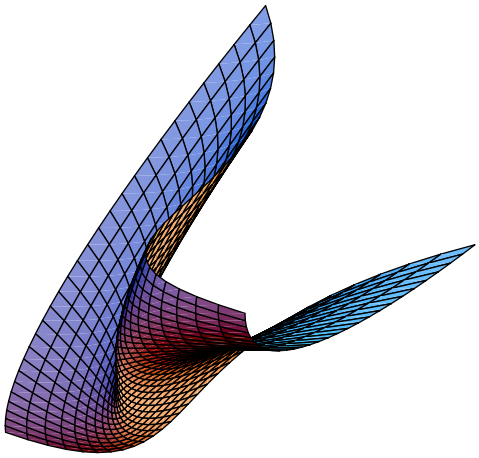
Match the parametric surfaces with their parameterization. No justification is needed.

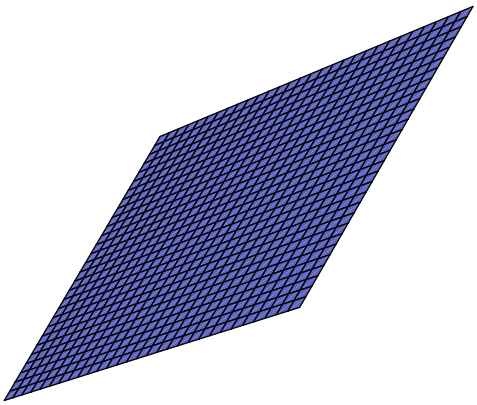


I



II





III

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u,v) = \langle u, v, u+v \rangle$
	$\vec{r}(u,v) = \langle u, v, \sin(uv)) \rangle$
	$\vec{r}(u,v) = \langle 0.2 + u(1-u^2)\cos(v), (0.2 + u(1-u^2))\sin(v), u \rangle$
	$\vec{r}(u,v) = \langle u^3, (u-v)^2, v \rangle$

Solution:

Enter I,II,III,IV here	Parameterization
IV	$\vec{r}(u,v) = \langle u, v, u+v\rangle$
I	$\vec{r}(u,v) = \langle u, v, \sin(uv))\rangle$
II	$\vec{r}(u,v) = \langle (0.2+u(1-u^2))\cos(v), (0.2+u(1-u^2))\sin(v), u\rangle$
III	$\vec{r}(u,v) = \langle u^3, (u-v)^2, v\rangle$

Surface I is a graph.
Surface II is a surface of revolution.
Surface III is algebraic. One of the traces is (u^3, u^2), an other trace is the parabola (v^2, v).
Surface IV is a plane.

Problem 3) (10 points)

a) (6 points) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.

b) (4 points) Find a plane perpendicular to that line of intersection.

Solution:
a) The line of intersection has the direction $(3,-2,1) \times (1,2,3) = 8(-1,-1,1)$. The parameterization is $\boxed{\vec{r}(t) = (1,-2,0) + t(-1,-1,1)}$.
b) If a line contains the point $(x_0, y_0, z_0) = (1,-2,0)$ and a vector $\langle a,b,c\rangle = \langle -1,-1,1\rangle$. One plane is $-x - y + z = 1$.

Problem 4) (10 points)

a) (4 points) Find the area of the parallelogram with vertices $P = (1,0,0)$ $Q = (0,2,0)$, $R = (0,0,3)$ and $S = (-1,2,3)$.

b) (3 points) Verify that the triple scalar product has the property $[\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]$.

c) (3 points) Verify that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ has the property

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||$$

Solution:
a) One has to realize which vectors form the sides of the parallelogram. The solution is $|\vec{PQ} \times \vec{PR}| = 7$.
b) $[u+v, v+w, w+u] = [u,v,w] + [u,v,u] + [u,w,w] + [u,w,u] + [v,v,w] + [v,v,u] + [v,w,w] + [v,w,u]$. Any term, where two parallel vectors appear is zero. So, only $2[u,v,w]$ remains on the right hand side.
c) Build the parallelepiped spanned by u, v, w and note that one can shear it in such a way that it is contained in the box of size $||\vec{u}||$ and $||\vec{v}||$ and $||\vec{w}||$. You can also see the identity by using angle formulas for the dot product $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha)$ and the the length of the cross produc $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\beta)$

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq |\vec{u}||\vec{v}||\vec{w}||\cos(\alpha)||\sin(\beta)|$$

where β is the angle between $\vec{v}$ and $\vec{w}$ and where α is the angle $\vec{v} \times \vec{w}$ and $\vec{u}$.

Problem 5) (10 points)

Find the distance between the two lines

$$\vec{r}_1(t) = \langle t, 2t, -t \rangle$$

and

$$\vec{r}_2(t) = \langle 1+t, t, t \rangle \ .$$

Solution:
The point $P = (0,0,0)$ is on the first line. The point $Q = (1,0,0)$ on the second line. The vector $\vec{v} = \langle 1, 2, -1 \rangle$ in the first line and $\vec{w} = \langle 1, 1, 1 \rangle$ in the second line. We have $\vec{n} = \langle 3, -2, -1 \rangle$. Now, the distance is $3/\sqrt{14}$. $(Q-P) \cdot \vec{n}/|\vec{n}| = \langle 1,0,0 \rangle \cdot \langle 3, -2, -1 \rangle / |n| = 3/\sqrt{14}$.

Problem 6) (10 points)

Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

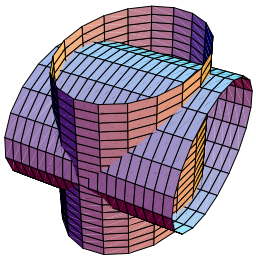
Solution:
The line of intersection is parallel to the cross product of $\vec{v} = \langle 2, 1, 1\rangle$ and $\vec{w} = \langle 1, 3, 1\rangle$ which is $\langle -2, -1, 5\rangle$. This vector is perpendicular to the plane we are looking for. The equation of the plane is $-2x - y + 5z = 0$.

Problem 7) (10 points)

The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
c) (3 points) What is the arc length of this curve?

Solution:
a) Fix first $x(t), y(t)$ to satisfy the first equation then get $z(t)^2 = \cos^2(t)$ and $z = \pm\cos(t)$by solving the second equation for z. $\vec{r}(t) = (\cos(t), \sqrt{2}\sin(t), \pm\cos(t))$.
b) We find the velocity $\vec{r}'(t) = (-\sin(t), \sqrt{2}\cos(t), -\sin(t))$ and then the speed $|\vec{r}'(t) = \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)} = \sqrt{2}$. The length is $\int_0^{2\pi} |\vec{r}'(t)|\ dt = \int_0^{2\pi} \sqrt{2}\ dt$. Also an expression like $\boxed{\int_0^{2\pi} \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)}\ dt}$ is here correct at this stage.
c) Evaluate the integral $\boxed{2\sqrt{2}\pi}$.



Problem 8) (10 points)

a) (6 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(t), \sin(t), -2t\rangle$ at the point $\vec{r}(0)$.

b) (4 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(5t), \sin(5t), -10t\rangle$ at the point $\vec{r}(0)$.

Hint. Use one of the two formulas for the curvature

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}\ ,$$

where $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$. The curvatures in b) can be derived from the curvature in a).

There is no need to redo the calculation in b) if you give a proper justification.

Solution:
a) We use the second formula for the curvature: $\vec{r}\,'(t) = \langle \sin(t), \cos(t), -2\rangle$. $\vec{r}\,''(t) = \langle \cos(t), -\sin(t), 0\rangle$. The speed of the curve satisfies $|\vec{r}\,'(t)| = \sqrt{5}$. The vector $\vec{r}\,'(t) \times \vec{r}\,''(t)$ is $(-2\sin(t), -2\cos(t), -1)$ which has length $\sqrt{5}$. therefore, the curvature is constant $\boxed{\kappa(t) = 1/5}$.
b) Because the curvature is independent of the parametrization, the curvature is agin $1/5$.

Problem 9) (10 points)

For each of the following, fill in the blank with $<$ (less than), $>$ (greater than), or $=$ (equal). Justify your answer completely.

1.	The arc length of the curve parameterized by $\vec{f}(t) = \langle \cos 2t, 0, \sin 2t\rangle$, $0 \le t \le \pi$.	_____	The arc length of the curve parameterized by $\vec{g}(u) = \langle 3, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le \sqrt{\pi}$.
2.	The arc length of the curve parameterized by $\vec{f}(t) = \langle t^2, 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.	_____	The arc length of the curve parameterized by $\vec{g}(u) = \langle u^4, 2\cos u^2, 2\sin u^2\rangle$, $0 \le u \le 2\pi$.
3.	The arc length of the curve parameterized by $\vec{f}(t) = \langle 1 + 3t^2, 2 - t^2, 5 + 2t^2\rangle$, $0 \le t \le 1$.	_____	The arc length of the curve parameterized by $\vec{g}(u) = \left\langle \frac{1}{2}u^2, u, \frac{2\sqrt{2}}{3}u^{3/2}\right\rangle$, $0 \le u \le 2$.
4.	The arc length of the curve parameterized by $\vec{f}(t) = \langle \sin t, \cos t, t\rangle$, $1 \le t \le 5$.	_____	The arc length of the curve parameterized by $\vec{g}(u) = \langle u\sin u, u\cos u, u\rangle$, $1 \le u \le 5$.

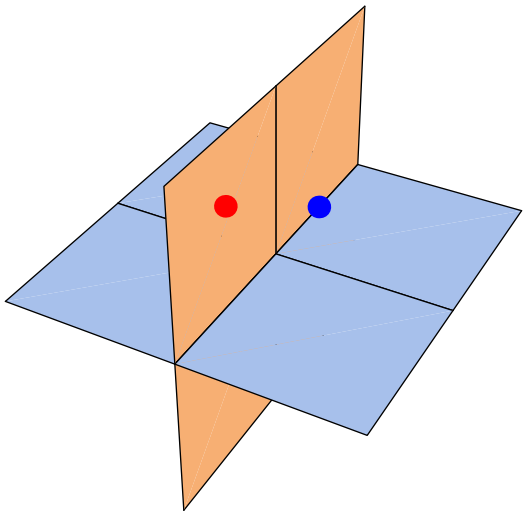
Solution:

1. The left curve is a circle of radius 1 so has arc length 2π. The right curve is half of a circle of radius 2, so it also has arc length 2π. The lengths are the same.
2. The left curve is a subset of the right curve; it is the portion of the right curve with $0 \le u \le \sqrt{2\pi}$. Therefore, the right curve has greater arc length.
3. The left curve is a line segment from $(1, 2, 5)$ to $(4, 1, 7)$, which has length $\sqrt{14}$. To find the length of the right curve, we use the arc length formula, which says the length is $\int_0^2 \|\vec{g}'(u)\|\, du$. We calculate $\vec{g}'(u) = \langle u, 1, \sqrt{2u}\rangle$, so $\|\vec{g}'(u)\| = \sqrt{u^2 + 2u + 1} = u+1$, and the arc length is $\int_0^2 (u + 1)\, du = 4$, which is greater than $\sqrt{14}$.
4. Both curves spiral upward the same amount, but the coils of the right curve are always wider, so the right curve has greater arc length. Alternatively, it's easy to see that $\|f'(t)\| < \|g'(t)\|$ whenever $t > 1$, so $\int_1^5 \|f'(t)\|\, dt$ must be smaller than $\int_1^5 \|g'(t)\|\, dt$.

Problem 10) (10 points)

Given the plane $x + y + z = 6$ containing the point $P = (2, 2, 2)$. Given is also a second point $Q = (3, -2, 2)$.

Find the equation $ax + by + cz = d$ for the plane through P and Q which is perpendicular to the plane $x + y + z = 6$.



Solution:
The vector $\vec{v} = \langle 1, 1, 1 \rangle$ is perpendicular to the first plane and so parallel to the second plane. The vector $\vec{w} = \vec{QP} = \langle 1, -4, 0 \rangle$ is also in the second plane. Therefore, $\vec{n} = \vec{v} \times \vec{w} = \langle 4, 1, -5 \rangle$ is perpendicular to the second plane. The plane has the equation $\boxed{4x + y - 5z = d = 0}$. The constant $d = 0$ was obtained here by plugging in a point like $P = (2, 2, 2)$.

9/27/2017 FIRST HOURLY PRACTICE 10 **Math 21a, Fall 2017**

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TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] For $\vec{N}$ i
$\vec{B}$ ·

Solution:
The three vectors have length 1 and are perpendicular to each other.

2) [T] [F] For any three points P, Q, R in space, $\vec{PQ} \times \vec{PR} = \vec{QP} \times \vec{RP}$

Solution:
The two vectors are switching sign on the right hand side.

3) [T] [F] The triangle defined by the three points $(-1, 0, 2), (-4, 2, 1), (1, -1, 2)$ has a right angle.

Solution:
It would be inefficient to compute the angles. Better is to look at the squares of the lengths of the triangle which are $14, 5$ and 35. If there was a right angle, Pythagoras would apply.

4) [T] [F] The function $f(x, y, z) = x^2 + y^2 + z^2 / \sin(x^2 + y^2 + z^2)$ is continuous everywhere in space.

Solution:
The problem is not $(0, 0, 0)$. The function is continuous there becomes 1 as one could see in spherical coordinates $f(\rho) = \rho / \sin(\rho)$ is continuous at 0 (use Hopital's rule). Note however that there are other values like on the sphere $\rho = \pi$, where the function is not continuous. The function blows up there.

5) [T] [F] $\vec{u} \times \vec{u} = 0$ implies $\vec{u} = \vec{0}$.

Solution:
The left hand side is always true. To see that it is false, take $\vec{u} = \langle 1, 0, 0 \rangle$. It is not the zero vector, but still $\vec{u} \times \vec{u} = 0$.

6) [T] [F] The level curves $f(x,y) = 1$ and $f(x,y) = 2$ of a smooth function f never intersect.

Solution:
If they would intersect in a point (x,y), then f would take two values 1 and 2, at the point which is not possible.

7) [T] [F] For any vector $\vec{v}$, we have $\text{proj}_{\vec{i}}(\text{proj}_{\vec{j}}(\vec{v})) = \vec{0}$.

Solution:
The vector $proj_{\vec{j}}(\vec{v}))$ is parallel to $\vec{j}$ which is perpendicular to $\vec{i}$ and the projection onto $\vec{i}$ is therefore the zero vector.

8) [T] [F] $(\vec{j} \times \vec{i}) \times \vec{i} = \vec{k} \times (\vec{i} \times \vec{k})$

Solution:
The left vector is parallel to j, the right vector is parallel to i.

9) [T] [F] If a parametrized curve $\vec{r}(t)$ lies in a plane and the velocity $\vec{r}\,'(t)$ is never zero, then the normal vector $\vec{N}(t)$ also lies in that plane.

Solution:
This is intuitively clear.

10) [T] [F] The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

Solution:
It is true for circles, but false in general. For example, on a line, the acceleration parallel to the velocity.

11) [T] [F] If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $proj_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

Solution:
The projection vector has length $|\vec{v} \cdot \vec{w}|/|\vec{w}|$ which has length smaller or equal to $\vec{v}$ (use the cos formula).

12) [T] [F] A line intersects an ellipsoid in at most 2 distinct points.

Solution:
One can see this geometrically. Here is an argument: we know it for a sphere. When streching the picture with the sphere and line the number of intersections does not change.

13) [T] [F] For any vectors $\vec{v}$ and $\vec{w}$, the formula $(\vec{v} - \vec{w}) \cdot \vec{P}_{\vec{w}}(\vec{v}) = 0$ holds.

Solution:
Take $\vec{v} = \vec{i}$ and $\vec{w} = \vec{j}$.

14) [T] [F] Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on S. If $\vec{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.

Solution:
The condition $\vec{n} \cdot \overrightarrow{PQ} = 0$ implies that the vector $\vec{PQ}$ is parallel to the plane.

15) [T] [F] The vectors $\langle 2, 2, 1\rangle$ and $\langle 1, 1, -4\rangle$ are perpendicular.

Solution:
The dot product vanishes

16) [T] [F] $\|\vec{v} \times \vec{w}\| = \|v\|\|w\| \cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$.

Solution:
It is sin not cos.

17) [T] [F] The vector $\vec{i} \times (\vec{j} \times \vec{k})$ has length 1.

Solution:
It is the zero vector.

18) [T] [F] The distance between the z-axis and the line $x - 1 = y = 0$ is 1.

Solution:
You can see that geometrically.

19) [T] [F] There is a quadric surface which both hyperbola and parabola appear as traces. Traces are intersections of the surface with the coordinate planes $x = 0$, $y = 0$, or $z = 0$.

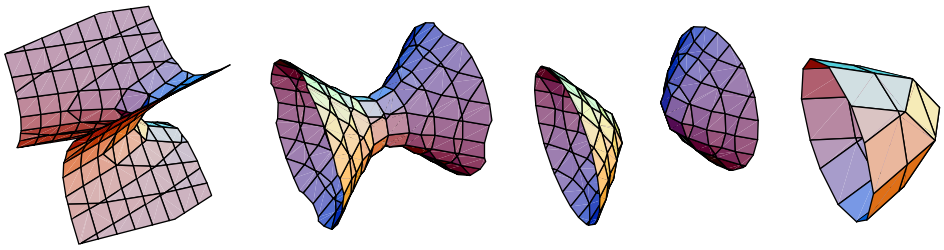
Solution:
The hyperbolic paraboloid has that.

20) [T] [F] The equation $x^2 + y^2 - z^2 = -1$ defines a one-sheeted hyperboloid.

Solution:
$f(x, y, z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

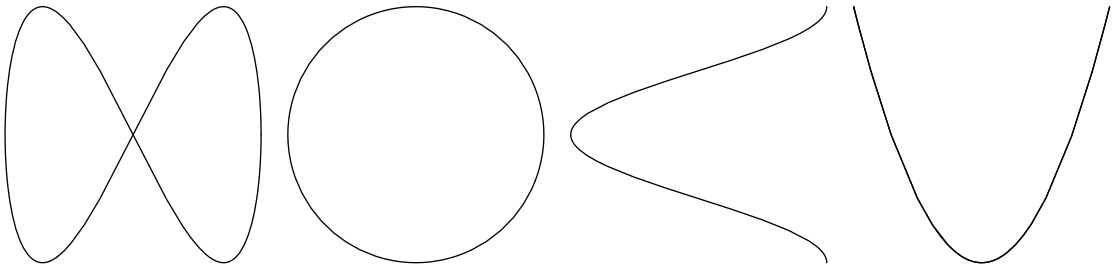
Problem 2) (10 points)

Match the equation with the pictures. No justifications are necessary in this problem.



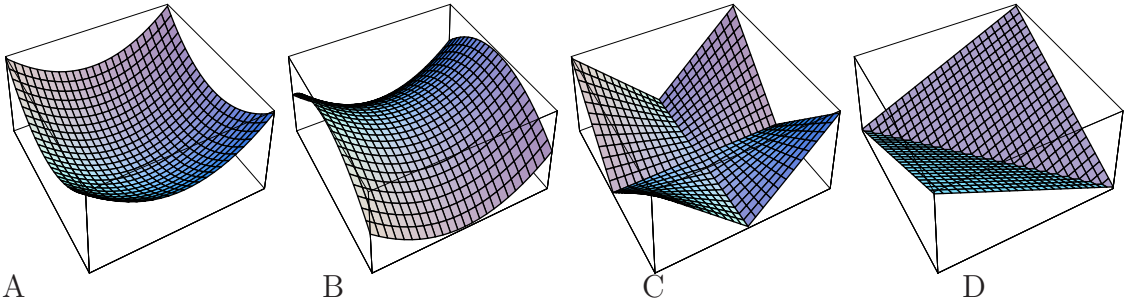
I II III IV

Enter I,II,III,IV here	Equation
	$x + y^2 - z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 + 1 = 0$
	$-x + y^2 + z^2 + 1 = 0$



1 2 3 4

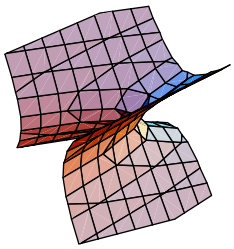
Enter 1,2,3,4 here	Equation
	$\langle \cos(t), \sin(t) \rangle$
	$\langle \cos(t), t \rangle$
	$\langle \cos(t), \cos^2(t) \rangle$
	$\langle \cos(t), \sin(2t) \rangle$

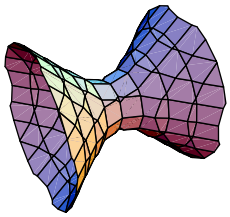


A B C D

Enter A,B,C,D here	Equation
	$f(x, y) = x^2 - y^2$
	$f(x, y) = \lvert x + y \rvert$
	$f(x, y) = x^2 + y^2$
	$f(x, y) = \lvert xy \rvert$

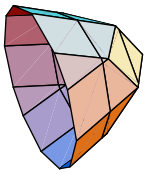
Solution:


I

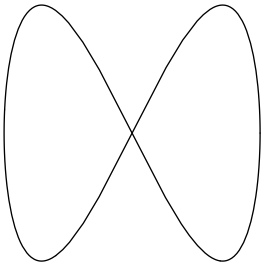

II

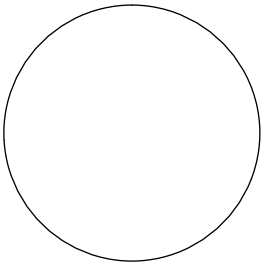


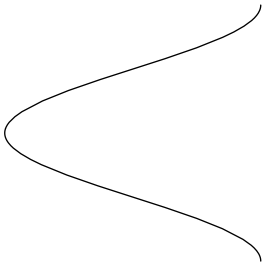
III

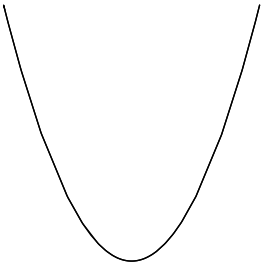

IV

Enter I,II,III,IV here	Equation
I	$x + y^2 - z^2 - 1 = 0$
II	$-x^2 + y^2 + z^2 - 1 = 0$
III	$-x^2 + y^2 + z^2 + 1 = 0$
IV	$-x + y^2 + z^2 + 1 = 0$

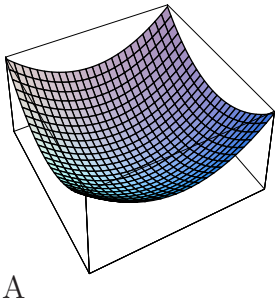

1

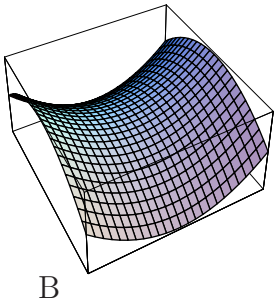

2

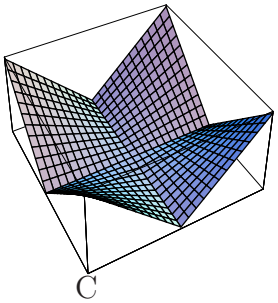

3

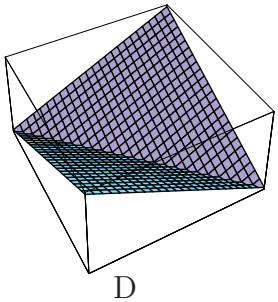

4

Enter 1,2,3,4 here	Equation
2	$\langle \cos(t), \sin(t) \rangle$
3	$\langle \cos(t), t \rangle$
4	$\langle \cos(t), \cos^2(t) \rangle$
1	$\langle \cos(t), \sin(2t) \rangle$


A


B


C


D

Enter A,B,C,D here	Equation
B	$f(x, y) = x^2 - y^2$
D	$f(x, y) = \lvert x + y \rvert$
A	$f(x, y) = x^2 + y^2$
C	$f(x, y) = \lvert xy \rvert$

Problem 3) (10 points)

Imagine the planet Earth as the unit sphere in 3D space centered at the origin. An asteroid is approaching from the point $P = (0, 4, 3)$ along the path

$$\vec{r}(t) = \langle (4-t)\sin(t), (4-t)\cos(t), 3-t \rangle \ .$$

a) When and where will it first hit the Earth?

b) What velocity will it have at the impact?



Solution:
a) The distance to the origin $|\vec{r}(t)| = \sqrt{(4-t)^2 + (3-t)^2} = \sqrt{25 + 2t^2 - 14t}$ is equal 1 for $t = 3$ or $t = 4$.
b) The velocity is $\vec{r}\,'(t) = \langle (4-t)\cos(t) - \sin(t), -\cos(t) - (4-t)\sin(t), -1 \rangle$. The velocity at time $t = 3$ is $\langle \cos(3) - \sin(3), -\sin(3) - \cos(3), -1 \rangle$. (The speed at time $t = 3$ is $\sqrt{3}$.)

Problem 4) (10 points)

Find the distance between the cylinder $x^2 + y^2 = 1$ and the line

$$L : \frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2}.$$

Solution:
We first compute the distance between the z axes and the line L. The z axes can be parametrized as

$$\vec{r}(t) = P + t\vec{v} = \langle 0, 0, 0 \rangle + t\langle 0, 0, 1 \rangle$$

The line L can be parametrized as

$$\vec{r}(t) = Q + t\vec{w} = \langle -2, 1, 0 \rangle + t\langle 4, 3, 2 \rangle$$

The distance is the length of the projection of $\vec{PQ} = \langle -2, 1, 0 \rangle$ onto the normal vector $\vec{n} = \vec{v} \times \vec{w} = \langle -3, 4, 0 \rangle$. This is

$$d = \frac{|\langle -2, 1, 0 \rangle \cdot \langle -3, 4, 0 \rangle|}{|\langle -3, 4, 0 \rangle|} = 10/5 = 2 \ .$$

The distance between the line L and the cylinder is by 1 smaller. The answer is $\boxed{1}$.

Problem 5) (10 points)

a) Find a parametrization $\vec{r}(t)$ of the line which is the intersection of the two planes

$$4x + 6y - z = 1$$

and

$$4x + z = 0 \ .$$

b) Find the point on the line which is closest to the origin.

Solution:
a) In order to find the line of intersection, we have to find a point Q in the intersection as well as the direction of intersection. We get a point in the intersection by setting one variable zero. Lets take $x = 0$. Then $6y - z = 1, z = 0$ so that $Q = (0, 1/6, 0)$.
The cross product of the normal vectors between two vectors is perpendicular to the normal vectors of the plane. The vector $\vec{v} = \langle 4, 6, -1 \rangle$ is perpendicular to the plane $4x + 6y - z = 1$. The vector $\vec{w} = \langle 4, 0, 1 \rangle$ is perpendicular to the plane $4x + z = 0$. The vector $\vec{u} = \vec{v} \times \vec{w} = \langle 6, -8, -24 \rangle$ is a vector in the direction of the line. b) The vectors $\vec{r}(t)$ and $\vec{u}$ must be perpendicular, that is the dot product between $\vec{r}(t) = \langle 6t, 1/6 - 8t, -24t \rangle$ and $\vec{u} = \langle 6, -8, -24 \rangle$ is zero. This gives $t = 1/507$. The closest point is $\vec{r}(t) = (0, 1/6, 0) + (6, -8, -24)/507$.

Problem 6) (10 points)

Consider the parameterized curve

$$\vec{r}(t) = \langle e^t + e^{-t}, 2\cos(t), 2\sin(t) \rangle \ .$$

Find the arc length of this curve from $t = 0$ to $t = 4$.

Solution:
The velocity is $\vec{r}\,'(t) = \langle e^t - e^{-t}, -2\sin(t), 2\cos(t) \rangle$. The speed is $\sqrt{2 + e^{-2t}, +e^{2t}} = (e^t + e^{-t})$. The integral

$$L = \int_0^4 (e^t + e^{-t}) \, dt$$

gives $e^4 - e^{-4} = 2\sinh(4)$.

Problem 7) (10 points)

The set of points P for which the distance from P to $A = (1, 2, 3)$ is equal to the distance from P to $B = (5, 8, 5)$ forms a plane S.

a) Find the equation $ax + by + cz = d$ of the plane S.

b) Find the distance from A to S.

Solution:
a) The key insight is that the point $Q = (A + B)/2 = (3, 5, 4)$ is in the middle of the two points. The plane has to pass through this point. The normal vector is parallel to $\vec{n} = \langle 4, 6, 2 \rangle$. The equation of the plane is

$$4x + 6y + 2z = 50 \, .$$

b) The distance from A to S is half the distance from A to B which is $|\vec{AB}|/2 = |\langle 4, 6, 2 \rangle|/2 = \sqrt{56}/2$.

Problem 8) (10 points)

The Swiss tennis player Roger Federer hits the ball at the point $\vec{r}(0) = (0, 0, 3)$. The initial velocity is $\vec{r}\,'(0) = \langle 100, 10, 13 \rangle$. The tennis ball experiences a constant acceleration $\vec{r}\,''(t) = \langle 2, 0, -32 \rangle$ which is due to the combined force of gravity and a constant wind in the x direction.



a) Where does the tennis ball hit the ground $z = 0$?

b) What is the z-component = (projection onto z vector) $proj_{\vec{k}}(\vec{r}\,'(t))$ of the ball velocity at the impact?

Solution:
a) This is a typical free fall problem. After integrating twice the equation $\vec{r}\,''(t) = \langle 2, 0, -32\rangle$, we get

$$\vec{r}(t) = (0,0,3) + t(100,10,13) + t^2(1,0,-16)$$

$$\vec{r}\,'(t) = (100,10,13) + 2t(1,0,-16)$$

We get an impact with the ground $z = 0$ at time $t = 1$. This is at the position $\vec{r}(1) = (101,10,0)$.
b) The velocity at time $t = 1$ is $(102,10,-19)$. The projection onto the vector $\vec{k}$ is $(0,0,-19)$. Note that this is a vector. The z-component of this vector is the third component of this vector which is -19.

Problem 9) (10 points)

a) (4 points) Parameterize the intersection of the ellipsoid

$$\frac{x^2}{4} + \frac{(y-5)^2}{4} + \frac{z^2}{9} = 2$$

with the plane $z = 3$.

b) (3 points) Parametrize the ellipsoid itself in the form

$$\vec{r}(\theta,\phi) = ...$$

c) (3 points) What is the curvature of the curve at the point $(2,5,3)$?

Hint. While you can use the curvature formula $\kappa(t) = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$ you are also allowed to cite a fact which you know about the curvature.

Solution:
a) The parametrization is

$$\vec{r}(t) = \langle 2\cos(t), 5 + 2\sin(t), 3\rangle .$$

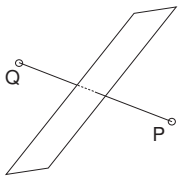
This is a circle of radius 2.
b) The parametrization is

$$\vec{r}(\theta,\phi) = \langle 2\sqrt{2}\cos(\theta)\sin(\phi), 5 + 2\sqrt{2}\sin(\theta)\sin(\phi), \sqrt{2}3\cos(\phi)\rangle .$$

c) The curvature is $1/2$ at all points.

Problem 10) (10 points)

Find an equation $ax + by + cz = d$ for the plane which has the property that $Q = (5, 4, 5)$ is the reflection of $P = (1, 2, 3)$ through that plane.



Solution:
The plane contains the point $(P + Q)/2 = (6, 6, 8)/2 = (3, 3, 4)$ which is the midpoint between P and Q. The direction of the normal vector to the plane is $\vec{n} = (Q - P) = (4, 2, 2)$. The equation is $4x + 2y + 2z = 12 + 6 + 8 = 26$ or $2x + y + z = 13$.

9/27/2017 FIRST HOURLY PRACTICE 11 **Math 21a, Fall 2017**

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TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
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6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The $(\rho, \theta$

Solution:
Make a picture and look at the angles. The θ angle is false.

2) [T] [F] If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$.

Solution:
No, the vectors can be parallel without beeing zero.

3) [T] [F] The surface $z^2 + 4y^2 = x^2 + 1$ is a two sheeted hyperboloid.

Solution:
It is a deformed one-sheeted hyperboloid.

4) [T] [F] The surface $4x^2 - 4x + y^2 - 2y - 120 = -z^2$ is an ellipsoid.

Solution:
Complete the square

5) [T] [F] The parametrized lines $\vec{u}(t) = \langle 1 + 2t, 2 - 5t, 1 + t \rangle$ and $\vec{v}(t) = \langle 3 - 4t, -3 + 10t, 2 - 2t \rangle$ are the same line.

Solution:
The vectors are parllel, and both lines go through the same point.

6) [T] [F] The surface $\sin(x) = z$ contains lines which are parallel to the y-axis.

Solution:
One can translate the surface in the y direction.

7) [T] [F] If $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{w} = 0$ and $\vec{v}$ is not the zero vector, then $\vec{u} \cdot \vec{w} = 0$.

Solution:
The assumption means that $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{w}$. But that does not mean that $\vec{u}$ and $\vec{w}$ are perpendicular.

8) [T] [F] The curvature of a curve depends upon the speed at which one travels upon it.

Solution:
The curvature does not depend on the parametrization.

9) [T] [F] Two lines in space that do not intersect must be parallel.

Solution:
They can be skew.

10) [T] [F] A line in space can intersect an elliptic paraboloid in 4 points.

Solution:
It can only intersect it in 2 points or 1 point or avoid it at all.

11) [T] [F] If $\vec{u} \times \vec{v} = 0$ and $\vec{u} \cdot \vec{v} = 0$, then one of the vectors $\vec{u}$ and $\vec{v}$ is zero.

Solution:
A vector which is both parallel and perpendicular to an other vector can only be the zero vector.

12) [T] [F] If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel at time $t = 1$, then the curvature $\kappa(t)$ of the curve is zero at time $t = 1$.

Solution:
You can see this from the formula $\kappa = |r'(t) \times r''(t)|/|r'(t)|^3$. You can also think about it as follows. Assume the curvature were $\kappa = 1/r$. Then you as well locally move on a circle with radius r. But the acceleration has now a component perpendicular to your velocity vector. But we assumed there is no such acceleration.

13) [T] [F] If the speed of a parametrized curve is constant over time, then the curvature of the curve $\vec{r}(t)$ is zero.

Solution:
It would be true if the velocity would be constant over time. But we can move on a circle with constant speed.

14) [T] [F] The length of the vector projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the length of the vector projection of $\vec{w}$ onto $\vec{v}$.

Solution:
If the lengths of $\vec{v}$ and $\vec{w}$ are the same, then the statement is true. In general, it is not.

15) [T] [F] A quadric $ax^2 + by^2 + cz^2 = 1$ is contained in the interior of a sphere $x^2 + y^2 + z^2 < 100$, then the constants a, b, c are all positive and the quadric is an ellipsoid.

Solution:
If any of the constants would become negative, the quadric becomes unbounded.

16) [T] [F] There is a hyperboloid of the form $ax^2 + by^2 - cz^2 = 1$ which has a trace which is a parabola.

Solution:
Traces are either hyperbola or ellipses.

17) [T] [F] The set of points in space which have distance 1 from the line $x = y = z$ form a cylinder.

Solution:
Yes, if the the line is the z-axis, then $x^2 + y^2 = 1$ is the equation of the cylinder.

18) [T] [F] The velocity vector of a parametric curve $\vec{r}(t)$ always has constant length.

Solution:
This is only true for an arc length parametrization.

19) [T] [F] The volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u} \times \vec{v}) \times \vec{w}|$.

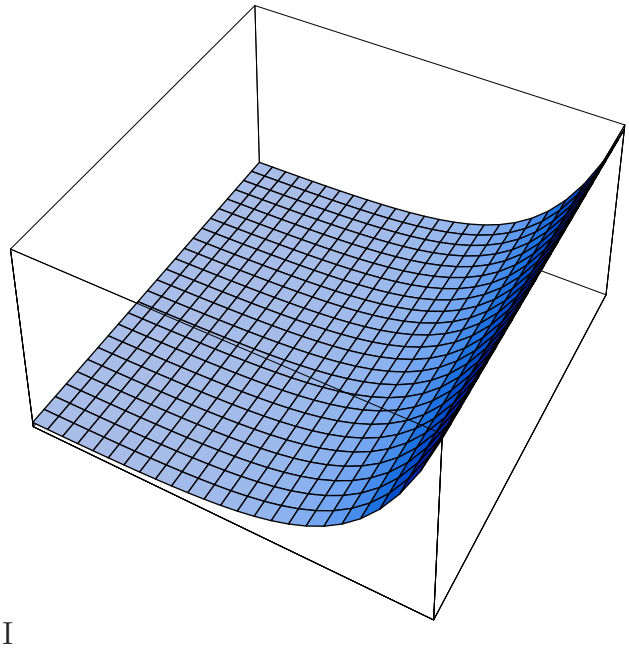
Solution:
The triple scalar product contains also a dot product.

20) [T] [F] The equation $x^2 + y^2/4 = 1$ in space describes an ellipsoid.

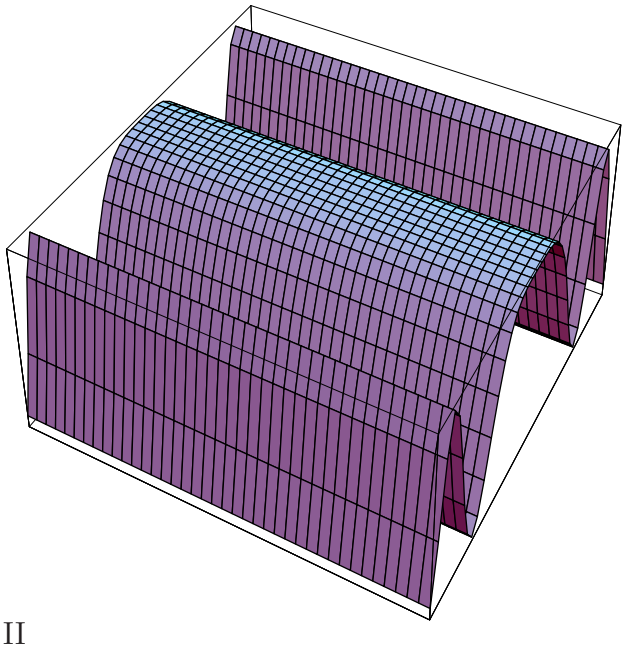
Solution:
The equation describes an elliptical cylinder.

Problem 2a) (3 points)

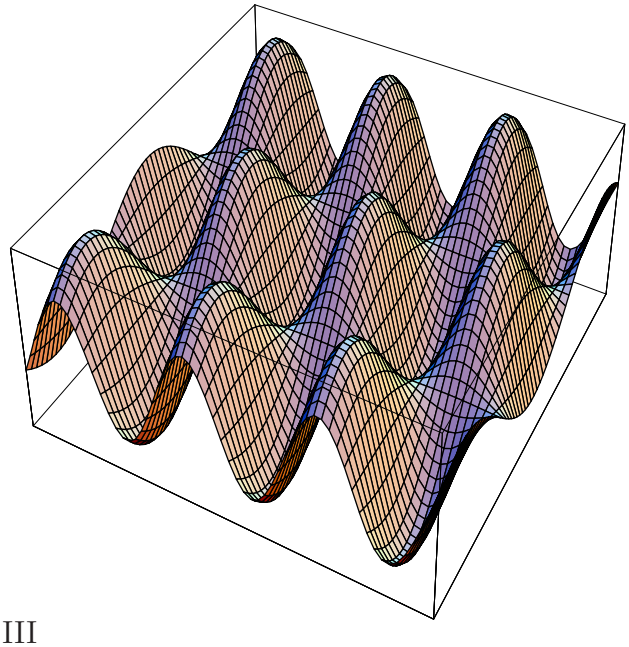
Match the equation with their graphs. No justifications are needed.



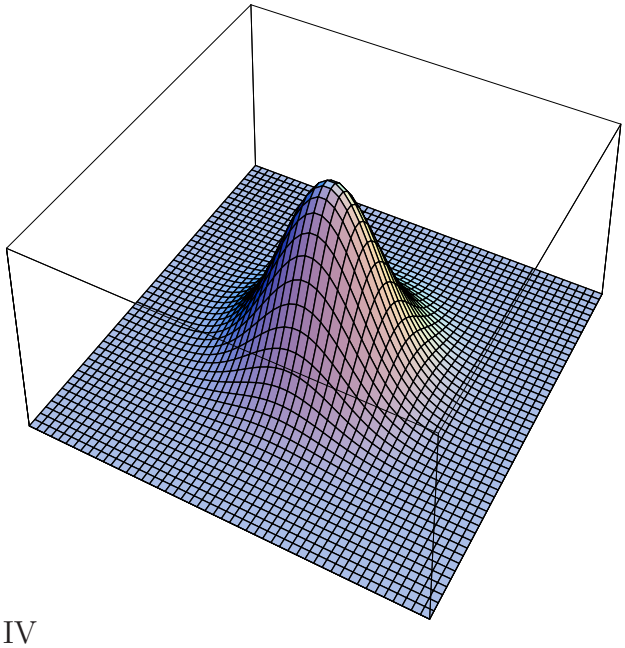
I



II



III



IV

Enter I,II,III,IV here	Equation
	$z = \sin(5x)\cos(2y)$
	$z = \cos(y^2)$
	$z = e^{-x^2-y^2}$
	$z = e^x$

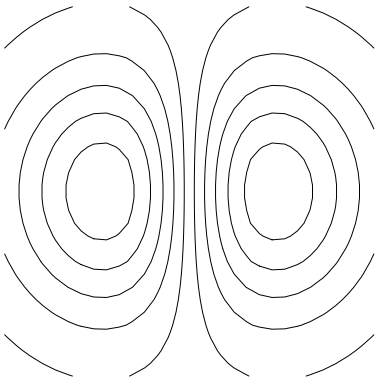
Solution:

Enter I,II,III,IV here	Equation	Justification
III	$z = \sin(5x)\cos(2y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
IV	$z = e^{-x^2-y^2}$	has a maximum at (0,0)
I	$z = e^x$	no y dependence, monotone in x

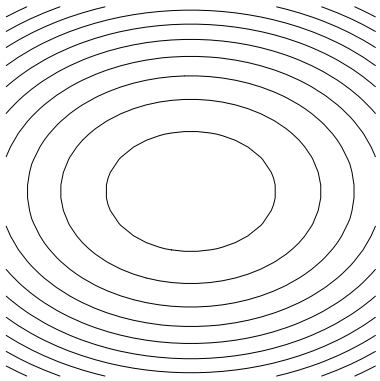
Problem 2b) (4 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

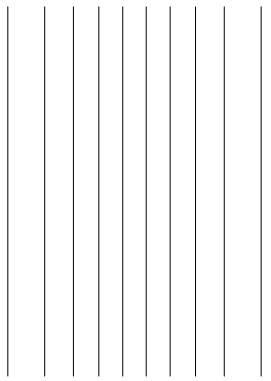
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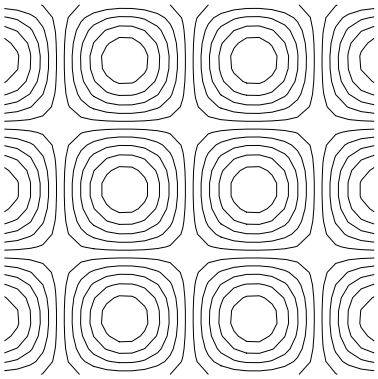
II



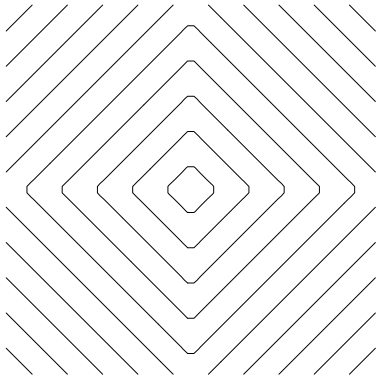
III



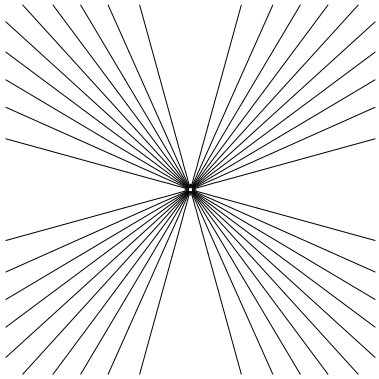
IV



V



VI



Enter I,II,III,IV,V or VI here	Function $f(x, y)$
	$f(x, y) = \sin(x)$
	$f(x, y) = x^2 + 2y^2$
	$f(x, y) = \lvert x \rvert + \lvert y \rvert$
	$f(x, y) = \sin(x)\cos(y)$
	$f(x, y) = xe^{-x^2-y^2}$
	$f(x, y) = x^2/(x^2 + y^2)$

Solution:

Enter I,II,III,IV,V or VI here	Function $f(x, y)$
III	$f(x, y) = \sin(x)$
II	$f(x, y) = x^2 + 2y^2$
V	$f(x, y) = \|x\| + \|y\|$
IV	$f(x, y) = \sin(x)\cos(y)$
I	$f(x, y) = xe^{-x^2-y^2}$
VI	$f(x, y) = x^2/(x^2 + y^2)$

Problem 2c) (3 points)

Match the following points in cartesian coordinates with the points in spherical coordinates:

a) $(x, y, z) = (\sqrt{2}, 0, 0)$
b) $(x, y, z) = (0, \sqrt{2}, 0)$
c) $(x, y, z) = (0, 0, \sqrt{2})$
d) $(x, y, z) = (1, 1, 0)$
e) $(x, y, z) = (1, 0, 1)$
f) $(x, y, z) = (0, 1, 1)$

1) $(\rho, \phi, \theta) = (\sqrt{2}, 0, 0)$.
2) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/4)$.
3) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, 0)$.
4) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/2)$.
5) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$.
6) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, 0)$.

Solution:
a = 3, b = 4, c = 1, d = 2, e = 6, f = 5

Problem 3) (10 points)

a) (7 points) Find a parametric equation for the line which is the intersection of the two planes $2x - y + 3z = 9$ and $x + 2y + 3z = -7$.

b) (3 points) Find a plane perpendicular to both planes given in a) which has the additional property that it passes through the point $P = (1, 1, 1)$.

Solution:
a) We get the direction of the line by taking the cross product of $\langle 2, -1, 3\rangle$ and $\langle 1, 2, 3\rangle$ which is $\langle -9, -3, 5\rangle$. To find a point in both lines, subtract one from the other to get $x - 3y = 16$. If $z = 0$, then $2x - y = 9$ and $x + 2y = -7$ so that $x = 11/5, y = -23/5$. The parametric equations are $(x, y, z) = (11/5, -23/5, 0) + t(-9, -3, 5)$.
b) Plug in the coordinates $(x, y, z) = (1, 1, 1)$ of the point to get the constant $-9x - 3y + 5z = -7$.

Problem 4) (10 points)

Given the vectors $\vec{v} = \langle 1, 1, 0\rangle$ and $\vec{w} = \langle 0, 0, 1\rangle$ and the point $P = (2, 4, -2)$. Let Σ be the plane which goes through the origin $(0, 0, 0)$ and which contains the vectors $\vec{v}$ and $\vec{w}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

a) (6 points) Compute the distance from P to the plane Σ.

b) (4 points) Find the shortest distance from P to the sphere S.

Solution:
a) $\Sigma : x - y = 0$, $n = (1, -1, 0)$. The point $Q = (0, 0, 0)$ is on the plane. $\vec{PQ} \cdot \vec{n}/|\vec{n}| = \langle 2, 4, -2\rangle \cdot \langle 1, -1, 0\rangle/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$ is the distance.
b) $\sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$ is the distance to the origin. So the distance to the sphere is 1 less. The answer is $\sqrt{24} - 1$.

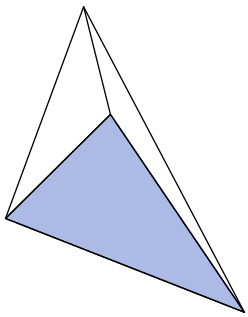
Problem 5) (10 points)

a) (6 points) Find an equation for the plane through the points $A = (0, 1, 0), B = (1, 2, 1)$ and $C = (2, 4, 5)$.

b) (4 points) Given an additional point $P = (-1, 2, 3)$, what is the volume of the tetrahedron

which has A, B, C, P among its vertices.

A useful fact which you can use without justification in b): the volume of the tetrahedron is 1/6 of the volume of the parallelepiped which has AB, AC, and AP among its edges.



Solution:
a) The vectors $\vec{v} = \vec{AB} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \vec{AC} = \langle 2, 3, 5 \rangle$ are in the plane. Their cross product is $\vec{n} = \langle 2, -3, 1 \rangle$. This vector is perpendicular to the plane. The equation of the plane is therefore $2x - 3y + z = d$. Plugging in one point like A, gives $d = -3$.
b) With the vector $\vec{u} = \vec{AP} = \langle -1, 1, 3 \rangle$, one can express the volume of the parallelepiped as $|[\vec{u}, \vec{v}, \vec{w}]| = |\vec{u} \cdot \vec{n}| = |\langle -1, 1, 3 \rangle \cdot \langle 2, -3, 1 \rangle| = |2| = 2$. The volume of the tetrahedron is $2/6 = 1/3$.

Problem 6) (10 points)

The parametrized curve $\vec{u}(t) =< t, t^2, t^3 >$ (known as the "twisted cubic") intersects the parametrized line $\vec{v}(s) = \langle 1 + 3s, 1 - s, 1 + 2s \rangle$ at a point P. Find the angle of intersection.

Solution:
The curves intersect at $P = (1, 1, 1)$ with $t = 1$, $s = 0$ So, it remains to find the angle between the velocity vectors $\vec{u}\,'(= \langle 1, 2, 3 \rangle$ and $\vec{v}\,'(0) = \langle 3, -1, 2 \rangle$, which is 60 degrees.

Problem 7) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (\log(t), 2t, t^2)$, where $\log(t)$ is the natural logarithm (denoted by $\ln(t)$ in some textbooks).

a) What is the velocity and what is the acceleration at time $t = 1$?

b) Find the length of the curve from $t = 1$ to $t = 2$.

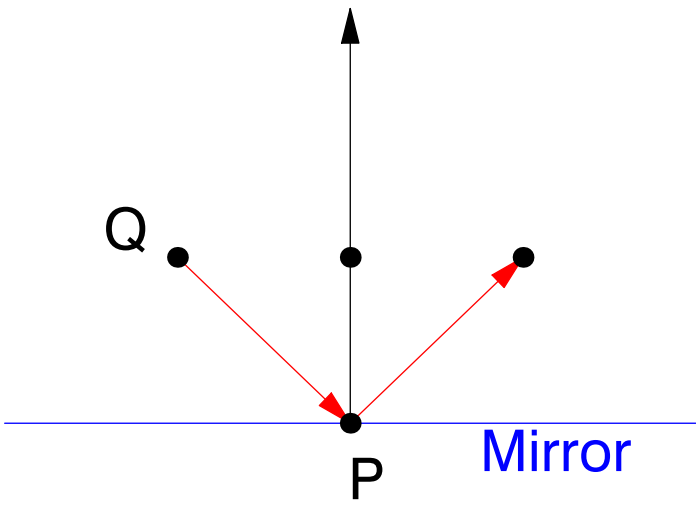
Solution:
a) $\vec{v}(t) = \vec{r}\,'(t) = \langle 1/t, 2, 2t\rangle$.
$\vec{v}(1) = \langle 1, 2, 2\rangle$
$\vec{a}(t) = \vec{r}\,''(t) = \langle -1/t^2, 0, 2\rangle$.
$\vec{a}(1) = \langle -1, 0, 2\rangle$.
b) $\int_1^2 \sqrt{1/t^2 + 4 + 4t^2}\, dt = \int_1^2 1/t + 2t\, dt = \log(t) + t^2|_1^2 = \log(2) + 3$.

Problem 8) (10 points)

A planar mirror in space contains the point $P = (4, 1, 5)$ and is perpendicular to the vector $\vec{n} = \langle 1, 2, -3\rangle$. The light ray $\vec{QP} = \vec{v} = \langle -3, 1, -2\rangle$ with source $Q = (7, 0, 7)$ hits the mirror plane at the point P.

a) (4 points) Compute the projection $\vec{u} = \vec{\mathrm{P}}_{\vec{n}}(\vec{v})$ of $\vec{v}$ onto $\vec{n}$.

b) (6 points) Identify $\vec{u}$ in the figure and use it to find a vector parallel to the reflected ray.



Solution:
a) $\mathrm{P}_{\vec{n}}(\vec{v}) = \frac{(\vec{n}\cdot\vec{v})}{|\vec{n}|^2}\vec{n} = (\langle 1,2,-3\rangle \cdot \langle -3,1,-2\rangle)/14\vec{n} = (5/14)\langle 1,2,-3\rangle$.
b) With $\vec{u}$ we can get the reflected vector $\vec{w}$ because $\vec{w} - \vec{v} = -2\vec{u}$ so that $\vec{w} = \vec{v} - 2\vec{u}$. Note that $\vec{u}$ points down towards the mirror.

Problem 9) (10 points)

We know the acceleration $\vec{r}\,''(t) = \langle 2,1,3\rangle + t\langle 1,-1,1\rangle$ and the initial position $\vec{r}(0) = \langle 0,0,0\rangle$ and initial velocity $\vec{r}\,'(0) = \langle 11,7,0\rangle$ of an unknown curve $\vec{r}(t)$. Find $\vec{r}(6)$.

Solution:

$$\vec{r}\,'(t) = \int_0^t \langle 2+t, 1-t, 3+t\rangle \, dt + \vec{r}\,'(0) = \langle 11 + 2t + t^2/2, 7 + t - t^2/2, 3t + t^2/2\rangle$$

$$\vec{r}\,(t) = \int_0^t \langle 2t+t^2/2, t-t^2/2, 3t+t^2/2\rangle \, dt + \vec{r}(0) = \langle 11t+t^2+t^3/6, 7t+t^2/2-t^3/6, 3t^2/2+t^3/6\rangle$$

Plug in the time $t = 6$ gives $= \boxed{\langle 138, 24, 90\rangle}$.

Problem 10) (10 points)

Intersecting the elliptic cylinder $x^2 + y^2/4 = 1$ with the plane $z = \sqrt{3}x$ gives a curve in space.

a) (3 points) Find the parametrization of the curve.

b) (3 points) Compute the unit tangent vector $\vec{T}$ to the curve at the point $(0,2,0)$.

c) (4 points) Write down the arc length integral and evaluate the arc length of the curve.

Solution:
a) With $x = \sin(t), y = 2\cos(t), z = \sqrt{3}\sin(t)$, we check $x^2 + y^2/4 = \sin^2(t) + \cos^2(t) = 1$. The parametrization is $\vec{r}(t) = \langle \sin(t), 2\cos(t), \sqrt{3}\sin(t)\rangle$.
b) Compute $\vec{r}\,'(t) = \langle \cos(t), -2\sin(t), \sqrt{3}\cos(t)\rangle$, the speed $|\vec{r}\,'(t)| = 2$ and $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)| = \langle \cos(t)/2, -\sin(t), \sqrt{3}\cos(t)/2\rangle$.
c) $|\vec{r}\,'(t)| = 2$. The length is $\int_0^{2\pi} |\vec{r}\,'(t)| \, dt = \int_0^{2\pi} 2 \, dt = \boxed{4\pi}$.

9/27/2017 FIRST HOURLY PRACTICE 12 **Math 21a, Fall 2017**

Name:

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MWF 11 Chenglong Yu
MWF 12 Stepan Paul
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10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | | F | The

Solution:
The xy-plane is $\phi = \pi/2$. Note that $\phi = \pi/4$ is the upper part of a cone.

2) | T | | F | The length of the unit tangent vector $\vec{T}$ for a curve $\vec{r}(t)$ is independent of t.

Solution:
It has length 1.

3) | T | | F | For all vectors $\vec{v}$ and $\vec{w}$ the vector $\vec{w} \times (\vec{w} \times \vec{v})$ is perpendicular to $\vec{v}$.

Solution:
Take an example like $\vec{v} = \vec{j}$ and $\vec{w} = i$.

4) | T | | F | There is a point (x, y, z) in space, for which the cylindrical coordinates (r, θ, z) and spherical coorindates (ρ, θ, ϕ) satisfy $(r, \theta, z) = (\rho, \theta, \phi - \pi/2)$.

Solution:
The first component mans $z = 0$. This implies $\phi = \pi/2$. Every point on the x, y plane satisfies this.

5) | T | | F | The two planes $x + y - z = 1$ and $-x - y + z = 2$ intersect in a line.

Solution:
Their intersection is empty because their normal vectors are the same and the equations are not just a scalar multiple of each other.

6) | T | | F | $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$ implies $|u| = |v|$.

Solution:
Multiply out using the distributive law. The mixed terms cancel and we see $|\vec{u}|^2 = |\vec{v}|^2$.

7) [T] [F] The contour curves $\sin(x) + y = 1$ and $\sin(x) + y = 2$ do not intersect.

Solution:
If these curves would intersect in a point (x, y), then f would take two values 1 and 2 at the same point, which is not possible. There are functions for which the contour lines intersect like $f(x, y) = x^2 - y^2/(x^2 + y^2$ but then this function is not continuous at 0.

8) [T] [F] There is a vector $\vec{v}$ for which the vector projection $\text{proj}_{\vec{v}}(\vec{j})$ is equal to $2\vec{j}$.

Solution:
The vector projection is parallel to $\vec{v}$ not $\vec{j}$.

9) [T] [F] $(\vec{k} \times \vec{i}) \times \vec{i} = \vec{j} \times (\vec{i} \times \vec{k})$

Solution:
The right hand side is zero, the left hand side not.

10) [T] [F] If a curve $\vec{r}(t)$ lies in a plane, goes through the point $(1, 1, 1)$, and has the binormal vector $\vec{B}(t) = \langle 3, 4, 5 \rangle$, then the plane is $3x + 4y + 5z = 12$.

Solution:
The unit tangent and normal vector are in the plane. The binormal vector is perpendicular.

11) [T] [F] The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

Solution:
It is true for circles, but false in general. For example, on a line, the acceleration parallel to the velocity.

12) [T] [F] A line intersects a hyperbolic paraboloid always in 2 distinct points.

Solution:
It can intersect in 1 point. Just take the z-axes for example.

13) [T] [F] There is a quadric surface, each of whose intersections with the coordinate planes is either an ellipse or a parabola.

Solution:
The elliptic paraboloid does the job.

14) [T] [F] The equation $x^2 - y^2 - z^2 = 1$ defines a one-sheeted hyperboloid.

Solution:
$f(x, y, z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

15) [T] [F] The function $f(x, y) = 1/(1 + x^2 + y^2)$ is continuous everywhere.

Solution:
Take the derivatives.

16) [T] [F] If the number $\vec{u} \cdot (\vec{v} \times \vec{w})$ is positive, then $(\vec{w} \times \vec{v}) \cdot \vec{u}$ is positive.

Solution:
Use that the cross product is anti commutative, and the dot product is commutative. You can find counter examples with simple choices like $\vec{u} = \vec{k}, \vec{v} = \vec{i}, \vec{w} = \vec{j}$.

17) [T] [F] The number $|\vec{u} \times (\vec{v} \times \vec{w})|$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:
The volume is the triple scalar product not the triple crossed product.

18) [T] [F] The set of points P for which the distance of P to the point $(0,0,0)$ is 1 less than the distance to the point $(0,0,2)$ is a paraboloid.

Solution:
This was a homework problem. Remember the GPS problem? It is a hyperboloid.

19) [T] [F] If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $\text{proj}_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

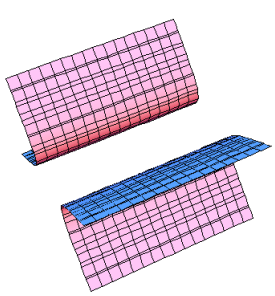
Solution:
The projection vector has length $|\vec{v} \cdot \vec{w}|/|\vec{w}|$ which has length smaller or equal to $\vec{v}$ (use the cos formula).

20) [T] [F] The number $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

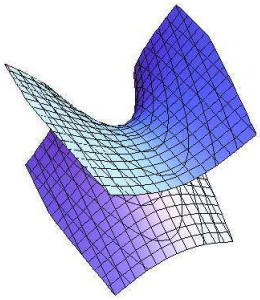
Solution:
Thats an important fact.

Problem 2a) (5 points)

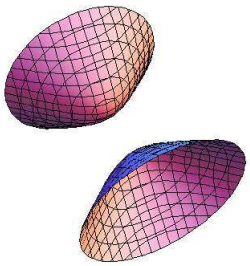
Match the equations with the pictures. No justifications are necessary in this problem.



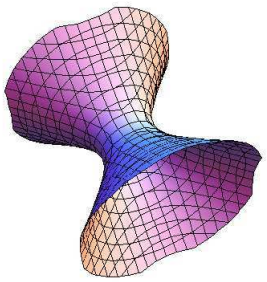
I



II

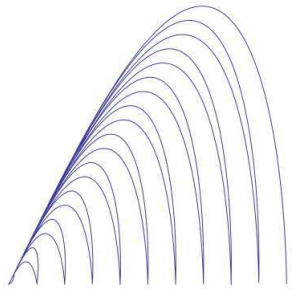


III

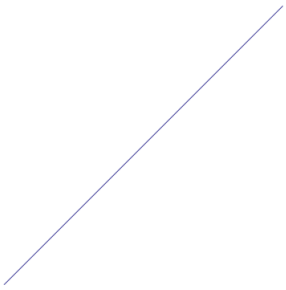


IV

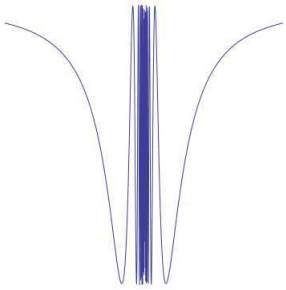
Enter I,II,III,IV here	Equation
	$x^2 + y - z^2 - 1 = 0$
	$y^2 - 2z^2 - 1 = 0$
	$x^2 - y^2 + z^2 + 1 = 0$
	$x^2 - y^2 + z^2 - 1 = 0$



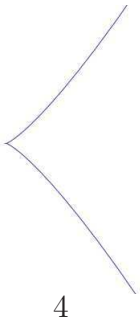
1



2

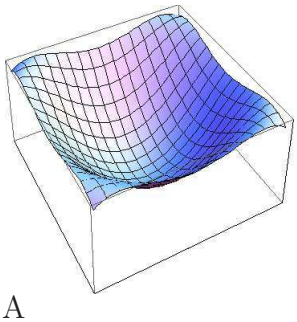


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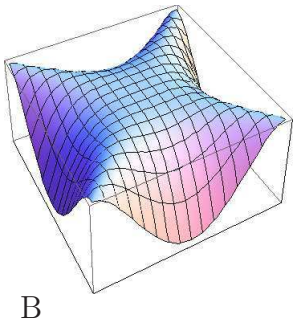


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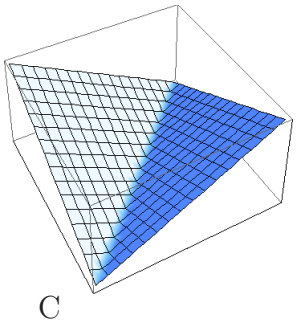
Enter 1,2,3,4 here	Equation
	$\langle t^4, 1 + t^5 \rangle$
	$\langle t\cos(5t), t\cos(5t) \rangle$
	$\langle \lvert t\cos(5t)\rvert, \lvert t\sin(10t)\rvert \rangle$
	$\langle 3 + 2t, \cos(1/t) \rangle$



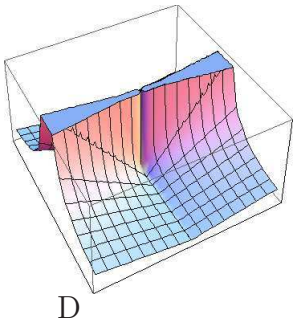
A



B



C



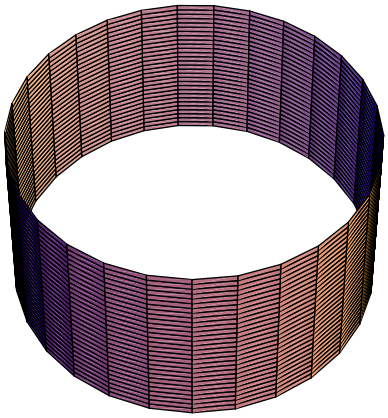
D

Enter A,B,C,D here	Equation
	$f(x, y) = \lvert x/y\rvert$
	$f(x, y) = \sin(x^2 + y^2)$
	$f(x, y) = \lvert x - y\rvert$
	$f(x, y) = \cos(x^2 - y^2)$

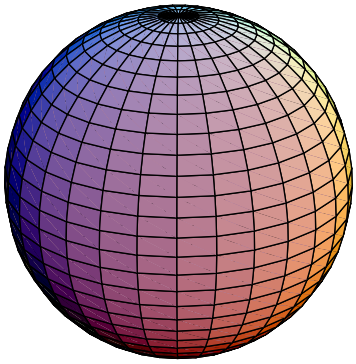
Solution:
II,I,III,IV, 4,2,1,3, D,A,C,B

Problem 2b) (5 points)

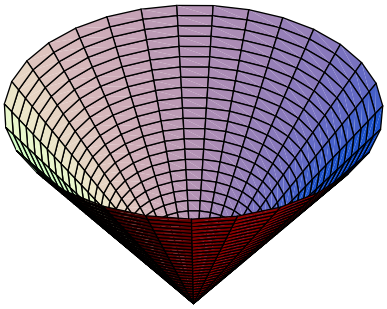
Match the surfaces with their parametrizations as well as with the description either in cylindrical coordinates (r, θ, z) or in spherical coordinates (ρ, ϕ, θ).



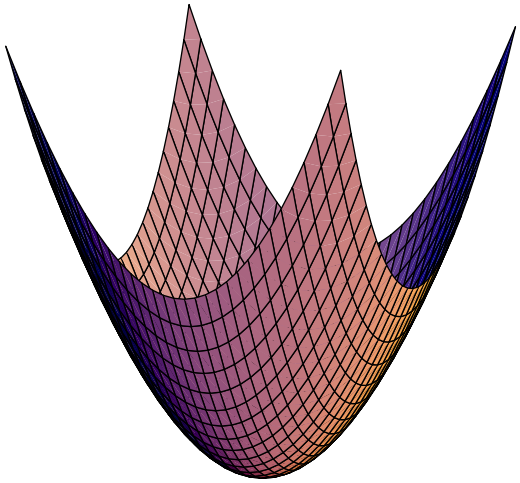
I



II



III



IV

Enter I,II,III,IV here	Parametrization of the surface
	$\langle 3\cos(\theta), 3\sin(\theta), 2z\rangle$.
	$\langle x, y, 3x^2 + 3y^2\rangle$
	$\langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$
	$\langle 3z\cos(\theta), 3z\sin(\theta), 2z\rangle$

Enter I,II,III,IV here	Description in cylindrical or spherical coordinates
	$z = 3r^2$
	$3z = 2r$
	$\rho = 3$
	$r = 3$

Solution:

Enter I,II,III,IV here	Parametrization of the surface
I	$\langle 3\cos(\theta), 3\sin(\theta), 2z\rangle$.
IV	$\langle x, y, 3x^2+3y^2\rangle$
II	$\langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$
III	$\langle 3z\cos(\theta), 3z\sin(\theta), 2z\rangle$

Enter I,II,III,IV here	Description in cylindrical or spherical coordinates
IV	$z = 3r^2$
III	$3z = 2r$
II	$\rho = 3$
I	$r = 3$

Problem 3) (10 points)

A tetrahedron has the vertices $A = (1,1,0), B = (3,2,0), C = (2,1,1), D = (3,2,1)$ with base triangle A, B, C.

a) (5 points) Find the height of the tetrahedron.

b) (5 points) The volume of a tetrahedron is the base area times height divided by 3. What is the volume of the tetrahedron with vertices A,B,C,D.

Solution:
a) The height is the distance of the point D to the plane spanned by $\vec{AB} = \langle 2,1,0\rangle$ and $\vec{AC} = \langle 1,0,1\rangle$. The normal vector $\vec{n} = \vec{AB} \times \vec{AC}$ is $\langle 1,-2,-1\rangle$. The distance is the scalare projection of $\vec{AD} = \langle 2,1,1\rangle$ onto $\vec{n}$ which is

$$d = \frac{\vec{n}\cdot\vec{AD}}{|\vec{n}|} = 1/\sqrt{6}\ .$$

b) The area of the base is the length of the vector $\vec{n}$ because the crossed product gives the area of a parallelepiped. The area is $\sqrt{6}/2$ because the triangle has half the area. The volume of the parallelepiped the height times the base area divided by 3 which is $\boxed{1/6}$.

Problem 4) (10 points)

Find a parametrization of the line containing the two planes

$$2x + y + z = 4$$

and

$$x - y + 2z = 5 \ .$$

Solution:
We can find the direction of the intersection by taking the cross product of $\langle 2, 1, 1\rangle$ with $\langle 1, -1, 2\rangle$ which is $\langle 3, -3, -3\rangle$. One point of intersection is $(3, -2, 0)$. We have therefore the parametrization $\vec{r}(t) = \langle 3, -2, 0\rangle + t\langle 3, -3, -3\rangle = \langle 3 + 3t, -2 - 3t, -3t\rangle$.

Problem 5) (10 points)

What is the distance between the two cylinders $x^2 + y^2 = 1$ and $(z - 2)^2 + (x - 5)^2 = 4$?

Solution:
The parametrization of the first cylinder axis is $\vec{r}(t) = \langle 0, 0, 0\rangle + t\langle 0, 0, 1\rangle$. The parametrization of the second cylinder axis is $\vec{r}(t) = \langle 5, 0, 2\rangle + t\langle 0, 1, 0\rangle$. The distance is $(\langle 5, 0, 2\rangle - \langle 0, 0, 0\rangle) \cdot (\langle 0, 0, 1\rangle \times \langle 0, 1, 0\rangle / |\langle 1, 0, 0\rangle| = \langle 5, 0, 2\rangle \cdot \langle 1, 0, 0\rangle = 5$. The distance between the cylinders is $5 - 1 - 2 = 2$. The final answer is $\boxed{2}$.

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle 2\sin(t), \frac{t^4}{4} + \frac{1}{2t^2}, 2\cos(t)\rangle$$

from $t = 1$ to $t = 2$.

Solution:
The velocity is $\vec{r}\,'(t) = \langle 2\cos(t), t^3 + t^{-3} - 2\sin(t)\rangle$. The speed is $\sqrt{2 + t^3, +1/t^3} = (t^3 + 1/t^3)$. The integral

$$L = \int_1^2 (t^3 + 1/t^3)\ dt$$

gives $t^4/4 - 1/(2t^2|_1^2 = \boxed{33/8}$.

Problem 7) (10 points)

At time $t = 0$ two trapeze artists have positions $\vec{r}(0) = \langle 0, 0, 25\rangle$ and $\vec{s}(0) = \langle 10, 0, 23\rangle$ and velocities $\vec{r}\,'(0) = \langle 2, 0, 1\rangle$ and $\vec{s}\,'(0) = \langle -3, 0, 2\rangle$. They both experience a constant gravitational acceleration $\langle 0, 0, -10\rangle$. Find the paths $\vec{r}(t), \vec{s}(t)$ and determine at which point the artists meet.



Solution:
This is a typical free fall problem. The main difficulty in this problem is to imagine, what happens when the two artists collide. The two artists actually were shot by cannons but we didn't want to scare you...
To the solution: Integrate $\vec{r}\,''(t)$ twice.

$$\vec{r}(t) = \langle 2t, 0, 25 + t - 5t^2\rangle \ .$$

$$\vec{s}(t) = \langle 10 - 3t, 0, 23 + 2t - 5t^2\rangle \ .$$

In order to find the intersection, we solve $\vec{r}(t) = \vec{s}(t)$. We can solve for t in any of the three coordinates. The artists meet at $t = 2$ at the point $\boxed{\langle 4, 0, 7\rangle}$.



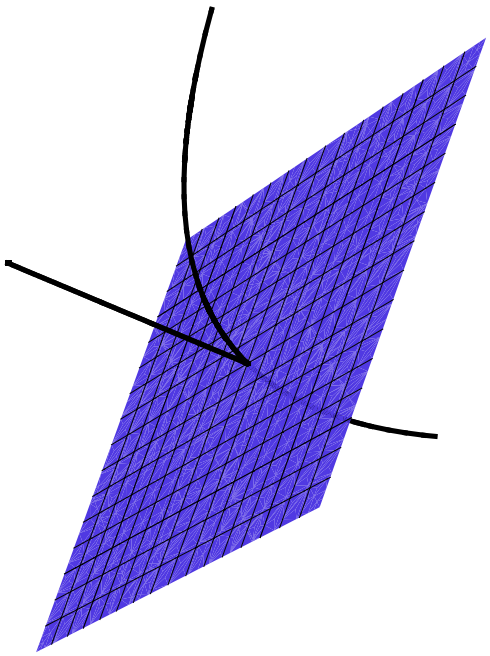
Problem 8) (10 points)

The angle between a curve and a plane is defined as $\pi/2 - \alpha$, where α is the angle between the normal vector to the plane and the velocity vector of the curve at the point of intersection.

a) (3 points) Find a normal vector to the plane $x + y - z/2 = 1$.

b) (3 points) What is the velocity vector to the curve $C : \vec{r}(t) = \langle 1, 0, 0\rangle + t\langle 1, 1, 3\rangle + t^2\langle 1, 1, 1\rangle$ at time $t = 0$?

c) (4 points) Find the angle (in radians) between the plane $x + y - z/2 = 1$ and the curve C at the point of intersection $\vec{r}(0)$.



Solution:
a) The normal vector to the plane is $\vec{n} = \langle 1, 1, -1/2 \rangle$.
b) The velocity vector is $\vec{r}\,'(t) = \langle 1, 1, 3 \rangle$.
c) We use the cos-formula: $\cos(\alpha) = \vec{r}\,'(t) \cdot \vec{n} / |\vec{r}\,'(t)||\vec{n}|$ and solve for

$$\cos(\alpha) = \frac{|\langle 1, 1, 3 \rangle \cdot \langle 1, 1, -1/2 \rangle|}{||\langle 1, 1, 3 \rangle| \cdot |\langle 1, 1, -1/2 \rangle|} = 1/(3\sqrt{11}\,.$$

Therefore, $\alpha = \arccos(1/3\sqrt{11})$ and the final answer is $\boxed{\pi/2 - \arccos(\frac{1}{3\sqrt{11}})}$.

Problem 9) (10 points)

The intersection of the paraboloid

$$x^2 + y^2 - z = 5$$

with the plane

$$x + y = 5$$

is a curve. Find the parametrization of this curve.

Solution:
From the second equation we can get $x = t$, $y = 5 - t$. From the first equation, we get $z = -5 + t^2 + (5 - t)^2$ which gives

$$\vec{r}(t) = \langle t, 5 - t, -5 + t^2 + (5 - t)^2 \rangle \ .$$

Problem 10) (10 points)

a) (3 points) Parametrize the plane containing the three points $A = (1, 1, 1), B = (1, 3, 2)$ and $C = (3, 4, 5)$.
b) (4 points) Parametrize the sphere which is centered at $(1, 1, 1)$ and has radius 3.
c) (3 points) Parametrize the surface which is given in spherical coordinates as $\rho = 3 + \sin(\phi)\sin(\theta)$.

Solution:
a) $\vec{r}(s, t) = \langle 1, 1, 1 \rangle + t\langle 0, 2, 1 \rangle + s\langle 2, 3, 4 \rangle$.
b) $\vec{r}(\theta, \phi) = \langle 1 + 3\cos(\theta)\sin(\phi), 1 + 3\sin(\theta)\sin(\phi), 1 + 3\cos(\phi) \rangle$.
c) $\vec{r}(\theta, \phi) = (3 + \sin(\phi)\sin(\theta))\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$.

11/1/2017 Second Hourly **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The identity $f_{yxyx} = f_{xyxy}$ holds for all smooth functions $f(x, y)$.

Solution:
This is Clairaut

2) [T] [F] Using linearization we can estimate $(1.003)^2(1.0001)^4 \approx 2 \cdot 0.003 + 4 \cdot 0.0001$.

Solution:
The constant is missing.

3) [T] [F] We have $d/dt(x^2(t)y(t)) = \langle 2x(t)y(t), x^2(t) \rangle \cdot \langle x'(t), y'(t) \rangle$.

Solution:
This is a direct application of the chain rule.

4) [T] [F] The function $f(x, y) = 3y^2 - 2x^3$ takes no maximal value on the "squircle" $x^4 + y^4 = 8$.

Solution:
We can find it using Lagrange. Also Bolzano assures that it exists.

5) [T] [F] If $f(x, t)$ solves the heat equation then $f(x, -t)$ solves the heat equation.

Solution:
The sign of f_t changes sign while f_{xx} does not.

6) [T] [F] If $f(x, t)$ solves the wave equation, then $f(x, -t)$ solves the wave equation.

Solution:
Both the sign of f_{tt} and f_{xx} change sign.

7) [T] [F] There exists a smooth function on the region $x^2 + y^2 < 1$ so that it has exactly two local minima and no other critical points.

Solution:
You have worked on that in a homework.

8) [T] [F] For a function $f(x, y)$, the vector $\langle f_x(0,0), f_y(0,0), -1\rangle$ is perpendicular to the graph $f(x, y) = z$ at $(0, 0, f(0, 0))$.

Solution:
Write $g(x, y, z) = f(x, y) - z$ and find the gradient.

9) [T] [F] If a function $f(x, y)$ is equal to its linearization $L(x, y)$ at some point, then $f_{xx}(x, y) = f_{yy}(x, y)$ at every point.

Solution:
Both sides are zero.

10) [T] [F] The equation $f(x, y) = 9x - 5x^2 - y^2 = -9$ implicitly defines $y(x)$ near $(0, 3)$ and $y'(0) = f_x(0, 3)/f_y(0, 3)$.

Solution:
The sign is off

11) [T] [F] If a tangent plane to a surface S intersects S at infinitely many points, then S must be a plane.

Solution:
Take the surface $\sin(x + y)$ for example and the tangent plane $z = 1$ at the point $(\pi/2, 0)$

12) [T] [F] If $\vec{u} = \langle 1, 0, 0\rangle$ and $\vec{v} = \langle 0, 1, 0\rangle$ then $(D_{\vec{u}}D_{\vec{v}} - D_{\vec{v}}D_{\vec{u}})f = D_{\vec{u}\times\vec{v}}f$.

Solution:
The left hand side is zero. The right hand side not necessarily.

13) [T] [F] The surface area of the parametrized surface $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ is $\int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta| r dr d\theta$.

Solution:
There is an r too much.

14) [T] [F] Let D be the unit disk $x^2 + y^2 \le 1$. Any function $f(x,y)$ which satisfies $|\iint_D f(x,y)\,dA| = \iint_D |f(x,y)|\,dA$ must have $f(x,y) \ge 0$ on D.

Solution:
Take a function $f = -1$ for example.

15) [T] [F] The iterated integral $\int_{-1}^{1}\int_{10}^{20} e^{x^2} y^{11}\,dx\,dy$ is zero.

Solution:
Symmetry

16) [T] [F] The tangent plane to the graph of $f(x,y) = xy$ at $(2,3,6)$ is given by $6 + 3(x-2) + 3(y-3) = 0$.

Solution:
There should also be a z-part.

17) [T] [F] If the gradient of $f(x,y)$ at $(1,2)$ is zero, then $f(1,2)$ must be either a local minimum or maximum value of $f(x,y)$ at $(1,2)$.

Solution:
It could be a saddle point

18) [T] [F] If $\vec{r}(t)$ is a parametrization of the level curve $f(x,y) = 5$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Yes because $\vec{r}'$ is perpendicular to the gradient.

19) [T] [F] The function $f(x,y) = (x^3+y^3)/(x^2+y^2)^2$ has a limiting value at $(0,0)$ so that it is continuous everywhere.

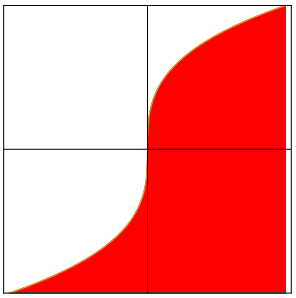
Solution:
Use polar coordinates to see this.

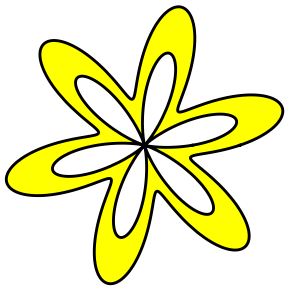
20) [T] [F] If the contour curves $f(x,y) = 1$ and $g(x,y) = 1$ have a common tangent line at $(1,2)$ and $|\nabla f(1,2)| = 1 = |\nabla g(1,2)| = 1$, then $(1,2)$ is a solution to the Lagrange equations for extremizing f under the constraint $g = 1$.

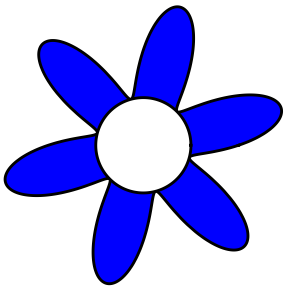
Solution:
The gradients are parallel. That is what the Lagrange equations tell.

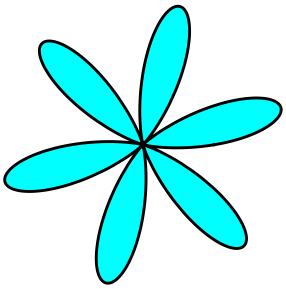
Problem 2) (10 points) No justifications needed

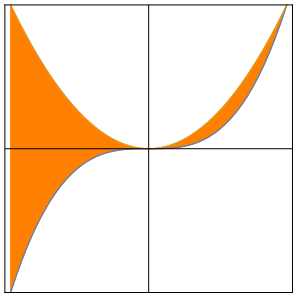
a) (6 points) Please match each picture below with the double integral that computes the area of the region:

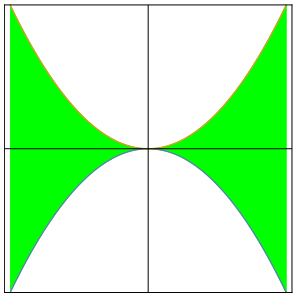
A 

B 

C 

D 

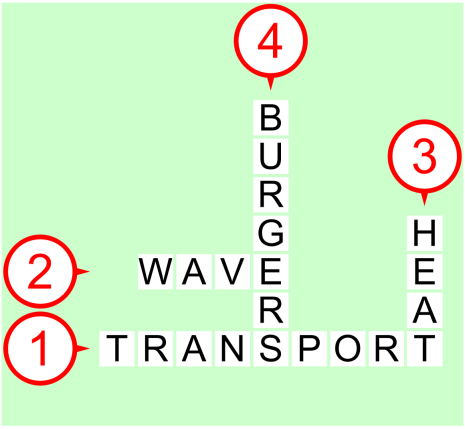
E 

F 

Enter A-F	Integral
	$\int_0^{2\pi}\int_{1+\sin(6\theta)}^{2+\sin(6\theta)} r\ drd\theta$
	$\int_{-1}^{1}\int_{-x^2}^{x^2} 1\ dydx$
	$\int_0^{2\pi}\int_0^{1+\sin(6\theta)} r\ drd\theta$
	$\int_{-1}^{1}\int_{y^3}^{1} 1\ dxdy$
	$\int_{-1}^{1}\int_{x^3}^{x^2} 1\ dydx$
	$\int_0^{2\pi}\int_1^{2+\sin(6\theta)} r\ drd\theta$

b) (4 points) We design a crossword puzzle. Match the PDEs:

Enter 1-4	
	$u_t = u_x$
	$u_t = u_{xx}$
	$u_{tt} = u_{xx}$
	$u_t + uu_x = u_{xx}$



Solution:
a) BFDAEC
b) 1324

Problem 3) (10 points)

3a) (7 points) Fill in the points A-G. There is an exact match. You see the level curves of a function $f(x, y)$ inspired from one of your homework submissions. The circular curve is $g(x, y) = x^2 + y^2 = 1$.

a) At the point [ ], the function f is a global maximum on $g = 1$.

b) At the point [ ], the function f is a global minimum on $g = 1$.

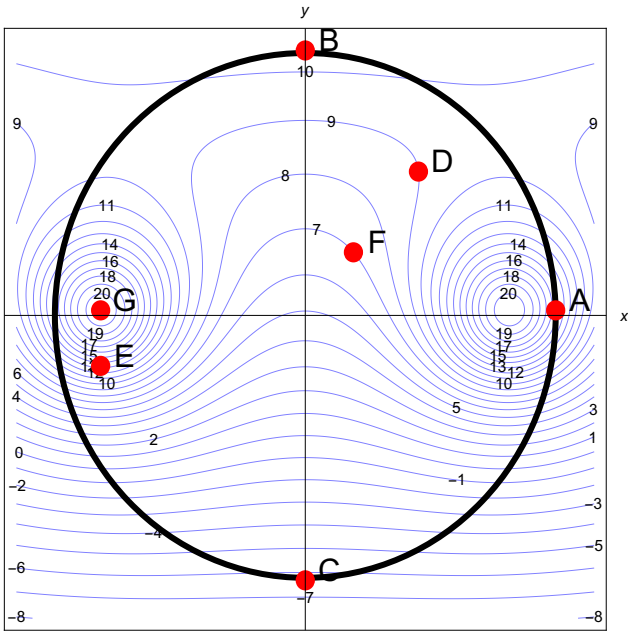
c) At the point [ ], $|f_y|$ is maximal among all points A-G.

d) At the point [ ], $f_x > 0$ and $f_y = 0$.

e) At the point [ ], $f_x > 0$ and $f_y > 0$.

f) At the point [ ], $\nabla f = \lambda \nabla g$ and $g = 1$ for some $\lambda > 0$.

g) At the point [ ], $|\nabla f|$ is minimal among all points A-G.



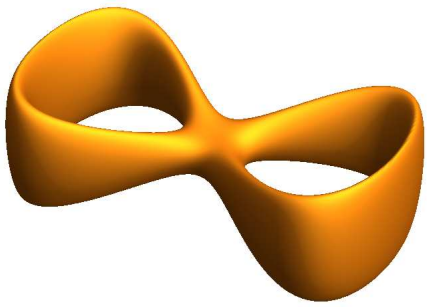
3b) (3 points) Fill in the numbers $1, -1$, or 0. In all cases, the vector $\vec{v}$ is a general unit vector.

a) At a maximum point of $f(x,y)$, we have	$D_{\vec{v}}f =$	
b) At any point (x,y), we have	$\lvert D_{\vec{v}}f\rvert/\lvert\nabla f\rvert \leq$	
c) If $D_{\vec{v}}f = 1$, then	$D_{-\vec{v}}f =$	

Solution:
a) ACEDFBG
b) 0,1,-1

Problem 4) (10 points)

The surface $f(x,y,z) = 1/10$ for $f(x,y,z) = 10z^2 - x^2 - y^2 + 100x^4 - 200x^6 + 100x^8 - 200x^2y^2 + 200x^4y^2 + 100y^4$ is a blueprint for a new sour-sweet gelatin candy brand.



a) (4 points) Find the equation $ax + by + cz = d$ for the tangent plane of f at $(0,0,1/10)$.

b) (3 points) Find the linearization $L(x,y,z)$ of f at $(0,0,1/10)$.

c) (3 points) Estimate $f(0.01, 0.001, 0.10001)$.

Solution:
a) The gradient is $\langle 0,0,2\rangle$ at the point. The equation is $2z = d$. The constant is obtained by plugging in the point. We get $2z = 2/10$ or $z = 1/10$.
b) The linearization is $1/20 + 2(z - 1/10)$.
c) We can estimate $1/10 + 2 \cdot 0.00001$.

Problem 5) (10 points)

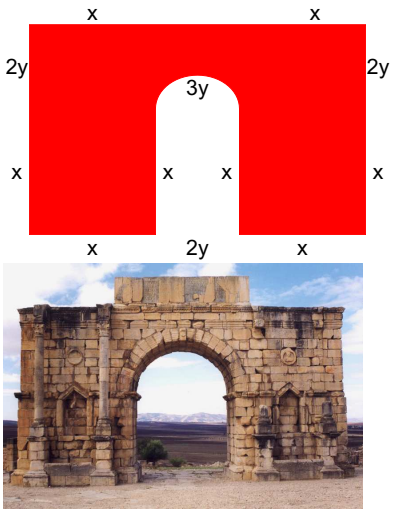
The **marble arch of Caracalla** is a Roman monument, built in the year 211. We look at a region modelling the arch. Using the Lagrange optimization method, find the parameters (x, y) for which the area

$$f(x, y) = 2x^2 + 4xy + 3y^2$$

is minimal, while the perimeter

$$g(x, y) = 8x + 9y = 33$$

is fixed.



Solution:
First write down the Lagrange equations $\nabla f = \lambda \nabla g$ and eliminate λ.

$$\begin{aligned} 4x + 4y &= \lambda 8 \\ 4x + 6y &= \lambda 9 \\ 8x + 9y &= 33\ . \end{aligned}$$

We end up with $(4x + 4y)9 = 8(4x + 6y)\lambda$. Eliminating λ gives $y = x/3$. Plugging this into the constraint gives $\boxed{x = 3, y = 1}$.

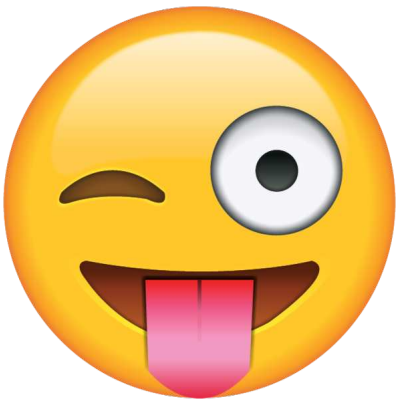
Problem 6) (10 points)

a) (8 points) Find and classify the critical points of the function

$$f(x, y) = x^2 - y^2 - xy^3\ .$$

b) (2 points) Decide whether f has a global maximum or minimum on the entire 2D plane.

We don't know of any application for f. But if you read out the function aloud, it rolls beautifully off your tongue!

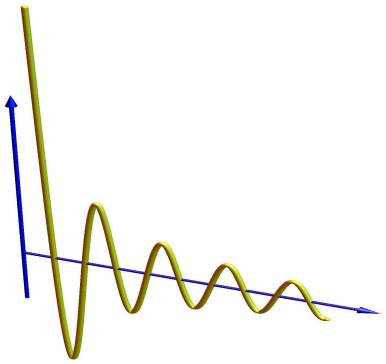


Solution:
The critical points are the solution of $2x - y^3 = 0, -2y - 3xy^2 = 0$. There is only one solution $(0, 0)$. Computing $D = -1$ shows that the point is a saddle point.

Problem 7) (10 points)

We look at the integral

$$\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} \sin(x)/x^2 \, dxdy \, .$$

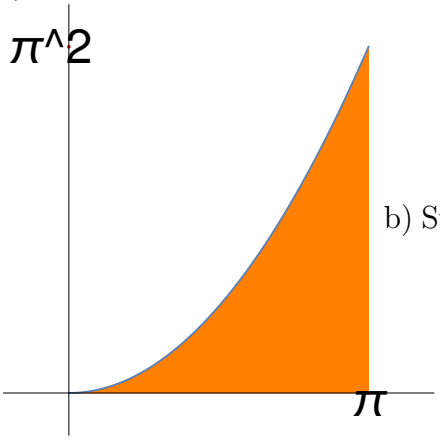


Just for illustration, we have drawn the graph of the function $f(x) = \sin(x)/x^2$.

a) (5 points) Draw the region over which the double integral is taken.

b) (5 points) Find the value of the integral.

Solution:
a) It is the region below the graph of $f(x) = x^2$ on the interval $[0, \pi]$.

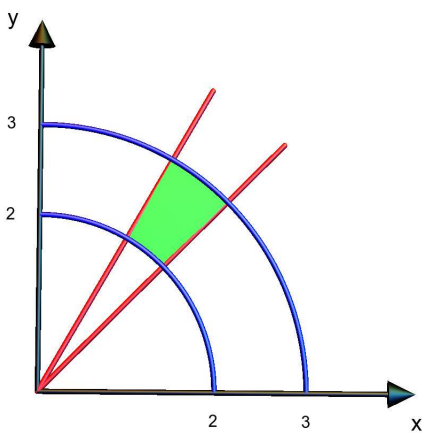


b) Switch the order of integration $\int_0^{\pi} \int_0^{x^2} \sin(x)/x^2 \, dydx = 2$.

Problem 8) (10 points)

"Heat-assisted magnetic recording" (HAMR) promises high density hard drives like 20 TB drives in 2019. The information is stored on sectors, now typically 4KB per sector. Let's assume that the magnetisation density on the drive is given by a function $f(x, y) = \sin(x^2 + y^2)$. We are interested in the total magnetization on the sector R in the first quadrant bounded by the lines $x = y$, $y = \sqrt{3}x$ and the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. In other words, find the integral

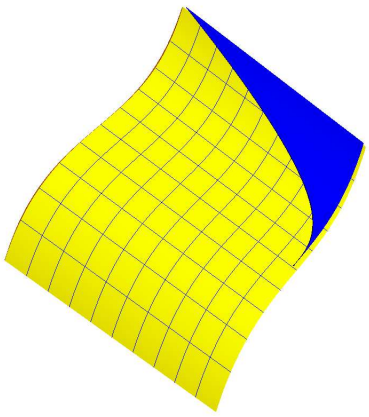
$$\int \int_R \sin(x^2 + y^2) \, dxdy \, .$$



Solution:

$$\int_{\pi/4}^{\pi/3} \sin(r^2) r \; dr d\theta \; = (\pi/3 - \pi/4)[-\cos(r^2)/2|_2^3 = (\pi/24)(\cos(4) - \cos(9))$$

Problem 9) (10 points)



a) (4 points) Write down the double integral for the surface area of

$$\vec{r}(x, y) = \langle 2x, y, x^3/3 \; + y \rangle$$

with $0 \leq x \leq 2$ and $0 \leq y \leq x^3$.

b) (6 points) Find the surface area.

Solution:
a) $\int_0^2 \int_0^{x^3} \sqrt{x^4 + 8} \, dy dx$.
b) $(24^{3/2} - 8^{3/2})/6$.

11/1/2017 Second Hourly Practice 1 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The function $f(x,y) = x^3y/(x^6+y^5)$ can be filled in at the origin with a value $f(0,0) = a$ so that f is continuous everywhere.

Solution:
Go to polar coordinates. The function diverges at $(0,0)$.

2) [T] [F] The chain rule assures that $\int_0^1 (\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(1)) - f(\vec{r}(0))$.

Solution:
Just integrate the chain rule and use the fundamental theorem of calculus.

3) [T] [F] The formula $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 \int_0^1 f(y,x)\, dy\, dx$ holds.

Solution:
Just change the variables x, y first on the left hand side to get $\int_0^1 \int_0^1 f(y,x)\, dx\, dy$, then use Fubini to get the right hand side.

4) [T] [F] If $u(x,t)$ solves the partial differential equation $u_t = u_x$, then so does the function u_x.

Solution:
We can use Clairaut to see that.

5) [T] [F] There is a surface S containing the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for which the tangent plane to S at $(0,0,0)$ is $x + 2y + 3z = 0$.

Solution:
The velocity vector is perpendicular to the surface, not parallel to the surface.

6) [T] [F] For any two unit vectors $\vec{u}$ and $\vec{v}$, and any f, we have $D_{\vec{u}} D_{\vec{v}} f = D_{\vec{v}} D_{\vec{u}} f$.

Solution:
Write down the definitions, we have in both cases $f_{xx}u_1v_1 + f_{yy}u_2v_2 + f_{xy}u_1v_2 + f_{yx}u_2v_1$.

7) [T] [F] If the tangent plane to $z = f(x, y)$ at $(0, 0, f(0, 0))$ is $4 + 3x + 2y + z = 0$, then $L(x, y) = 4 + 3x + 2y$ is the linearization of $f(x, y)$ at $(0, 0)$.

Solution:
This is fast because $L(x, y) = -4 - 3x - 2y$

8) [T] [F] For $f(x, y) = x^3 e^{y^2 \cos y} - x^4 \cos y$ the function $f_{xyxyxyxyxy}$ is zero everywhere.

Solution:
Use Clairaut. We differentiate 5 times with respect to x.

9) [T] [F] The point $(0, 0)$ is a critical point of $f(x, y) = x^3y^2$.

Solution:
There are many critical points but $(0, 0)$ belongs there.

10) [T] [F] The gradient of $f(x, y) = x^2 + y^2$ is a vector perpendicular to the surface $z = f(x, y)$.

Solution:
It is not a 3-vector.

11) [T] [F] If the function $f(x, y)$ attains an absolute maximum on the region $x^2+y^2 \leq 4$ at the point $(2, 0)$, then we must have $f_{xx}(2, 0) \leq 0$.

Solution:
The maximum does not need to be in the interior.

12) [T] [F] If $f(x, y) \leq 5$ for all values of (x, y), then $\int_0^{2\pi} \int_0^7 f(r\cos\theta, r\sin\theta) r \, dr \, d\theta \leq 5\pi(7^2)$.

Solution:
The integral is smaller or equal to $\int_0^{2\pi}\int_0^7 5r\,dr\,d\theta$ which is 5 times the area of the disk.

13) [T] [F] For any constant a, we have $\int_{-a}^{a}\int_0^a \left(e^{x^2}\sin y\right)\,dx\,dy = 0$.

Solution:
Use the symmetry

14) [T] [F] The linearization of the function $f(x,y) = e^{x^2+y}$ at the point $(0,0)$ is the function $L(x,y) = 1 + 2x^2e^{x^2+y} + ye^{x^2+y}$.

Solution:
The linearization is linear

15) [T] [F] Let $\vec{u}$ be the unit vector in the direction $\langle 1,1\rangle/\sqrt{2}$. Then $D_{\vec{u}}f = f_{xy}$.

Solution:
The directional derivative only invokes the first derivatives of f.

16) [T] [F] The integral of $f(x,y) = \sqrt{x^2+y^2}$ over the unit disk is $\int_0^{2\pi}\int_0^1 r\;drd\theta$.

Solution:
We have forgotten the factor r.

17) [T] [F] There is a function $f(x,y)$ for which $D_{\vec{v}}f(0,0) = 1$ for all directions $\vec{v}$.

Solution:
Switching the direction gives a negative value.

18) [T] [F] Given $f(x,y(x)) = 0$, then $f_x + f_y\frac{dy}{dx} = 0$.

Solution:
This is implicit differentiation

19) [T] [F] Any function on a closed and bounded region must have a critical point.

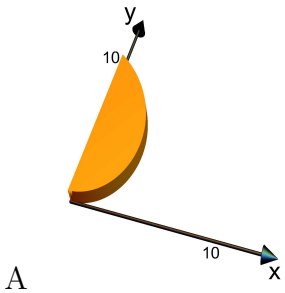
Solution:
Take a linear non-constant function like $x + y$.

20) [T] [F] The integral $\iint_{x^2+y^2\leq 1} |f(x,y)|\, dxdy$ computes the surface area of the surface $z = f(x,y), x^2 + y^2 \leq 1$.

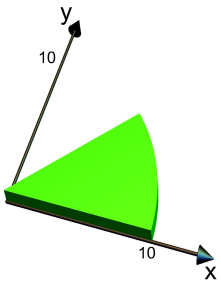
Solution:
The surface area is $\sqrt{1 + f_x^2 + f_y^2}$. In general the areas are different. Take $f = 2$ for example, then the integral under consideration is 2π while the surface area is π.

Problem 2) (10 points) No justifications needed

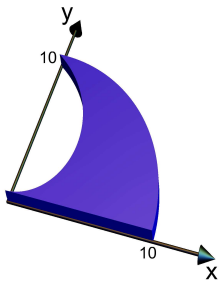
a) (6 points) Double integrals like $\iint_R 1\, dxdy$ or $\iint_R r\, drd\theta$ can be interpreted both as the **area** of the region R as well as the **volume** of the solid under the graph of the constant function $f(x,y) = 1$ or $f(\theta, r) = 1$. Match the regions with the integrals:



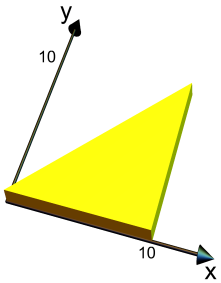
A



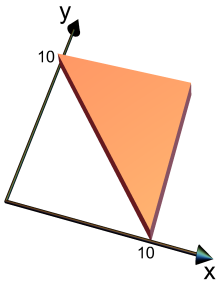
B



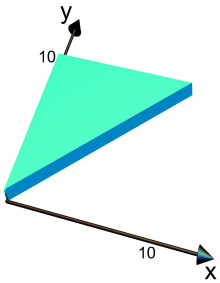
C



D



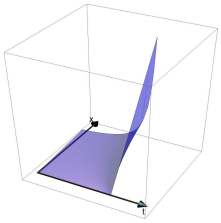
E



F

Enter A-F	Integral
	$\int_0^{10}\int_0^{x} 1\, dydx$
	$\int_0^{10}\int_0^{\pi/4} r\, d\theta dr$
	$\int_0^{\pi/2}\int_0^{20\theta/\pi} r\, drd\theta$
	$\int_0^{\pi/2}\int_{20\theta/\pi}^{10} r\, drd\theta$
	$\int_0^{10}\int_0^{y} 1\, dxdy$
	$\int_0^{10}\int_{10-x}^{10} 1\, dydx$

b) (4 points)



You know the **Transport**, **Wave**, **Heat**, or **Burgers** equation. Given in a possibly different order, these differential equations are $u_t = u_{xx}, u_t = u_x, u_{tt} = u_{xx}, u_t + uu_x = u_{xx}$. Check all the boxes where the given function solves the given PDE.

$f(x,t) = x/(1+t)$ solves	Name
	Burgers
	Transport

$f(x,t) = xt$ solves	Name
	Heat
	Wave

Solution:
a) The order is $DBACFE$.
b) $x/(1+t)$ solves the Burgers equation and xt solves the wave equation.

Problem 3) (10 points)

3a) (7 points) All the parts of this problem refer to the labeled points and the differentiable function $f(x, y)$ whose level curves are shown in the following plot:

a) At the point $\square$, the gradient ∇f has maximal length

b) At the point $\square$, $f_x > 0$ and $f_y = 0$

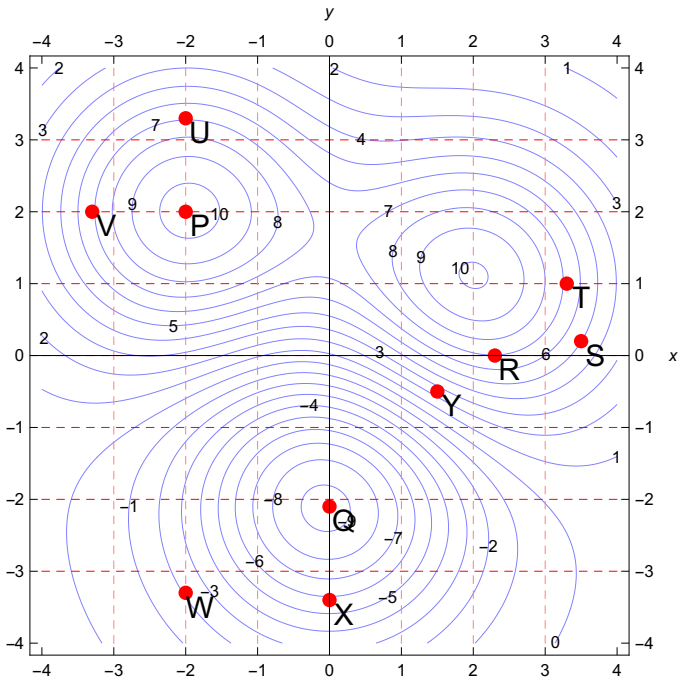
c) At the point $\square$, $f_x < 0$ and $f_y < 0$

d) At the point $\square$, $D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f = 0$ and $f_x \neq 0$

e) At the point $\square$, f achieves a global min on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$

f) At the point $\square$, $\nabla f = \vec{0}$ and $f_{xx} < 0$

g) At the point $\square$, ∇f points straight toward the top of the page.



3b) (3 points) Check the cases where the maximum, minimum or saddle point of the function can be established **conclusively** using the second derivative test. Don't check the box if the test does not apply, (even if it might be a sort of minimum, maximum or saddle).

Critical Point	x^4+y^2	xy	x^2-y^4
Maximum			
Minimum			
Saddle point			

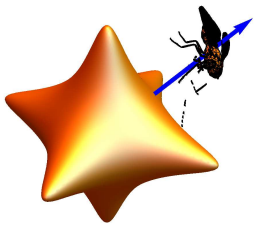
Solution:
a) The answer is $YVWSQPR$.
b) Only check "Saddle xy". The other cases are critical points but situations where $D=0$ and the second derivative test is inconclusive.

Problem 4) (10 points)

a) (4 points) A **math candy** of the form

$$f(x,y,z) = 3x^2y^2 + 3y^2z^2 + 3x^2z^2 + x^2 + y^2 + z^2 = 12$$

is leaning at $(1,1,1)$ at the plane tangent to it. Find that plane.



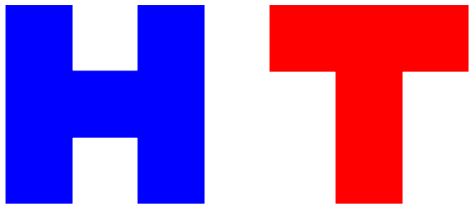
b) (3 points) Estimate $f(1.1, 1.01, 0.98)$ using linearization.

c) (3 points) A fruit fly just dipped some sugar from the candy at $(1,1,1)$ and moves along a path $\vec{r}(t)$ with constant speed 1 perpendicularly away from the candy. What is $\frac{d}{dt}f(\vec{r}(t))$ at the moment of take-off?

Solution:
a) $\nabla f(1,1,1)$ gives a normal vector to the tangent plane. As $\nabla f = \langle 6xy^2 + 6xz^2 + 2x, 6x^2y + 6yz^2 + 2y, 6y^2z + 6x^2z + 2z\rangle$, we have $\nabla f(1,1,1) = \langle 14, 14, 14\rangle$. The plane is $\boxed{14x + 14y + 14z = 42}$, where we got the constant on the right hand side by plugging in the point $(1,1,1)$.
b) The linearization $L(x,y,z)$ of $f(x,y,z)$ at $(1,1,1)$ is given by $L(x,y,z) = f(1,1,1) + \nabla f(1,1,1)\cdot\langle x-1, y-1, z-1\rangle = 12+14(x-1)+14(y-1)+14(z-1)$. So $f(1.1, 1.01, 0.98) \approx L(1.1, 1.01, 0.98) = 12 + 1.4 + 0.14 - 0.28 = 13.26$.
c) This is the length of the gradient as we know that this is the directional derivative in the direction of gradient. The length is $14\sqrt{3}$. If you should have forgotten the interpretation of the length of the gradient, here is the computation: by the chain rule, we have $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. Since the fly moves perpendicularly to the candy at the unit speed, $\vec{r}\,'(t)$ at the moment of take -off is the unit vector in the direction of $\nabla f(1,1,1) = \langle 14, 14, 14\rangle$, and thus it is $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. Note that $\nabla f(1,1,1)$ points outward, so this is the direction of the fly's path.) Hence $\frac{d}{dt} f(\vec{r}(t))$ at the moment of take-off is $\nabla f(1,1,1) \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = 14\sqrt{3}$.

Problem 5) (10 points)

In order to figure out the **Egos** x and y of the US presidential candidates, we want to minimize the sum of the perimeter of the letters H and T written in units x and y if the total area is fixed. The letter H has area $7x^2$ and perimeter $16x$, the letter T has area $5y^2$ and perimeter $12y$. Minimize



$$f(x,y) = 16x + 12y\ .$$

under the constraint

$$g(x,y) = 7x^2 + 5y^2 = 2016\ .$$

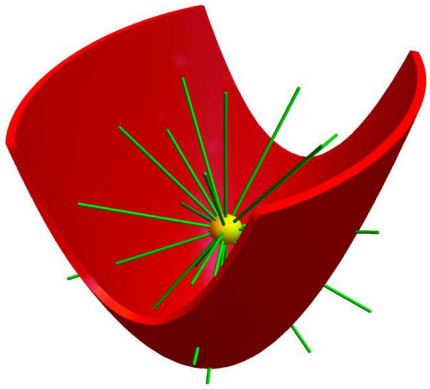
We don't actually need to know x and y. **As political pundits, we are only interested in the ratio y/x at the minimum.** Find this ratio!

Solution:
Write down the Lagrange equations $16 = \lambda 14x, 12 = \lambda 10y$, then eliminate λ to get $y/x = 21/20$. There is slightly more "Ego" for T.

Problem 6) (10 points)

With $F(x,y,z) = 2x^2 + y^2 + z^2$ and the surface S parametrized by $\vec{r}(x,y) = \langle 2x, y, 2x^2+y^2-1\rangle$, the function $f(x,y) = F(\vec{r}(x,y))$ giving the value F on S is

$$f(x,y) = 4x^4 + 4x^2y^2 + 4x^2 + y^4 - y^2 + 1 \ .$$



a) (8 points) Find all the critical points of f and classify them with the second derivative test. Please organize your work carefully so that we can see your method and your conclusions easily.

b) (2 points) The minimum could be obtained by minimizing $F(x,y,z)$ on the surface $G(x,y,z) = x^2/2 + y^2 - 1 - z = 0$. We would then use a method found by some mathematician. Which one? Just check the name. No additional work is needed in b).

Fubini	Burgers	Laplace	Lagrange	Bolzano	Clairaut

Solution:
a) This is a standard problem for exrema without constraint. The gradient of f is $\langle 8x\left(2x^2+y^2-1\right), 4y\left(2x^2+y^2-1\right)\rangle$. We have $f_{xx} = 8 + 48x^2 + 8y^2$. There are three critical points $(0,0), (0,\sqrt{2}/2), (0,-\sqrt{2}/2)$. The first is a saddle with $D = -16$, the other two are minima with $D = 48$ and $f_{xx} = 12$.

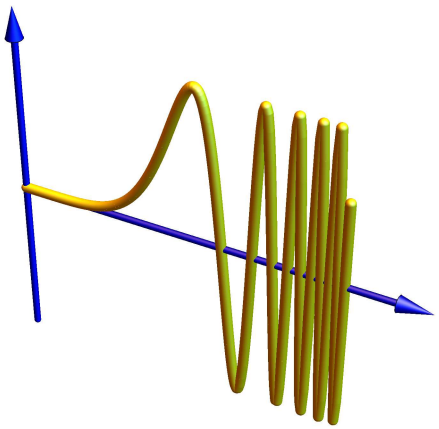
x	y	D	f_{xx}	Type
0	0	-16	8	saddle
0	$1/\sqrt{2}$	48	12	minimum
0	$-1/\sqrt{2}$	48	12	minimum

b) The problem has been reformulated as a Lagrange problem. Some were crossing Bolzano but since the surface is not bounded, one can not conclude the existence of a minimum from Bolzano.

Problem 7) (10 points)

Integrate

$$\int_0^1 \int_{(1-y)^{1/4}}^1 \sin(x^5)\, dxdy\ .$$



The figure just shows a fancy plot of the function $\sin(x^5)$.

Solution:
Change the order of integration to get

$$\int_0^1 \int_{1-x^4}^1 \sin(x^5)\, dy\, dx\ .$$

This simplifies to

$$\int_0^1 x^4 \sin(x^5)\, dx = (1-\cos(1))/5\ .$$

Problem 8) (10 points)

Integrate the double integral

$$\iint_R x^2\, dxdy\ ,$$

where R is the region

$$1 \le x^2 + y^2 \le 4$$

and

$$x \ge 0, y \ge 0, y \le x\ .$$



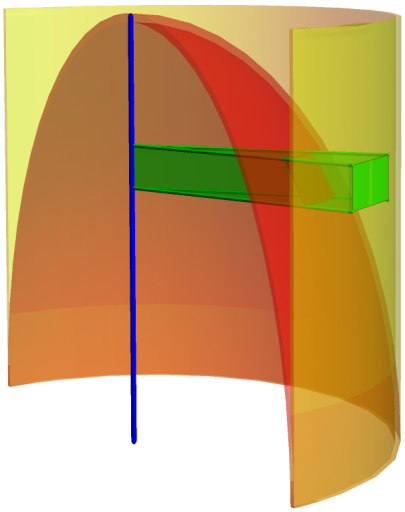
Solution:
Use polar coordinates. We integrate $\int_0^{\pi/4}(15/4)\cos^2(\theta)\ d\theta$. Now use the double angle formula. We end up with $\boxed{(15\pi+30)/32}$.

Problem 9) (10 points)

a) (7 points) Compute $A = |\vec{r}_\theta \times \vec{r}_\phi|$ for the half cylinder parametrized by

$$\vec{r}(\theta,\phi) = \langle \cos(\theta), \sin(\theta), \cos(\phi)\rangle\ .$$

with $0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi$ and use this to find the surface area of the half cylinder



b) (3 points) Compute $B = |\vec{r}_\theta \times \vec{r}_\phi|$ for the quarter sphere parametrized by

$$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$$

with $0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi$ to show that (remarkably!) it is the same factor than in part a).

Remark: The fact that the surface area elements A and B are the same has been realized by Archimedes already. It allowed him to compute the surface area of the sphere in terms of the surface area of the cylinder.

Solution:
a) The integration factor is $A = \sin(\phi)$. The integral is $\int_0^\pi \int_0^{\pi/2} \sin(\phi) = \pi$.
b) The integration factor is again $\sin(\phi)$. We did not ask for the integral as the integral is the same.
P.S. It is remarkable that one can project the sphere onto the cylinder to see that the areas are the same. You can do that also in a way how Archimedes might have derived it: compare a thin slice of size dz on each surface. The surface area of the cylinder is $2\pi dz$ which integrates to 2π, the area of the full cylinder of height 1 and radius 1. At height $z = \cos(\phi)$ the radius is $r\sin(\phi)$ and since the surface is slanted, the length of the cross section of the piece is $dz/\sin(\phi)$ Now the surface area piece is is $2\pi r \sin(\phi) dz/\sin(\phi) = 2\pi dz$ again.

11/1/2017 Second Hourly Practice 2 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The point $(-5, 3)$ is a critical point of $f(x, y) = 3x^2 + 5y^2$.

Solution:
The function f has only $(0, 0)$ as a critical point.

2) [T] [F] If a function $f(x, y, z)$ has gradient satisfying $|\nabla f| = 1$ everywhere, then the level surface $f(x, y, z) = 1$ is a sphere.

Solution:
Its no problem to get a plane for example with this property.

3) [T] [F] The chain rule assures that the vector $\nabla f(\vec{r}(t))$ and the velocity vector $\vec{r}\,'(t)$ for any curve $\vec{r}(t)$ on the level surface are perpendicular.

Solution:
Yes, this is the gradient theorem.

4) [T] [F] The function $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ satisfies the formula $D_{\vec{d}}D = \nabla D \cdot \vec{d}$ for any unit vector $\vec{d}$.

Solution:
This is true for all functions f, not only for the D.

5) [T] [F] The function $f(x, y) = x^4 - 1$ has infinitely many critical points.

Solution:
All points on $x = 0$ are critical points.

6) [T] [F] The points $(0, 1)$ and $(0, -1)$ are maxima of $f(x, y) = y^2$ under the constraint $g(x, y) = x^2 + y^2 = 1$.

Solution:
Yes, by inspection

7) [T] [F] The function $f(x,y) = y^2$ satisfies the partial differential equation $u_{yx}(x,y) = u_x(x,y)$.

Solution:
Just differentiate.

8) [T] [F] Let $f(x,y) = x^3y$. At every point (x,y) there is a direction $\vec{v}$ for which $D_{\vec{v}}f(x,y) = 0$.

Solution:
It is a general fact for directional derivatives as $D_{\vec{v}}f(x,y) = -D_{\vec{v}}f(x,y)$ implies that if the directional derivative is positive in one direction, it is negative in the opposite direction.

9) [T] [F] If $f_{xy} = f_{yx}$ then $f(x,y) = xy$.

Solution:
While Clairaut's theorem is always true, the second statement is not necessary. We can have for example $f(x,y) = \sin(xy) + y^5$ and Clairaut still holds.

10) [T] [F] $g(x,y) = \int_0^x \int_0^y f(s,t)\ dtds$ satisfies the partial differential equation $g_{xy}(x,y) = f(x,y)$.

Solution:
By the fundamental theorem of calculus we have $g_{xy} = f(x,y)$.

11) [T] [F] If $f(x,y) = g(x,y) = x^2 + y^4$, then the Lagrange problem for maximizing f under the constraint $g(x,y) = 1$ has infinitely many solutions.

Solution:
Yes, every point on the constraint is a maximum.

12) [T] [F] The number $|\nabla f(0,0)|$ is the maximal directional derivative $|D_{\vec{v}}f(0,0)|$ among all unit vectors $\vec{v}$.

Solution:
By the cos-formula.

13) [T] [F] Any continuous function $f(x,y)$ takes a global maximum as well as a global minimum on the region $0 \leq x^2 + y^2 \leq 1$.

Solution:
This is the Bolzano extremal value theorm.

14) [T] [F] For any continuous function, $\int_0^1 \int_0^1 f(r,\theta) r \, dr d\theta = \int_0^1 \int_0^1 f(x,y) \, dx dy$.

Solution:
The domain is not correctly integrated

15) [T] [F] If the Lagrange multiplier λ at a solution to a Lagrange problem is positive then this point is a minimum.

Solution:
The sign of λ has nothing to do with the nature of the critical point.

16) [T] [F] The equation $f_{xy} f_{xx} f_{yy} = 1$ is an example of a partial differential equation.

Solution:
Yes, it is an equation for a function f involving partial derivatives.

17) [T] [F] If the discriminant D appearing in the second derivative test of $f(x,y)$ is positive at $(0,0)$ then $|\nabla f(0,0)| > 0$.

Solution:
Take a minimum of $f(x,y) = x^2 + y^2$.

18) [T] [F] If $f(x,y)$ is a continuous function then $\int_7^9 \int_5^7 f(x,y) \, dx dy = \int_5^7 \int_7^9 f(x,y) \, dx dy$.

Solution:
Take a simple example like $f(x,y) = x$, then $f(y,x) = y$, which gives an other result.

19) [T] [F] If f has the critical point $(0,0)$, then $f_y + f_x$ has the critical point $(0,0)$.

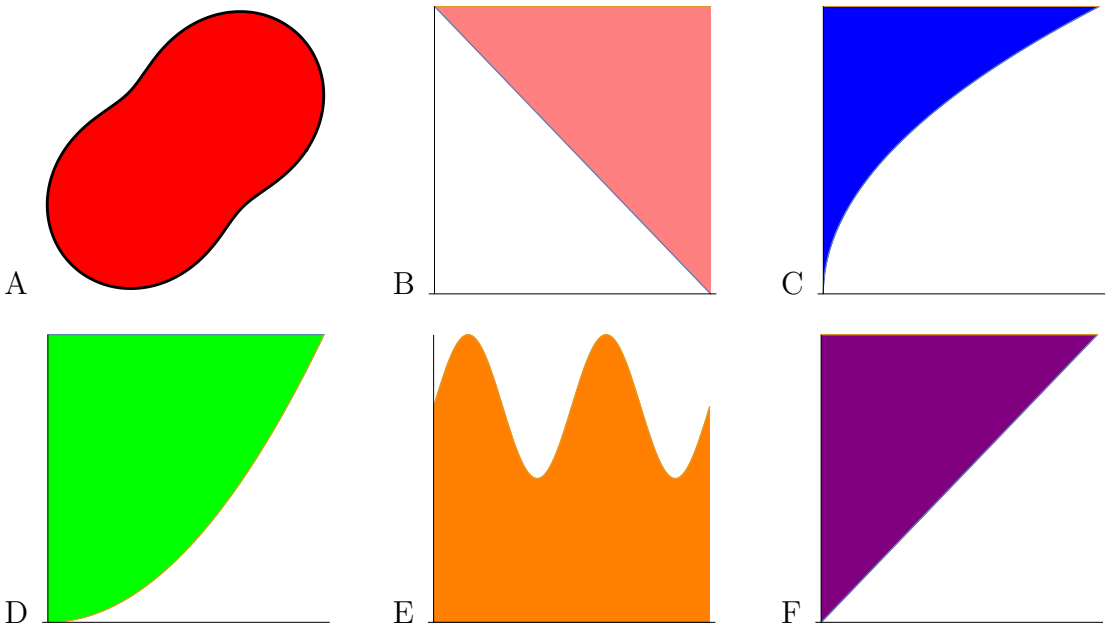
Solution:
Take an example $xy + y^2$

20) [T] [F] If $f(x,y)$ takes arbitrary large values, then $g(x,y) = |\nabla f(x,y)|$ takes arbitrary large values.

Solution:
A function can be unbounded without its derivative being unbounded. This already holds in single variable clalculus. A concrete counter example is $f(x,y) = x + y$.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A - F$.



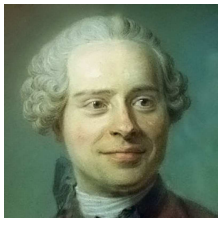
Enter A-F	Integral
	$\int_0^{2\pi} \int_0^{y} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_{2\pi-x}^{2\pi} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{y^2} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{3+\sin(2x)} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{3+\sin(2t)} f(r,t)\ rdrdt$

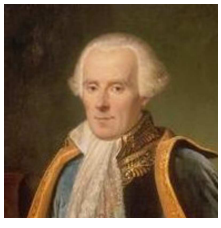
b) (4 points) We define the **complexity** of a partial differential equation for $u(t,x)$ or $u(x,y)$ as the number of derivatives appearing in total. For example, the partial differential equation $u_{xxx} = u_{tx}$ has complexity 5 because 5 derivatives have been taken in total. As an expert in PDEs, you know a few of them. Write down the complexities of the partial differential equations. These are integers ≥ 2 in each case.

Complexity	Name
	Laplace for $u(x,y)$
	Wave for $u(t,x)$
	Transport for $u(t,x)$
	Heat for $u(t,x)$


Taylor


Fourier


d'Alembert

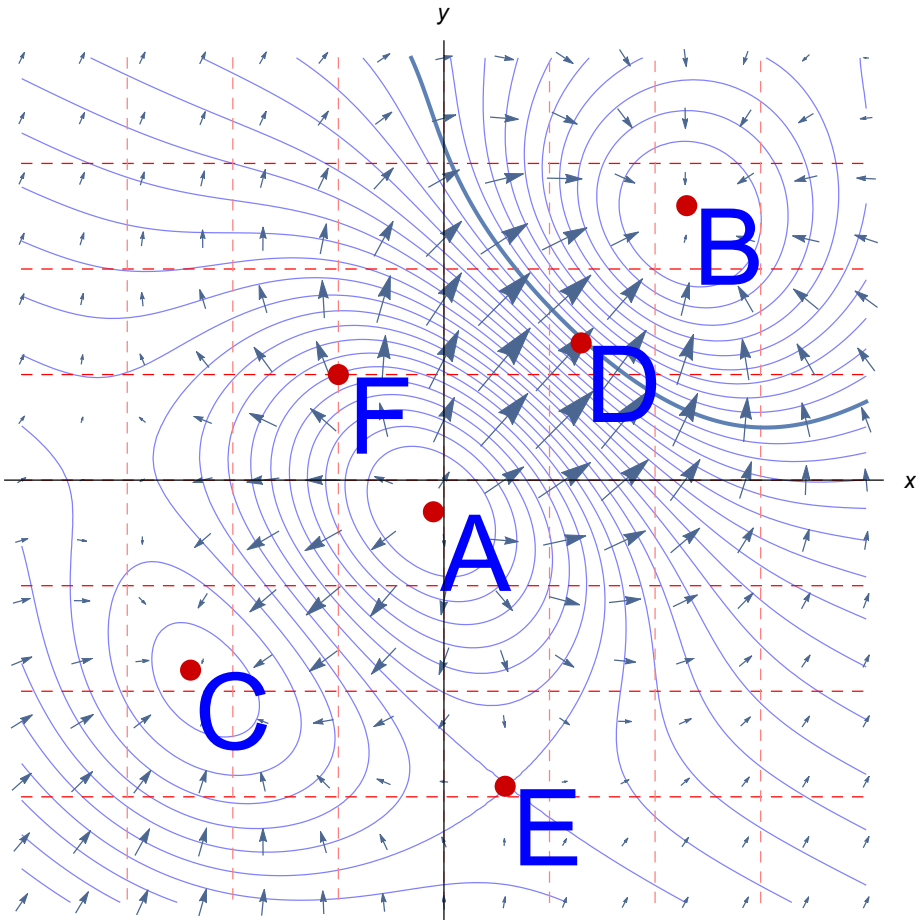

Laplace

Solution:
FDBCEA
4,4,2,3

Problem 3) (10 points)

3a) (5 points) In the following contour plot of a height function $f(x, y)$, neighboring contours $f(x, y) = c$ have height distance 1. The arrows indicate the gradient of f. Every point A-F occurs at most once.

Which of the points is the global maximum on the visible region?	
Which of the points is a global minimum on the visible region?	
Which of the points is a global maximum for the function $\|\nabla f(x, y)\|^2$	
Which of the points is a saddle point?	
Which of the points has the property that $f_x f_y < 0$ at this point?	



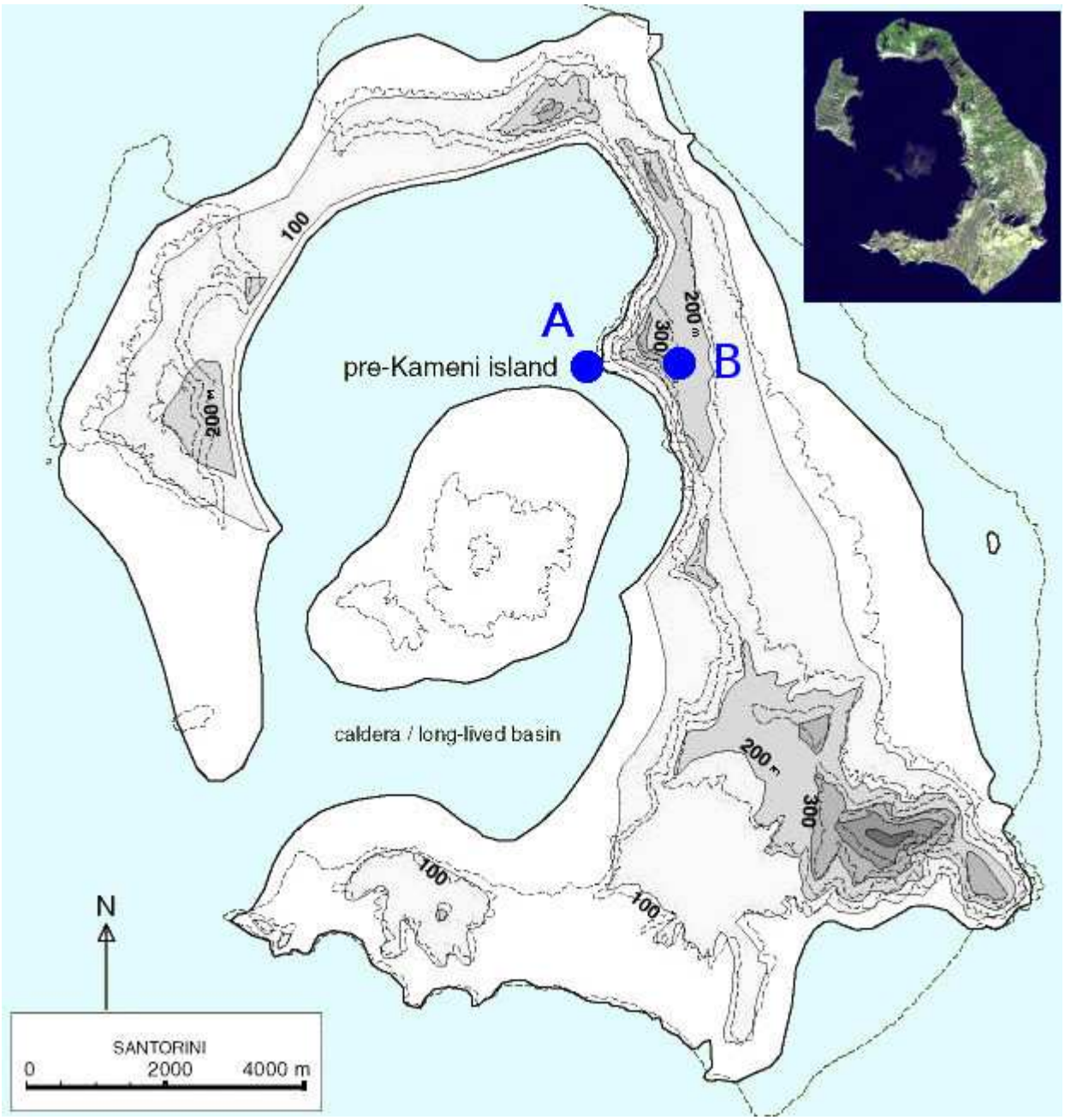
Part b) and c) of the problem are unrelated and on the new page.



Santorini panorama from Imerovigli with view onto Skaros rock, Caldera basin and volcanic island Nea Kameni. Photo: Oliver Knill, June 2015

3b) (2 points) You see a contour map of the Greek island of **Santorini**. Point A is on the water (0 elevation) Point B is **Skaros rock**, which used to be a fortification protecting merchants from pirates. Estimate the average directional derivative between A and B in the direction from A to B. Given elevation markers 100,200,300 are in meters.

Derivative	Check one
2	
0.2	
0.02	



Source: http://www.decadevolcano.net, the picture shows a reconstruction of pre-Minoan Thera done by Druitt and Francaviglia from 1991. The island of today is shown in dottet curves. A satelite picture of the Santorini Caldera with the Nea Kameni volcano in the center is seen in the upper right corner.

3c) (3 points) Which statements about a critical point with discriminant $D \neq 0$ always hold for a smooth function $f(x, y)$?

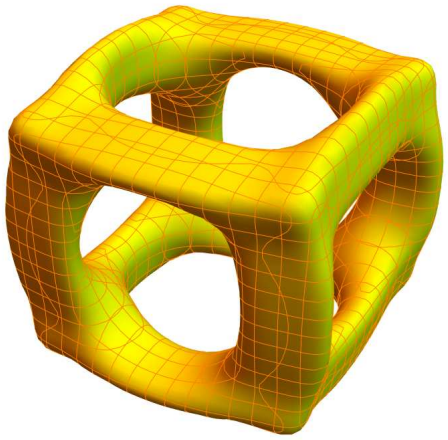
Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum				
Minimum				
Saddle point				

Solution:
a) B is the global maximum, A is the global minimum, D is the point with largest steepness $|\nabla f(x,y)|$. The point E is a saddle point. At the point F, the gradient is parallel to $\langle -1, 1\rangle$ showing that $f_x f_y < 0$.
b) The distance is about 1500 m and the height difference about 300 m which gives a value of $\boxed{0.2}$.
c) Since D is not zero, we have $f_{xx} > 0$ at a minimum and $f_{yy} < 0$ at a maximum since any three cases are critical points, the last two columns are empty. At a saddle point, we can have $f_{xx} > 0$ or $f_{yy} < 0$ but it does not need to be. Examples like $x^2 - y^2$ or $y^2 - x^2$ show that either case can occur. So:

Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum		X		
Minimum	X			
Saddle point				

Problem 4) (10 points)

a) (6 points) Let $g(x,y) = (6y^2-5)^2(x^2+y^2-1)^2$. Find the gradient of g at the points $(1,-1)$, $(-1,1)$ and $(1,1)$.



b) (4 points) A student from the **Harvard graduate school of design** contemplates the surface

$$f(x,y,z) = g(x,y) + g(y,z) + g(z,x) = 3$$

shown in the picture. She first discovers the formula

$$\begin{aligned} \nabla f(1,-1,1) &= \langle g_x(1,-1) + g_y(1,1)\ , \\ &\quad\ g_x(-1,1) + g_y(1,-1)\ , \\ &\quad\ g_x(1,1) + g_y(-1,1)\rangle\ . \end{aligned}$$

Without verifying this, find the tangent plane at $(1,-1,1)$.

Solution:
a) Compute the gradient

$$\nabla g(x,y) = \langle 4x(6y^2-5)^2(x^2+y^2-1), 24y(6y^2-5)(x^2+y^2-1)^2 + 4y(6y^2-5)^2(x^2+y^2-1)\rangle\ .$$

Plugging in the values gives $(4,-28)$, $(-4,28)$ and $(4,28)$.
b) As we have the g_x, g_y values from a), we have just to plug in the numbers and get $\nabla f = \langle 32, -32, 32\rangle$. The equation is $32x - 32y + 32z = d$. Now plug in the point $(1,-1,1$ to get $32 + 32 + 32 = 96$ and the plane is $\boxed{x - y + z = 3}$.

Problem 5) (10 points)



Octagons are used in architecture designs, in symbolism, for rugs or in traffic signs. Use the Lagrange method to find the octagon with maximal area

$$f(x,y) = (x+2y)^2 - 2y^2$$

if the circumference

$$g(x,y) = 4x + 4y\sqrt{2} = 8\ .$$

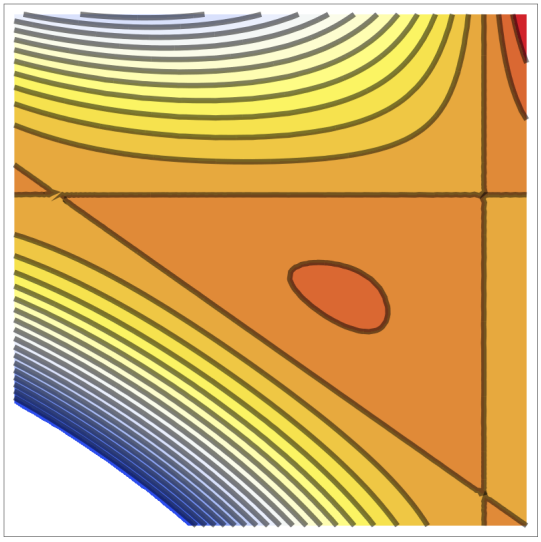
is fixed.

Solution:
Write down the Lagrange equations

$$\begin{aligned} 2(x+2y) &= \lambda 4 \\ 4(x+2y) - 4y &= \lambda 4\sqrt{2} \\ 4x + 4y\sqrt{2} &= 8 \end{aligned}$$

Dividing out λ gives $y = x/\sqrt{2}$. Plugging this into the third equation gives $x = 1$ and $y = 1/\sqrt{2}$.

Problem 6) (10 points)



a) (8 points) Find and classify the four critical points of the **"triangle function"**

$$f(x,y) = x^2y + y^2x - y^2 - y$$

using the second derivative test. There is no need to find the values of f.

b) (2 points) State whether any of the four points is a global maximum or minimum on the entire plane.

Solution:
a) Compute the gradient $\nabla f(x,y) = \langle 2xy + y^2, -1 + x^2 - 2y + 2xy \rangle$ and set it to $(0,0)$. Factor out the y in the first $y(2x+y=0)$ shows that either $y = 0$ leading to two solutions $x = \pm 1$ or $y = -2x$ which leads to a quadratic equation with $x = 1, x = 1/3$ as solutions. We compute also f_{xx} and D and get the classification

x	y	D	f_{xx}	Type	f
-1	0	-4	0	saddle	0
1/3	-2/3	4/3	-4/3	maximum	8/27
1	-2	-4	-4	saddle	0
1	0	-4	0	saddle	0

b) There is $\boxed{\text{no global maximum}}$ as $f(x,x) = 2x^3 - x^2 - x$ has no global maximum or minimum.

Problem 7) (10 points)

The region R defined by

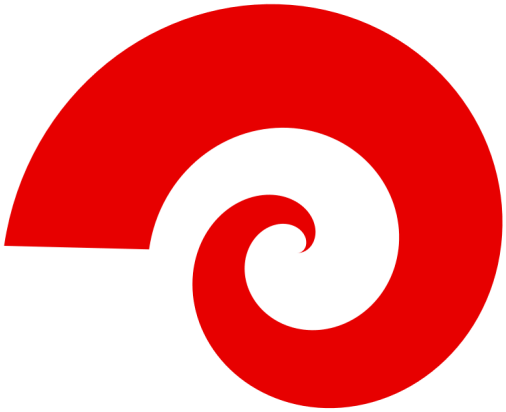
$$\theta \leq r(\theta) \leq 2\theta$$

with

$$0 \leq \theta \leq 3\pi$$

is shown in the picture. Compute its **moment of inertia**

$$\iint_R x^2 + y^2 \, dA \, .$$



Solution:
The integral is

$$\int_0^{3\pi} \int_\theta^{2\theta} r^2 \cdot r \, dr d\theta$$

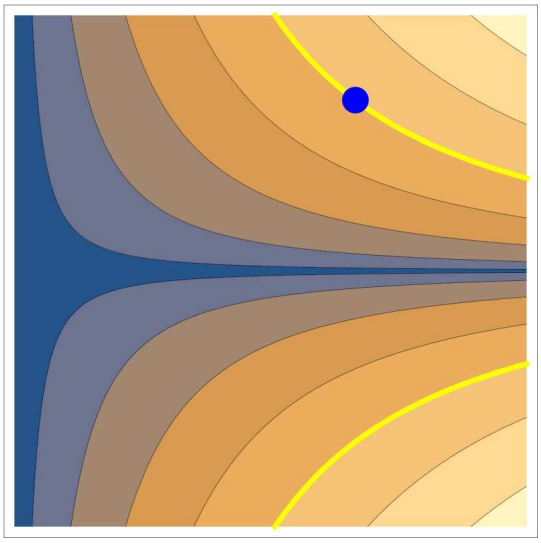
which is $729\pi^5/4$. As expected, most mistakes came from forgetting the integration factor r or (unexpectedly) that the lower bound θ morphed to 0 for many.

Problem 8) (10 points)

a) (5 points) Find a vector perpendicular to the **tangent line** of the curve

$$f(x,y) = 5(x^3y^2)^{1/5} = 100$$

at $(20, 20)$. The picture shows a contour map of f.



b) (5 points) Use the same function in a) to estimate

$$f(21,19) = 5(21^3 \cdot 19^2)^{1/5}$$

by **linearizing** f near $(20, 20)$.

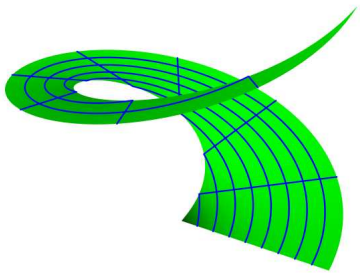
Solution:
a) Using the single variable chain rule and multiplication rule correctly to take the derivatives shows that the gradient is $\langle 3, 2\rangle$. b) $f_x(20,20) = 3$, $f_y(20,20) = 2$. We estimate $100 + 3 - 2 = 101$. The actual value is 100.88.

Problem 9) (10 points)

We compute the **surface area** of the surface

$$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), u\rangle$$

over the region $R : 0 \le u \le 2\pi, u \le v \le 2\pi$.



a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2}\, dudv\ .$$

b) (5 points) Now compute the surface area integral.

Solution:
a) Take the cross product of $\vec{r}_u = \langle -v\sin(u), v\cos(u), 1\rangle$ and $\vec{r}_v = \langle \cos(u), \sin(u), 0\rangle$ to get $\vec{r}_u \times \vec{r}_u = \langle -\sin(u), \cos(u), -v\rangle$ which has length $\sqrt{1+v^2}$.
b) The integral

$$\int_0^{2\pi}\int_u^{2\pi} \sqrt{1+v^2}\, dvdu$$

is not pleasant (doable as you have done in the homework). Better is to switch the order of integration. Draw the region which is a triangle and switch to a type II integral

$$\int_0^{2\pi}\int_0^{v} \sqrt{1+v^2}dudv = \int_0^{2\pi} v\sqrt{1+v^2}dv = (1/3)(1+v^2)^{3/2}|_0^{2\pi} = ((1+4\pi^2)^{3/2}-1)/3\ .$$

The answer is $\boxed{(1+4\pi^2)^{3/2}/3 - 1/3}$.

Problem 9) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$ is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

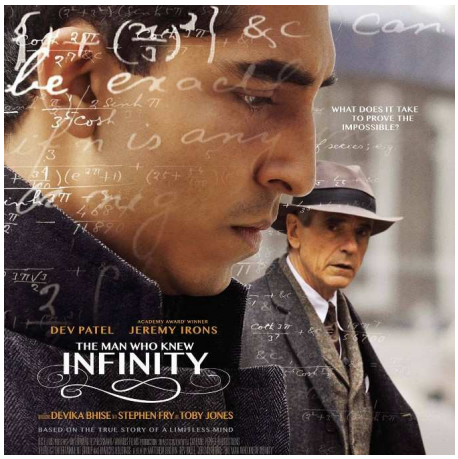
Let

$$f(x,y) = x\sqrt{y}\ .$$

Estimate

$$f(3.141, 163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



Ramanujan is featured in the movie: "The Man who knew infinity", 2015

Solution:
$\nabla f(x,y) = \langle \sqrt{y}, x/(2\sqrt{y})\rangle$ which is $\langle 13, 3/(2*13)\rangle = \langle 13, 3/26\rangle$. Now $L(3.141, 163) = 39 + 0.141*13 - 18/(2*13) = \boxed{40.1407}$. The actual result is 40.1016.

11/1/2017 Second Hourly Practice 3 **Math 21a, Fall 2017**

Name:

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MWF 12 Stepan Paul
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $|D_{\vec{v}} f(0,0)|$ among all unit vectors $\vec{v}$.

Solution:
By the cos-formula.

2) [T] [F] The relation $f_{xxyyxx} = f_{xxyxyx}$ holds everywhere for $f(x,y) = \cos(\exp(x^{10}) + \sin(x-y))$.

Solution:
By Clairot

3) [T] [F] $\int_0^4 \int_0^{4x} f(x,y)\, dydx = \int_0^{16} \int_{y/4}^{16} f(x,y)\, dxdy$.

Solution:
The inner integral on the right should be $\int_{y/4}^4$.

4) [T] [F] $g(x,y) = \int_y^0 \int_0^x f(s,t)\, dsdt$ satisfies $g_{xy} = -f(x,y)$.

Solution:
By the fundamental theorem of calculus.

5) [T] [F] If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$.

Solution:
Because the vector r_v is tangent to the surface.

6) [T] [F] If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(a,b) = 3$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(a,b) = 5$, then $D_{\vec{v}} f(a,b) \geq 0$ for all unit directions $\vec{v}$.

Solution:
If we change $\vec{v}$ to $-\vec{u}$ then the sign of the directional derivative changes.

7) [T] [F] Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x,y)$ we have $\frac{d}{dt}f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.

8) [T] [F] If $u(t,x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant c.

Solution:
Equal to u_{xx}.

9) [T] [F] If the Lagrange multiplier λ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.

Solution:
The sign of λ has nothing to say about the nature of the critical point.

10) [T] [F] The equation $f_x^2 + f_y^2 + f_z^2 = 1$ is an example of a partial differential equation.

Solution:
Write it out. It is an equation for a function f involving partial derivatives.

11) [T] [F] If the discriminant D of $f(x,y)$ is zero at $(0,0)$ then $\nabla f(0,0) = \langle 0,0\rangle$.

Solution:
We can have $f(x,y) = x + y$ for example, which has zero discriminant but no critical point.

12) [T] [F] If $f(x,y,z) = 0$ describes the unit sphere, then the gradient ∇f points outwards.

Solution:
the gradients of $x^2 = y^2 + z^2 - 1 = 0$ or $1 - x^2 - y^2 - z^2 = 0$ point in different directions.

13) [T] [F] If $f(x,y)$ is a continuous function then $\int_0^2 \int_0^1 f(x,y)\ dxdy = \int_0^2 \int_0^1 f(y,x)\ dxdy$.

Solution:
Take a simple example like $f(x,y) = x$, then $f(y,x) = y$, which gives an other result.

14) [T] [F] The point $(5,5,5)$ is a critical point of $f(x,y,z) = x+y+z$.

Solution:
the function f has no critical point.

15) [T] [F] Assume $\nabla f(0,0) = \langle 0,0\rangle$ with discriminant $D > 0$, then $-f(x,y)$ has the same critical point $(0,0)$ with discriminant $D < 0$.

Solution:
We still have $D > 0$

16) [T] [F] $\int\int_R |\nabla f|^2 \, dxdy$ is the surface area of the cubic paraboloid $z = f(x,y) = x^3 + y^3$ defined over the region R.

Solution:
It is not the formula.

17) [T] [F] If $D(x,y)$ is the discriminant of f at (x,y) then the following poetic formula of the **d**irectional **d**erivative of the **d**iscriminant holds: $D_{\langle 1,0\rangle} D = \partial_x D$.

Solution:
This is true for all functions f, not only for the D. The directional derivative in the x direction is the partial derivative f_x.

18) [T] [F] Assume $f(x,y) = -x^2 + y^4$ and a curve $\vec{r}(t)$ satisfying $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$ for all t.

Solution:
The assumption tells that the gradient is perpendicular to the velocity. By the chain rule, the expression is zero.

19) [T] [F] The Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = c$ have the same solutions as the Lagrange equations for extremizing $F = f + g$ under the constraint $g = c$.

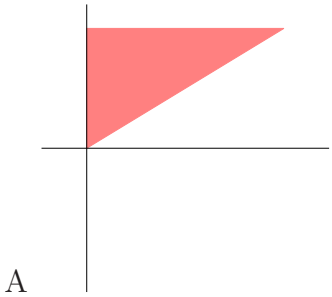
Solution:
Write down the Lagrange equations.

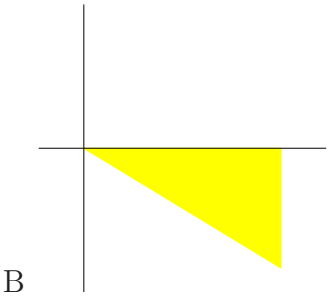
20) [T] [F] If f is a maximum under the constraint $g = 1$ at $(0, 0)$, and $(0, 0)$ is not a critical point for both f and g, then the level curves of f and g have the same tangent line at $(0, 0)$.

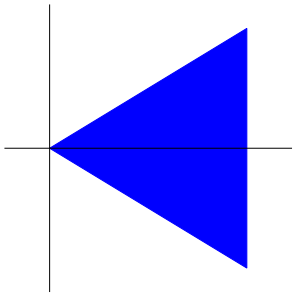
Solution:
The tangent line is determined by the point and the gradient. The tangent planes are the same if the gradients are parallel as long as they are not zero.

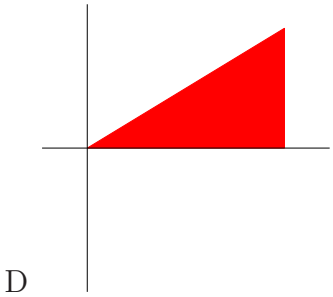
Problem 2) (10 points) No justifications needed

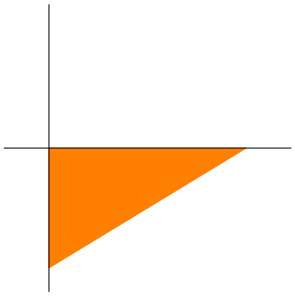
a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.

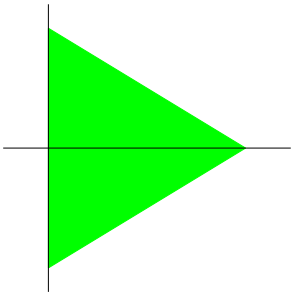

A


B


C


D


E


F

Enter A-F	Integral
	$\int_{-1}^{1}\int_{\lvert y\rvert}^{1} f(x,y)\,dxdy$
	$\int_{0}^{1}\int_{0}^{x} f(x,y)\,dydx$
	$\int_{-1}^{0}\int_{-y}^{1} f(x,y)\,dxdy$
	$\int_{0}^{1}\int_{x-1}^{0} f(x,y)\,dydx$
	$\int_{0}^{1}\int_{x-1}^{1-x} f(x,y)\,dydx$
	$\int_{0}^{1}\int_{x}^{1} f(x,y)\,dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation number	Formula for PDE
1	$\frac{\partial}{\partial_t}u - \frac{\partial}{\partial_y}u = 0$
2	$\frac{\partial}{\partial_t}u - \frac{\partial^2}{\partial_x^2}u = 0$
3	$\frac{\partial^2}{\partial_t^2}u - \frac{\partial^2}{\partial_x^2}u = 0$
4	$\frac{\partial^2}{\partial_x^2}u + \frac{\partial^2}{\partial_y^2}u = 0$

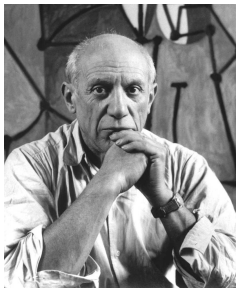
Solution:
CDBEFA
4,3,1,2

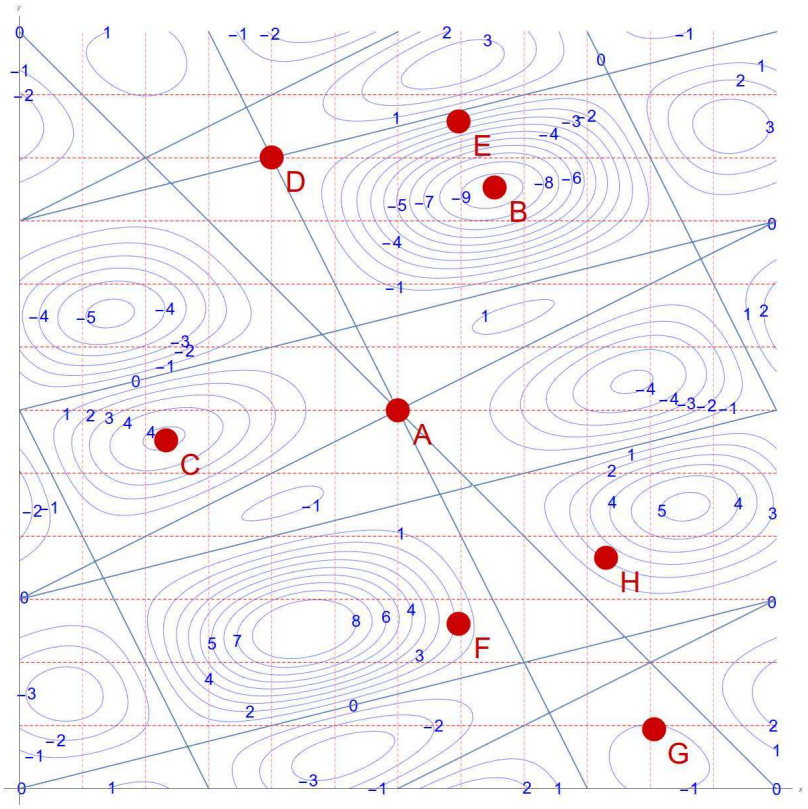
Problem 3) (10 points)

a) (7 points) The following contour map is inspired by a cubistic style of Picasso. Each of the points A-H fit exactly once

A point where $f_x \neq 0, f_y = 0$	
A saddle point of f	
A local maximum of f	
A critical point with $D = 0$	
A local minimum	
A point with $f_y \neq 0, f_x = 0$	
$\lvert\nabla f\rvert$ maximal among A-H	
Point where $D_{\langle -1,1\rangle/\sqrt{2}} f = 0$	







The painting "Pigeon with Green Peas" by Pablo Picasso was stolen in 2010. The thief got scared and disposed it to trash shortly after the theft. The garbage was emptied and taken away, the painting lost for ever. Or the thief had been clever ...

Solution:

b) (3 points) Given a function $f(x, y)$ and a curve $\vec{r}(t)$. Let L be the linearization of f at $\vec{r}(0)$. Each of the following 3 vectors $\vec{a}, \vec{b}, \vec{c}$ is placed exactly twice in the puzzle to the

right below.

$\vec{a} = \nabla f(\vec{r}(0))$	$\frac{d}{dt} f(\vec{r}(t))_{t=0} =$ $\square$ $\cdot$ $\square$
$\vec{b} = \vec{r}\,'(0)$	$L(\vec{r}(t)) =$ $f(\vec{r}(0)) +$ $\square$ $\cdot$ $\square$
$\vec{c} = \vec{r}(t) - \vec{r}(0)$	$\square$ $=$ $\lim_{t\to 0} \frac{1}{t}$ $\square$

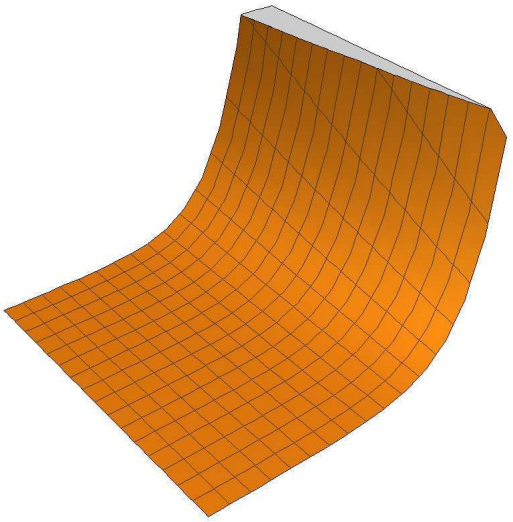
Solution:
FDCABGEH
ab,ac,bc

Problem 4) (10 points)

a) (5 points) Find the tangent plane to the **skate board ramp**

$$z - f(x, y) = z - \sqrt{x^{30}y^3 + x} = 0$$

at the point $(1, 2, 3)$.



b) (5 points) Estimate

$$f(1.006, 1.98) = \sqrt{1.006^{30} \cdot 1.98^3 + 1.006}$$

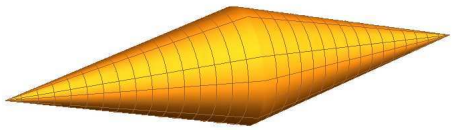
by linearizing the function $f(x, y)$ at $(1, 2)$.

Solution:
a) The gradient is $\langle(-30x^{29}y^3 + 1)/(2\sqrt{x^{30}y^3 + x}), (-3x^{30}y^2/(2\sqrt{x^{30}y^3 + x})), 1\rangle$. At $(1, 2, 3)$ this is $\langle -241/6, -2, 1\rangle$. Multiplying with -6 to make it easier shows that the equation of the plane is $241x + 12y - 6z = d$, where d is the constant obtained by plugging in the point. It is $d = 220$.
b) We have already computed the gradient $\langle 40, 8\rangle$ in a). The linearization is $L(x, y) = 3 + \langle 241/6, 2\rangle \cdot \langle (x - 1), (y - 2)\rangle$. We have $L(1.006, 1.98) = 3 + 241/6 \cdot 0.006 + 2(-0.02)$

Problem 5) (10 points)

A **croissant** of length $2h$ and radius r in the shape of two cones has fixed volume

$$V(r, h) = \frac{2\pi r^2 h}{3} = 18 \, .$$



Use Lagrange to find the values r and h for which the surface area

$$A(r, h) = 2\pi r\sqrt{r^2 + h^2}$$

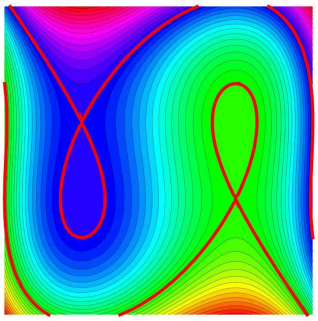
is minimal. **Hint:** as you have seen in homework, it is much more convenient to minimize $f(r, h) = A(r, h)^2$ instead.

Solution:
Write down the Lagrange equations and eliminate λ to get $h = \sqrt{2}r$. Plug that into the constraint to get $r = 3/(\pi^{(1/3)}2^{1/6})$, $h = 3\sqrt{2}/(\pi^{(1/3)}2^{1/6})$

Problem 6) (10 points)

Find the local maxima, minima and saddle points of the **tadpole function**

$$f(x, y) = 3y^2 + 4x^3 + 2y^3 - 12x \, .$$



First tadpole: "What is your favorite book?" Second tadpole: "Metamorphosis by Kafka. What is your favorite year?" First tadpole: "Leap year!" Both croak with laughter. Then, sick of frog jokes, they turn green.

Solution:
Write down the gradient $\nabla f(x,y) = \langle 12x^2 - 12, 6y + 6y^2 \rangle$. This shows that $x = \pm 1$ and $y = 0$ or -1. There are 4 solutions.

x	y	D	f_{xx}	Type f
-1	-1	144	-24	maximum 9
-1	0	-144	-24	saddle 8
1	-1	-144	24	saddle -7
1	0	144	24	minimum -8

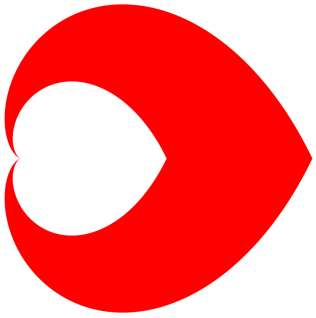
Problem 7) (10 points)

Find the area of the heart shaped polar region

$$(\theta - \pi)^2 \leq r \leq 2(\theta - \pi)^2$$

with $0 \leq \theta \leq 2\pi$.

Warning: Valentine cards displaying **"You are my** $r < (\theta - \pi)^2$**!"** *do not always work.*



Solution:
$(3/5)\pi^5$.

Problem 8) (10 points)

Mathematica 10 does not give an elementary expression for the integral

$$\int_0^1 \int_{\exp(y)}^{e} \frac{1}{\log(x)}\, dx\, dy\ ,$$

where log is the natural log. You can! "Humans are awesome 2014"! https://www.youtube.com/watch?v=ZBCOMG2F2Zk



The logarithmic integral $\mathrm{Li}(x) = \int_0^x dt/\log(t)$ is important in number theory. It was Gauss who proposed first that the number $\pi(x)$ of primes smaller than x is about $\mathrm{Li}(x)$. It is now known that $0.89\ \mathrm{Li}(x) \leq \pi(x) \leq 1.11\ \mathrm{Li}(x)$ for all large enough x. P.S. Mathematica can solve the double integral of course, but only if told to "FullSimplify".

Solution:
Switch the order of integration. It is pivotal to make a picture.

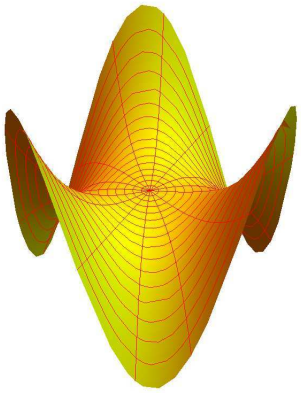
$$\int_1^e \int_0^{\log(x)} \frac{1}{\log(x)}\, dy\, dx = e - 1 \, .$$

Problem 9) (10 points)

Compute the weighted surface area

$$\int\int_R (u^2 + v^2)|\vec{r}_u \times \vec{r}_v|\, dudv$$

of the monkey saddle parametrized by $\vec{r}(u, v) = \langle u, v, u^3 - 3uv^2 \rangle$ over the domain $R : u^2 + v^2 \leq 1$. This quantity is also known as the moment of inertia of the surface. Spin that monkey!



Solution:
$\vec{r}_u = \langle 1, 0, 3u^2 - 3v^2 \rangle$ and $\vec{r}_v = \langle 0, 1, -6uv \rangle$ so that we get $2\pi \int_0^1 r^3\sqrt{1 + 9r^4}\, dr$. This can be integrated with substitution $u = r^4$. The answer is $\boxed{10\sqrt{10} - 1)\pi/27}$.

Problem 10) (10 points)

The following two integrals are called "Mad Max" integrals because they were written while watching that movie:



a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)}\, dxdy\ .$$

b) (5 points) Integrate the double integral

$$\int\int_R \sin(x^2+y^2)\, dxdy$$

where R is the disk of radius $\sqrt{\pi/2}$.

Solution:
a) Make a picture! Change the order of integration to get

$$\int_0^{\pi/2} \int_0^{\sin(x)} \frac{xy}{\sin(x)}\, dydx\ .$$

After solving the first integral, we get

$$\int_0^{\pi/2} x\sin(x)/2\ dx = 1/2\ .$$

1/2.
b)

$$2\pi \int_0^{\sqrt{\pi/2}} \sin(r^2) r\ dr = -\pi\cos(r^2)|_0^{\sqrt{\pi}} = \pi\ .$$

11/1/2017 Second Hourly Practice 4 **Math 21a, Fall 2017**

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- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] For any continuous function $f(x,y)$, we have $\int_0^1 \int_1^2 f(x,y)\,dx\,dy = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$.

Solution:
This is not Fubini

2) [T] [F] If $\vec{u}$ is a unit vector tangent to $f(x,y) = 1$ at $(0,0)$ and $f(0,0) = 1$, then $D_{\vec{u}} f(0,0)$ is zero.

3) [T] [F] Assume f is zero on $x = y$ and $x = -y$, then $(0,0)$ is a critical point of f.

4) [T] [F] If $(0,0)$ is the only local minimum of a function f and f has no local maxima, then $(0,0)$ is a global minimum.

5) [T] [F] If $(0,0)$ is a critical point for f, and $f_{yy}(0,0) < 0$ then $(0,0)$ is not a local minimum.

6) [T] [F] If $f(x,y)$ and $g(x,y)$ have the same non-constant linearization $L(x,y)$ at $(0,0)$ and $f(0,0) = g(0,0) = 0$, then the level sets $f = 0$ and $g = 0$ have the same tangent line at $(0,0)$.

7) [T] [F] There are saddle points with positive discriminant $D > 0$.

Solution:
No. By definition not

8) [T] [F] If R is the unit disc, then $\int\int_R x^2 - y^2\,dxdy$ is zero.

Solution:
Yes. This can be seen as the definition of area.

9) [T] [F] There is a nonzero function $f(x,y)$ for which the linearization $L(x,y)$ is equal to $2f(x,y)$.

Solution:
The condition means that the function is linear. For a linear function, the linearization is itself.

10) [T] [F] The directional derivative at a local minimum $(0,0)$ is positive in every direction.

Solution:
It is zero

11) [T] [F] If $\vec{r}(t)$ is a curve on the surface $g(x,y,z)=1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Use the chain rule and the fact that $g(\frac{r}{(}t))$ is constant so that $d/dtg(\frac{r}{(}t))$ is zero.

12) [T] [F] If $|\nabla f(0,0)| = 2$, there is a direction in which the directional derivative at $(0,0)$ is 2.

Solution:
Yes, we have seen this computation

13) [T] [F] If $D > 0$ at $(0,0)$ and $\nabla f(0,0) = 0$ and $f_{xx}(0,0) < 0$ then $f_{yy}(0,0) < 0$.

Solution:
We have seen this from $D = f_{xx}f_{yy} - f_{xy}^2$.

14) [T] [F] $\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_y^1 f(x,y)\, dxdy$.

Solution:
This is almost a correct correct switch but the $dxdy$ have also to be switched.

15) [T] [F] The surface area of the sphere of radius L is $\int_0^\pi L^2 \sin(\phi)\, d\phi$.

Solution:
The answer is 4π.

16) [T] [F] If $f(x,y) = g(x)$ is a function of x only, then $D = 0$ at every critical point.

Solution:
Indeed, then $f_{xy}, f_{xx}f_{yy}$ are both zero.

17) [T] [F] The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x,y)$.

Solution:
Nonsense

18) [T] [F] If $|\nabla f(0,0)| = 2$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}} f(0,0) = 1$.

Solution:
Intermediate value theorem.

19) [T] [F] The gradient of the function $f(x,y) = \int_x^y \sin(t)\, dt$ is $\langle -\sin(x), \sin(y)\rangle$.

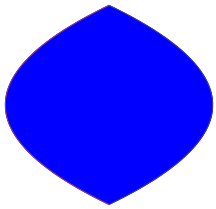
Solution:
Yes, this is the derivative by the fundamental theorem.

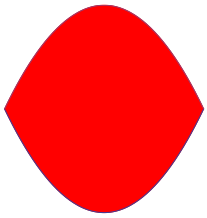
20) [T] [F] Assume $f(x,y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$.

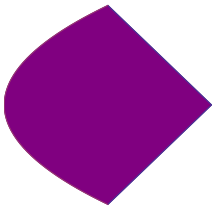
Solution:
FTTFTTFTFFTTTTFTFTTT

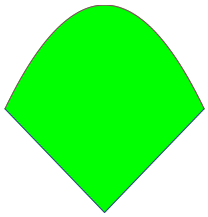
Problem 2) (10 points) No justifications needed

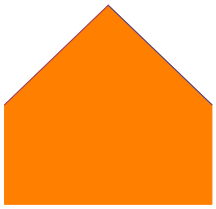
a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.

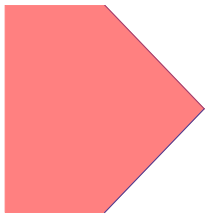
A 

B 

C 

D 

E 

F 

Enter A-F	Integral
	$\int_{-1}^{1}\int_{-1}^{2-\lvert y\rvert} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{y^2}^{2-\lvert y\rvert} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{x^2}^{2-x^2} f(x,y)\,dydx$
	$\int_{-1}^{1}\int_{\lvert x\rvert}^{2-x^2} f(x,y)\,dydx$
	$\int_{-1}^{1}\int_{y^2}^{2-y^2} f(x,y)\,dxdy$
	$\int_{-1}^{1}\int_{-1}^{2-\lvert x\rvert} f(x,y)\,dydx$

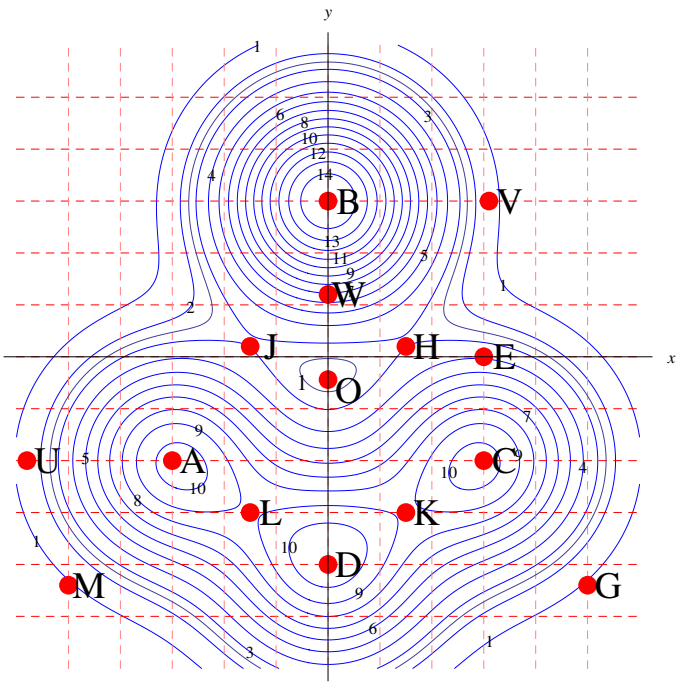
b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation Number	PDE
1	$g_x - g_y = 0$
2	$g_{xx} - g_{yy} = 0$
3	$g_x - g_{yy} = 0$
4	$g_{xx} + g_{yy} = 0$

Solution:
FCBDAE
4,2,1,3

Problem 3) (10 points)



a) (6 points) Enter one label into each of the boxes.

	At which point is the length of the gradient maximal?
	At which point is the global maximum?
	At which point is $f_x > 0, f_y = 0$?
	At which point is $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$, $D_{\langle 1,-1\rangle/\sqrt{2}} f < 0$?
	At which point is f maximal under the constraint $g(x,y) = y = 0$?
	At which point does f have a local minimum?

Solution:

b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

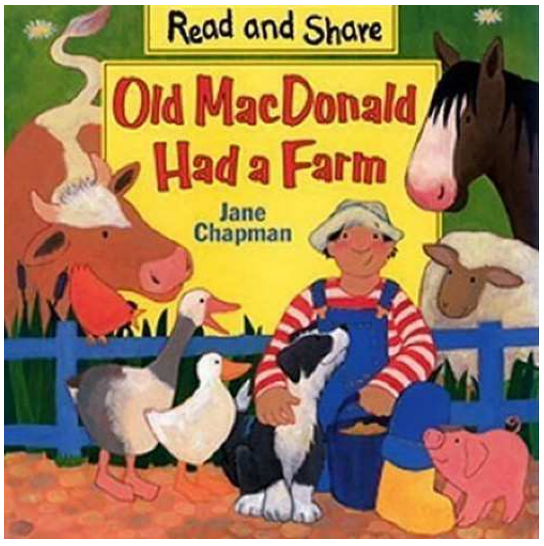
	parallel	perp	
The gradient ∇f is always			to the surface $f = c$.
For a Lagrange minimum, ∇g is			to ∇f.
If $(0, 0)$ is a min. of f then $\nabla f(0, 0)$ is			to $\langle 1, 0\rangle$.
If $(0, 0)$ is max. of f and $g = z - f(x, y)$ then ∇g is			to $\langle 0, 0, 1\rangle$.

Solution:
WBUGEO

	parallel	perp	
The gradient ∇f is always		*	to the surface $f = c$.
For a Lagrange minimum, ∇g is	*		to ∇f.
If $(0, 0)$ is a min. of f then $\nabla f(0, 0)$ is	*	*	to $\langle 1, 0\rangle$.
If $(0, 0)$ is max. of f and $g = z - f(x, y)$ then ∇g is	*		to $\langle 0, 0, 1\rangle$.

Problem 4) (10 points)

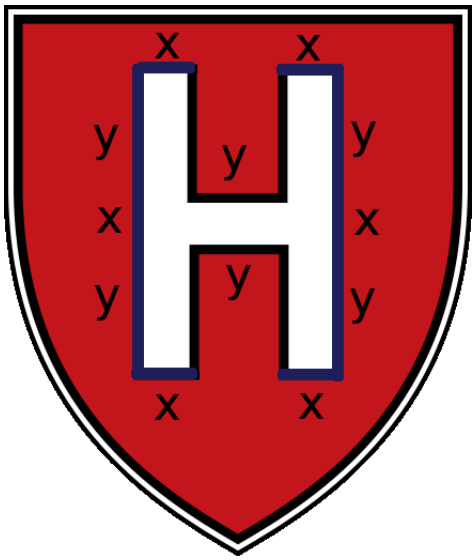
A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y. Estimate $f(12, 19)$.



"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"

Solution:
$f(10, 20) = 1000000$. The linearization is $L(12, 19) = f(10, 20)+2(10)(12-10)+(20)^2(19-20) = 1000000 + 2(20) - 400 = 999640$.

Problem 5) (10 points)



Find the Harvard H which has maximal area

$$f(x,y) = 5xy + 2x^2$$

with fixed exposed perimeter

$$6x + 4y = 88 \ .$$

Find the maximum using Lagrange.

Solution:
The Lagrange equations are

$$\begin{aligned} 5y + 4x &= 6\lambda \\ 5x &= 4\lambda \\ 6x + 4y &= 88 \end{aligned}$$

Eliminating λ gives $y = (7/10)x$. $x = 10, y = 7, f(10,7) = 550$.

Problem 6) (10 points)

a) (7 points) A minigolf on the cape has a hole at a local minimum of the function

$$f(x,y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 \ .$$

Find all the critical points and classify them.

b) (3 points) A golfer hits tangent to the level curve $f(x,y) = 2$ through $(1,1)$. Find this line.



About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.

Solution:

a)

-1	0	60	-6	max	1
-1	1	-180	-6	saddle	-2
0	0	-60	6	saddle	0
0	1	180	6	min	-3

b) We know the gradient at the point. $x = 1$.

Problem 7) (10 points)



A circular track near Salem is a circle of radius 500 which is centered at the origin $(0, 0)$. A go-kart goes counter-clockwise around the track $\vec{r}(t)$. The cheering intensity is given by a function $f(x, y)$. The go-kart passes the point $(300, 400)$ at time $t = 0$ with velocity $\langle -4, 3\rangle$. We know that $f_x(300, 400) = 2$ and $f_y(300, 400) = 10$. Find the rate of change

$$\frac{d}{dt} f(\vec{r}(t))$$

at $t = 0$.

Solution:
Use the chain rule:

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

At $t = 0$, we have $\langle 2, 10\rangle \cdot \langle -4, 3\rangle = -8 + 30 = 22$.

Problem 8) (10 points)

a) (6 points) Find the integral

$$\int_0^1 \int_y^{y^{1/5}} \frac{e^x + x^7}{x - x^5}\, dx\, dy\ .$$

b) (4 points) Integrate

$$\int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dx dy\ .$$

Solution:
a) Change the order of integration $e - 7/8$
b) Use polar coordinates $\pi/2(e-1)$. Note that the region is in the fourth quadrant so that we integrate from $-\pi/2$ to 0.

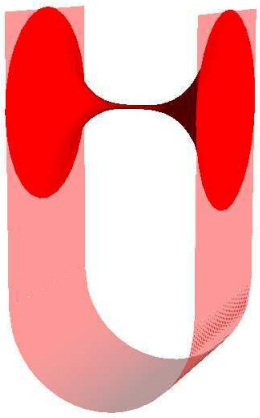
Problem 9) (10 points)

Find the surface area of the "wormhole"

$$\vec{r}(u,v) = \langle 3v^3, v^9\cos(u), v^9\sin(u)\rangle \ ,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

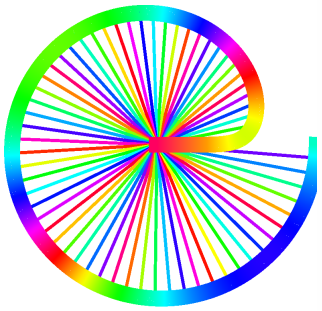
Einstein-Rosen bridges *are hypothetical topological constructions which would allow shortcuts through space-time. Tunnels connecting different parts of the universe appear frequently in science fiction.*



Solution:
Note that $\int\int |\vec{r}_u \times \vec{r}_v|\, dudv$ has a nonnegative integrand. In our case it is $9v^{11}\sqrt{1+v^{12}}$. When we integrate this from 0 to 2π we get $18\pi v^{11}\sqrt{1+v^{12}}$. But note that we have to take either square root so that the integrand is nonnegative or just take the absolute value. In any case, we can just take twice the integral from 0 to 1. The answer is not zero but $2\pi(\sqrt{8}-1)$.

Problem 10) (10 points)

a) (5 points) We become typographer and design new mathematically defined **typeface** of the alphabet. The new letter "e" in this "21a" design is given by a polar region $r(t) \leq t^{1/7}$, with $0 \leq t \leq 2\pi$. Find the area of this region.



b) (5 points) Integrate

$$\int_0^1 \int_0^{\arccos(y)} \frac{1}{\cos(x)}\, dxdy \ .$$



114

The Letter S

Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 25 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analytic geometry may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematics underlying printed S's, and also to give an example of the METAFONT language I have recently been developing. (A complete description of METAFONT, which is a computer system and language intended to aid in the design of letter shapes, appears in [3, part 2].

Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central reason is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of metals or of movable type. The task of making a plate for a printed page is now essentially that of constructing a gigantic matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I wanted the second edition of one of my books to look like the first edition, although the first edition had been typeset with the old hot-lead technology; and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and it also discusses the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct S's

.r., con quello tondo qual ch'i lo suo puncto de mezo fors del quadro, longe da la inferiore linea del quadro puncto mezo. Poi largo lo circino puncti .a., procedo una puncta dove finisti la inferiore parte del .S. qual fu facta a drita linea, cioè longe da la linea del spacio da parte drita puncti .a., e altri

puncti .q. da la [linea] inferiore del quadro. L'altra puncta longe da quella del spacio da parte sinistra puncti .a. descendenti in tondo verso man dritta tanto che giongi sopra la media linea. Poi con dicta larghezza de circino ponenda l'una puncta dove al presente finisti, l'altra puncta longe da la linea del spacio da parte sinistra puncti .a., venendo dal dicto ultimo loco del .S. tanto che sia lontano da la inferiore linea del quadro puncti .a. Poi da questa ultima parte in tondo vengni a drita linea a congiungere con lo inferiore tondo longe da la linea da parte sinistra del quadro puncti r. e sarne ocrasi; & sarà finita la littera .S., come apertamente si vede.

Fig. 1. Francesco Torniello's method of "squaring the S" in 1517. (This is page 45 of [4], reproduced by kind permission of Officina Bodoni in Verona, Italy.)

Remark: Computer scientist **Donald Knuth** once wrote an entire article about "The Letter S".

Solution:
a) We have $\int_0^{2\pi} t^{2/7}/2\ dt = (7/18)(2\pi)^{9/7}$.
b) Change the order of integration. We have

$$\int_0^{\pi/2} \int_0^{y} \frac{1}{\cos(x)}\, dydx = \int_0^{\pi/2} 1\ dy = \pi/2 \ .$$

11/1/2017 Second Hourly Practice 5 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] If $\vec{r}(t)$ is a space curve satisfying $\vec{r}\,'(0) = 0$ and $f(x, y, z)$ is a function of three variables then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at $t = 0$.

Solution:
This is the chain rule

2) [T] [F] The integral $\int\int_R 1 \, dxdy$ is the area of the region R in the xy-plane.

Solution:
Yes, it is the volume of the region below the function $f(x, y) = 1$.

3) [T] [F] If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}} f(x, y)$ is independent of $\vec{u}$.

Solution:
The gradient is constant $\langle a, b\rangle$ and the directional derivative $D_u f(x, y) = \langle a, b\rangle \cdot \langle \cos(t), \sin(t)\rangle$ depends on the direction.

4) [T] [F] If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}} f(x, y)$ is independent of (x, y).

Solution:
The gradient does not depend on (x, y) so that $D_u f(x, y) = \nabla f(x, y) \cdot u$ does not depend on (x, y).

5) [T] [F] If $(0, 0)$ is a saddle point of $f(x, y)$ it is possible that $(0, 0)$ is a minimum of $f(x, y)$ under the constraint $x = y$.

Solution:
The gradient is zero so that the directional derivative is zero in all direction.

6) [T] [F] The equation $f_{xy}(x, y) = 0$ is an example of a partial differential equation.

Solution:
Yes, it is an equation for a function and the equation contains partial derivatives with respect to different variables.

7) [T] [F] The linearization of $f(x,y) = 4 + x^3 + y^3$ at $(x_0, y_0) = (0,0)$ is $L(x,y) = 4 + 3x^2 + 3y^2$.

Solution:
The linearization is a linear function.

8) [T] [F] Assume $(1,1)$ is a saddle point of $f(x,y)$. Then $D_{\vec{v}} f(1,1)$ takes both positive and negative values as $\vec{v}$ varies over all directions.

Solution:
It is constant zero. Because the gradient is zero.

9) [T] [F] The integral $\int_{\pi/2}^{\pi} \int_0^2 r \, dr d\theta$ is equal to π.

Solution:
Yes, it is the area of a quarter of a disc of radius π.

10) [T] [F] If $|\nabla f(0,0)| = 1$, then there is a direction in which the slope of the graph of f at $(0,0)$ is 1.

Solution:
It is the direction of the gradient

11) [T] [F] The vector $\nabla f(a,b)$ is a vector in space orthogonal to surface defined by $z = f(x,y)$ at the point (a,b).

Solution:
Big misconception. This gradient vector is a vector in the plane, not in space.

12) [T] [F] If $f(x,y,z) = 1$ defines y as a function of x and z, then $\partial y(x,z)/\partial x = -f_x(x,y,z)/f_y(x,y,z)$.

13) [T] [F] In a constrained optimization problem it is possible that the Lagrange multiplier λ is 0.

14) [T] [F] The area $\int\int_R |\vec{r}_u \times \vec{r}_v| \, dudv$ of a surface is independent of the parametrization.

Solution:
Yes.

15) [T] [F] The function $f(x,y) = x^6 + y^6 - x^5$ has a global minimum in the plane.

16) [T] [F] The area of a graph $z = f(x,y)$ where (x,y) is in a region R is the integral $\int\int_R |f_x \times f_y| \, dxdy$.

Solution:
The equation does not make sense. f_x, f_y are both scalars, not vectors. If we want to compute the surface area, we have to parametrize the surface first as $\langle x, y, f(x,y)\rangle$.

17) [T] [F] The gradient of a function $f(x,y)$ of two variables can be written as $\langle D_{\vec{i}} f(x,y), D_{\vec{j}} f(x,y)\rangle$, where $\vec{i} = \langle 1,0\rangle$ and $\vec{j} = \langle 0,1\rangle$.

18) [T] [F] The length of the gradient of f at a critical point is positive if the discriminant $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ is strictly positive.

Solution:
The length of the gradient is zero.

19) [T] [F] If $f(0,0) = 0$ and $f(1,0) = 2$ then there is a point on the line segment between $(0,0)$ and $(1,0)$, where the gradient has length at least 2.

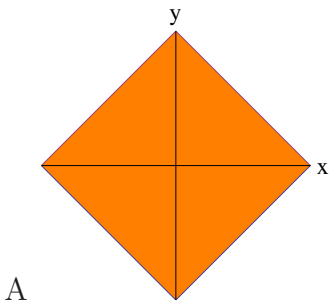
Solution:
This is the intermediate value theorem applied to the function $g(x) = f(x,0)$.

20) [T] [F] The tangent plane of the surface $-x^2 - y^2 + z^2 = 1$ at $(0,0,1)$ intersects the surface at exactly one point.

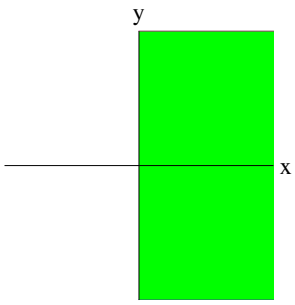
Solution:
The linearization is a linear function.

Problem 2) (10 points)

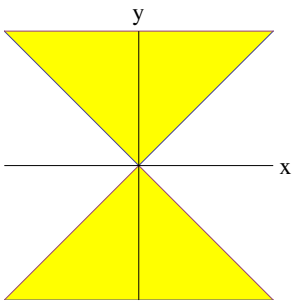
a) (6 points) Match the regions with the integrals. Each integral matches exactly one region $A - F$.



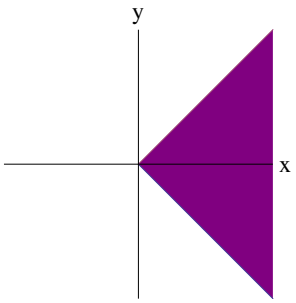
A



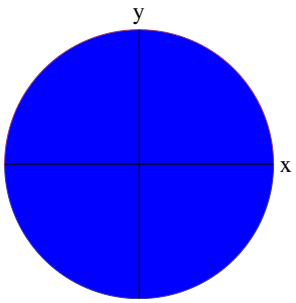
B



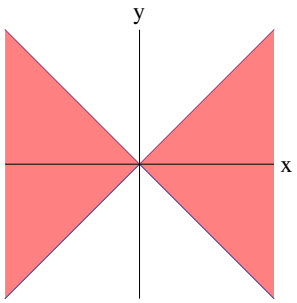
C



D



E



F

Enter A-F	Integral
	$\int_{-\pi}^{\pi} \int_{-\lvert y\rvert}^{\lvert y\rvert} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(r,\theta) r\, drd\theta$
	$\int_{-\pi}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{0}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{-\pi+\lvert x\rvert}^{\pi-\lvert x\rvert} f(x,y)\, dydx$

b) (4 points) Name the partial differential equations correctly. Each equation appears once to the left.

Fill in 1-4	Order
	Burgers
	Transport
	Heat
	Wave

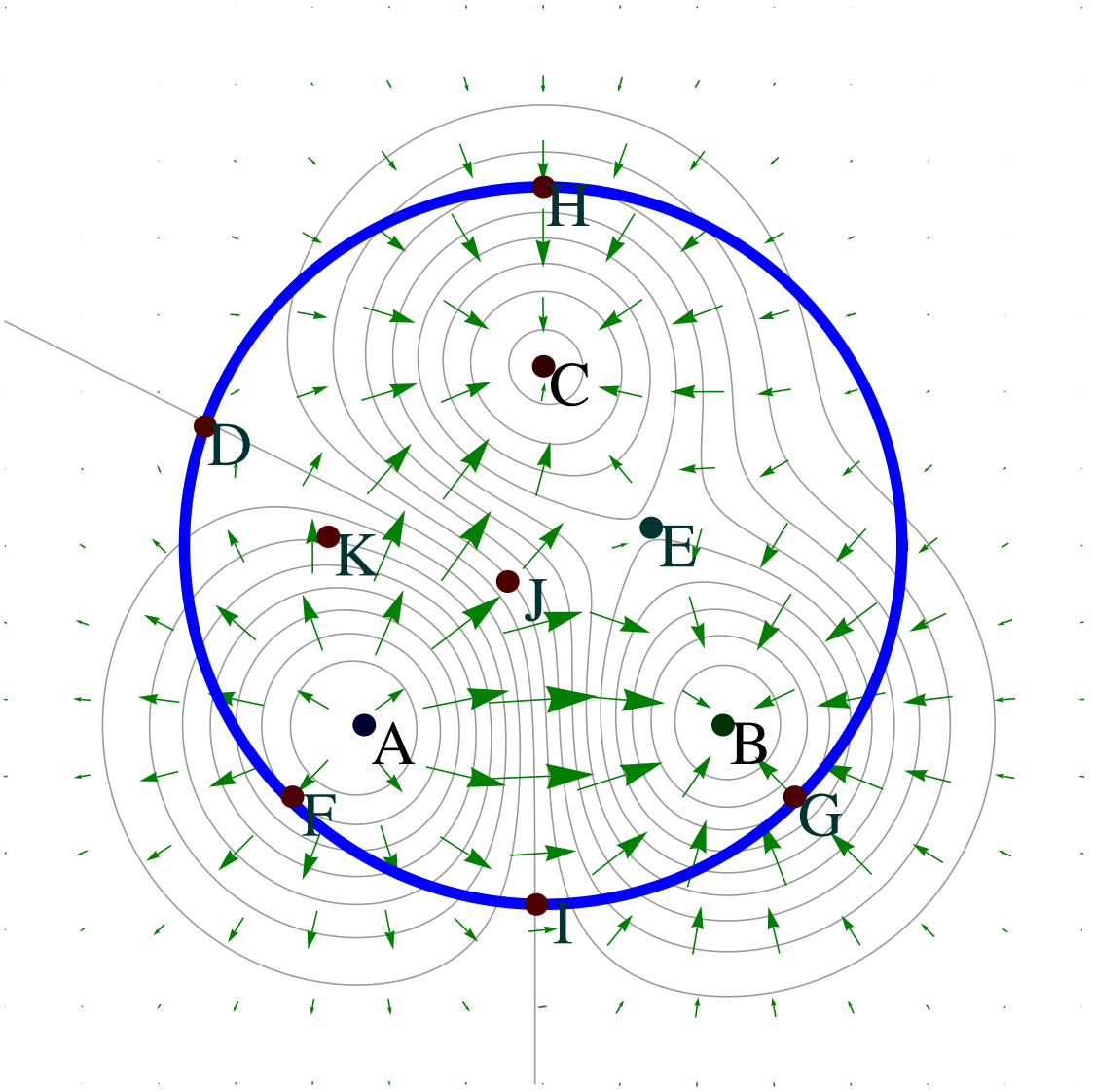
Equation Number	PDE
1	$u_x - u_y = 0$
2	$u_{xx} - u_{yy} = 0$
3	$u_x - u_{yy} = 0$
4	$u_x + uu_x - u_{xx} = 0$

Solution:
a) CEFDBA
b) 4132

Problem 3) (10 points)

(10 points) Let's label some points in the following contour map of a function $f(x, y)$ indicating the height of a region. The arrows indicate the gradient $\nabla f(x, y)$ at the point. Each of the 11 selected points appears each exactly once.

Enter A-K	description
	a local minimum of $f(x, y)$ inside the circle
	a saddle point of $f(x, y)$ inside the circle
	a point, where $f_x \neq 0$ and $f_y = 0$
	a point, where $f_x = 0$ and $f_y > 0$
	a point, where $f_x = 0$ and $f_y < 0$
	a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$.
	the lowest point on the circle
	the highest point on the circle
	the local but not global maximum inside or on the circle
	the global maximum inside or on the circle
	the steepest point inside the circle

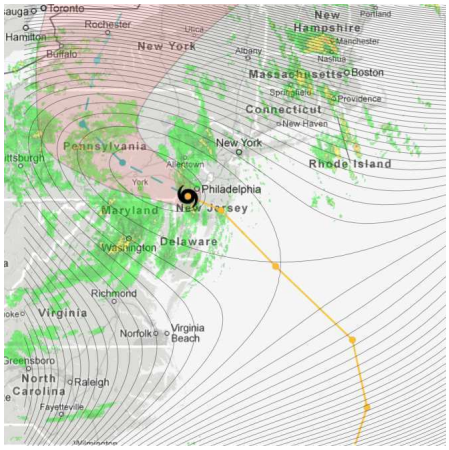


Solution:

Enter A-K	description
A	a local minimum of $f(x,y)$ inside the circle
E	a saddle point of $f(x,y)$ inside the circle
I	a point, where $f_x \neq 0$ and $f_y = 0$
K	a point, where $f_x = 0$ and $f_y > 0$
H	a point, where $f_x = 0$ and $f_y < 0$
D	a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$.
F	the lowest point on the circle
G	the highest point on the circle
C	the local but not global maximum inside or on the circle
B	the global maximum inside or on the circle
J	the steepest point inside the circle

Solution:
AEIKHDFGCBJ.

Problem 4) (10 points)



On October 30, 2012, the wind speed of Hurricane Sandy was given by the function

$$f(x,y) = 60 - x^3 + 3xy + y^3 \ .$$

Classify the critical points (maxima, minima and saddle points) of this function. Compute also the values of f at these points.

Solution:
Compute the gradient $\nabla f(x,y) = \langle -3x^2 + 3y, 3x + 3y^2 \rangle$, then $f_{xx} = -6x$ and $D = f_{xx}f_{yy} - f_{xy}^2 = -36xy - 3$.

Point	D	f_{xx}	$f(x,y)$	
$(-1,1)$	27	6	*minimum*	59
$(0,0)$	-9	0	*saddle*	60

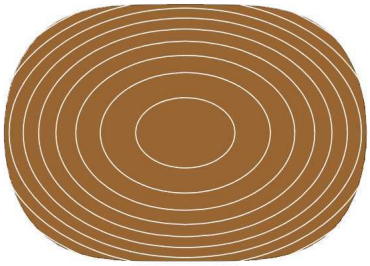
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Problem 5) (10 points)

Use the second derivative test and the method of Lagrange multipliers to find the global maximum and minimum of the sugar concentration $f(x,y) = 10 + x^2 + 2y^2$ on a cake given by

$$g(x,y) = x^4 + 4y^2 \leq 4 \; .$$

Note that this means you have to look both inside the cake and on the boundary.



Solution:
(i) We compute first the critical points in the interior. To do so, we find the gradient $\nabla f(x,y) = \langle 2x, 4y \rangle$. In the interior, there is the critical point $(0,0)$ and the value of the function is 10 at this point. It is a local minimum.
(ii) Now we compute the critical points on the boundary. This is a Lagrange problem. The Lagrange equations are

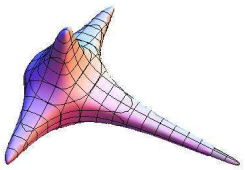
$$\begin{aligned} 2x &= \lambda 4x^3 \\ 4y &= \lambda 8y \; . \\ x^4 + 4y^2 &= 4 \; . \end{aligned}$$

Eliminating λ from the first two equations gives $16xy = 16yx^3$. This means either $x = 0$ or $y = 0$ or $x = \pm 1$. From the third equation we can then get the value of the other variable. There are 8 critical points $(0, \pm 1)$, $(\pm 1, \pm\sqrt{3}/2)$, $(\pm\sqrt{2}, 0)$ on the boundary. The function takes the value $12, 12.5, 12$. Therefore, the global maxima are at $(\pm 1, \pm\sqrt{3}/2)$ and the global minimum is at $(0,0)$.

Problem 6) (10 points)

a) (5 points) A seed of "Tribulus terrestris" has the shape

$$x^2 + y^2 + z^2 + x^4y^4 + x^4z^4 + y^4z^4 - 9z = 21$$



Find the tangent plane at $(1, 1, 2)$.

b) (5 points) The seed intersects with the xy-plane in a curve

$$x^2 + y^2 + x^4y^4 = 21 \ .$$

Find the tangent line to this curve at $(1, 2)$.

Solution:
a) Compute the gradient

$$\nabla f(x, y, z) = \langle 2x + 4x^3(y^4 + z^4), 2y + 4y^3(x^4 + z^4), 2z + 4z^3(x^4 + y^4) - 9\rangle \ .$$

Evaluated at $(1, 1, 2)$ it is $\langle 70, 70, 59\rangle$. The equation of the plane is $70x + 70y + 59z = d$. Plug in the point $(1, 1, 2)$ to find $d = 258$ so that $\boxed{70x + 70y + 59z = 258}$.
b) Compute the gradient

$$\nabla f(x, y) = \langle 2x + 4x^3y^4, 2y + 4x^4y^3\rangle \ .$$

At $(1, 2)$ it is $\langle 66, 36\rangle$. The equation of the line is $66x + 36y = d$. now plug in the point to get $d = 138$. The equation is $\boxed{11x + 6y = 23}$.

Problem 7) (10 points)

Let $f(x, y)$ model the time that it takes a rat to complete a maze of length x given that the rat has already run the maze y times. We know $f_y(10, 20) = -5$ and $f_x(10, 20) = 1$ as well as $f(10, 20) = 45$. Use this to estimate $f(11, 18)$.



Picture by Ellen van Deelen,

South west News service (UK)

Solution:
Use the linearization $L(x, y)$ to the function at $(10, 20)$:

$$f(11, 18) \sim L(11, 18) = 45 + 1(11 - 10) - 5(18 - 20) = 56 \, .$$

Problem 8) (10 points)

a) (5 points) Find the double integral

$$\int\int_R x \, dydx \, ,$$

where R is the region obtained by intersecting $x \leq |y|$ with $x^2 + y^2 \leq 1$.

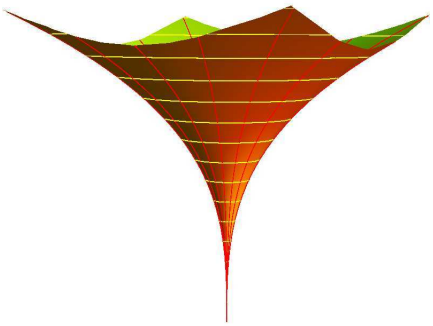
b) (5 points) The square $\sin^2(x)/x^2$ of the sync function $\sin(x)/x$ does not have a known antiderivative. Compute nevertheless the integral

$$\int_0^{\pi^2/4} \int_{\sqrt{y}}^{\pi/2} \frac{\sin^2(x)}{x^2} \, dx \, dy \, .$$

Solution:
a) Make a picture of the region. Use Polar coordinates: $\int_{\pi/4}^{7\pi/4} \int_0^1 r^2 \cos(\theta) \, drd\theta = -\sqrt{2}/3$.
b) Change the order of integration:

$$\int_0^{\pi/2} \int_0^{x^2} \frac{\sin^2(x)}{x^2} \, dydx = \frac{\pi}{4} \, .$$

Problem 9) (10 points)



Find the surface area of the surface of revolution $x^2 + y^2 = z^6$ where $0 \leq z \leq 1$. The surface is parametrized by

$$\vec{r}(t, z) = \langle z^3 \cos(t), z^3 \sin(t), z \rangle$$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$.

Solution:
Compute

$$\vec{r}_t = \langle -z^3 \sin 9t), z^3 \cos(t), 0 \rangle, \vec{r}_z = \langle 3z^2 \cos(t), 3z^2 \sin(t), 1 \rangle \ .$$

Now

$$|\vec{r}_t \times \vec{r}_z| = \sqrt{z^6 \cos^2(t) + z^6 \sin^2(t) + 9z^10} = z^3\sqrt{1 + 9z^4} \ .$$

We integrate

$$\int_0^{2\pi} \int_0^1 z^3\sqrt{1 + 9z^4} \, dz d\theta = (1 + 9z^4)^{3/2} \frac{1}{3} \frac{1}{9} 2\pi =$$

$\boxed{(10^{3/2} - 1)\frac{\pi}{27}.}$

Problem 10) (10 points)



It turns out that there is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{x^7}{\sqrt{x} - x^2} \, dx \, dy \ .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{10} \, dxdy \ .$$

Solution:
a) Change the order of integration and first switch also the order of the integral because y^2 is smaller than $\sqrt{y}$.

$$= -\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{x^7}{\sqrt{x} - x^2}\, dy\, dx = -\int_0^1 x^7\, dx = -1/8\ .$$

b) Write the integral in polar coordinates noticing that the region is a quarter circle.

$$\int_0^{\pi/2} \int_0^1 r^{21}\, dr\, d\theta = \frac{\pi}{44}\ .$$

The answers are $\boxed{-1/8}$ and $\boxed{\pi/44}$.

11/1/2017 Second Hourly Practice 6 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] There is a function $f(x,y)$ for which the linearization at $(0,0)$ is $L(x,y) = x^2 + y^2$.

Solution:
The linearization is a linear function and not quadratic.

2) [T] [F] For any two functions f, g and unit vector $\vec{u}$ we have $D_{\vec{u}}(f+g) = D_{\vec{u}}f + D_{\vec{u}}g$.

Solution:
This follows directly from the definition.

3) [T] [F] $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dydx = \int_0^2 \int_0^{\pi/2} r^2\, d\theta dr$.

Solution:
The r factor is forgotten.

4) [T] [F] If we solve $\sin(y) - xy^2 = 0$ for y, then $y' = -y^2/(\cos(y) - 2xy)$.

Solution:
This is an application of the implicit differentiation formula. But the sign is wrong.

5) [T] [F] If $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y, then $g(x,y) = \int_0^x \int_0^y f(s,t)\, dtds$ solves $g_{xy}(x,y) = f(x,y)$.

Solution:
Fundamental theorem of calculus. The assumption $f(x,0) = 0$ and $f(0,y) = 0$ is not necessary.

6) [T] [F] If $|\nabla f| = 1$ at $(0,0)$, then there exists a direction in which the slope of the graph of f at $(0,0)$ is 1.

Solution:
It is the direction of the gradient

7) [T] [F] The function $f(x,y) = x^2 + y^2$ satisfies the partial differential equation $f_{xx}f_{yy} - f_{xy}^2 = 4$.

Solution:
Yes, this is a computation.

8) [T] [F] The height of Mount Wachusett is $f(x,y) = 4 - 2x^2 - y^2$. On the trail $x^2 + y^2 = 1$, the point $(1,0)$ is a maximum.

Solution:
It is a local minimum.

9) [T] [F] Mount Wachusett has height $f(x,y) = 4-2x^2-y^2$. Except at the maximum $(0,0)$, the gradient vector is perpendicular to the graph of the function.

Solution:
The gradient vector is a vector with two components, not a vector in space.

Solution:
The gradient vector of f is a vector with two components and not in space.

10) [T] [F] If $f_x(a,b) > 0$ and $f_y(a,b) > 0$ then for any unit vector $\vec{u}$ we must have $D_{\vec{u}}f(a,b) > 0$.

Solution:
Take a unit vector $\langle -1, 0\rangle$ for example. The directional derivative in this direction is zero.

11) [T] [F] If $f(x,y)$ has two local minima, then f must have at least one local maximum.

Solution:
Take a function like $-x^2 \exp(-x^2 - y^2)$. It has two local minima and a saddle point.

12) [T] [F] If $\vec{r}(t)$ is a curve on the surface $g(x, y, z) = x^2 + y^2 - z^2 = 6$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
This is a recurring theme. The gradient vector is perpendicular to the surface. This fact is based on the chain rule and the fact that $g(r(t))$ is constant so that $d/dtg(r(t))$ is zero.

13) [T] [F] If f and g have the same trace $\{x = 5\}$ then $f_x(5, y) = g_x(5, y)$ for all y.

Solution:
We know $f(5, y) = g(5, y)$ but the x derivative can be different. Take $f(x, y) = x - 5$ and $g(x, y) = y(x - 5)$ then $f_x = 1$ and $g_x = y$.

14) [T] [F] If f and g have the same trace $\{x = 5\}$ then $f_y(5, y) = g_y(5, y)$ for all y.

Solution:
Because $f(5, y) = g(5, y)$, also the derivatives are the same.

15) [T] [F] The surface area of $\vec{r}_1(u, v) = \langle u\cos(v), u\sin(v), u^2 \rangle$ and $\vec{r}_2(u, v) = \langle \sqrt{u}\cos(v), \sqrt{u}\sin(v), u \rangle$ defined on $\{0 \leq u, v \leq 1\}$ are the same.

Solution:
Surface area does not depend on parametrization.

16) [T] [F] If $\vec{r}(t)$ is a curve on a graph $z = f(x, y)$ of a function $f(x, y)$, then the velocity vector of $\vec{r}$ is perpendicular to the vector $\langle f_x, f_y, -1 \rangle$.

Solution:
The vector $\langle f_x, f_y, -1 \rangle$ is the gradient of the function $g(x, y, z) = f(x, y) - z$ which has as a level surface the graph $g = 0$ of f.

17) [T] [F] A continuous function $f(x,y)$ on the closed disc $R = \{x^2 + y^2 \leq 51^2\}$ (of course, R is called **"area** 51π**"**) has a global maximum on R.

Solution:
We know that a continuous function has a maximum on a closed bounded domain. Interesting corollary: take for f the probability density that an alien has landed there. Then there is a point where the probability density is largest, proving so that aliens are likely in area 51.

18) [T] [F] Any continuous function $f(x,y)$ has a global minimum and maximum on the curve $y = x^2$.

Solution:
The curve is an unbounded parabola. The function $f(x,y) = x$ for example is unbounded on it.

19) [T] [F] Fubini's theorem assures that $\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \int_c^d f(x,y)\, dxdy$.

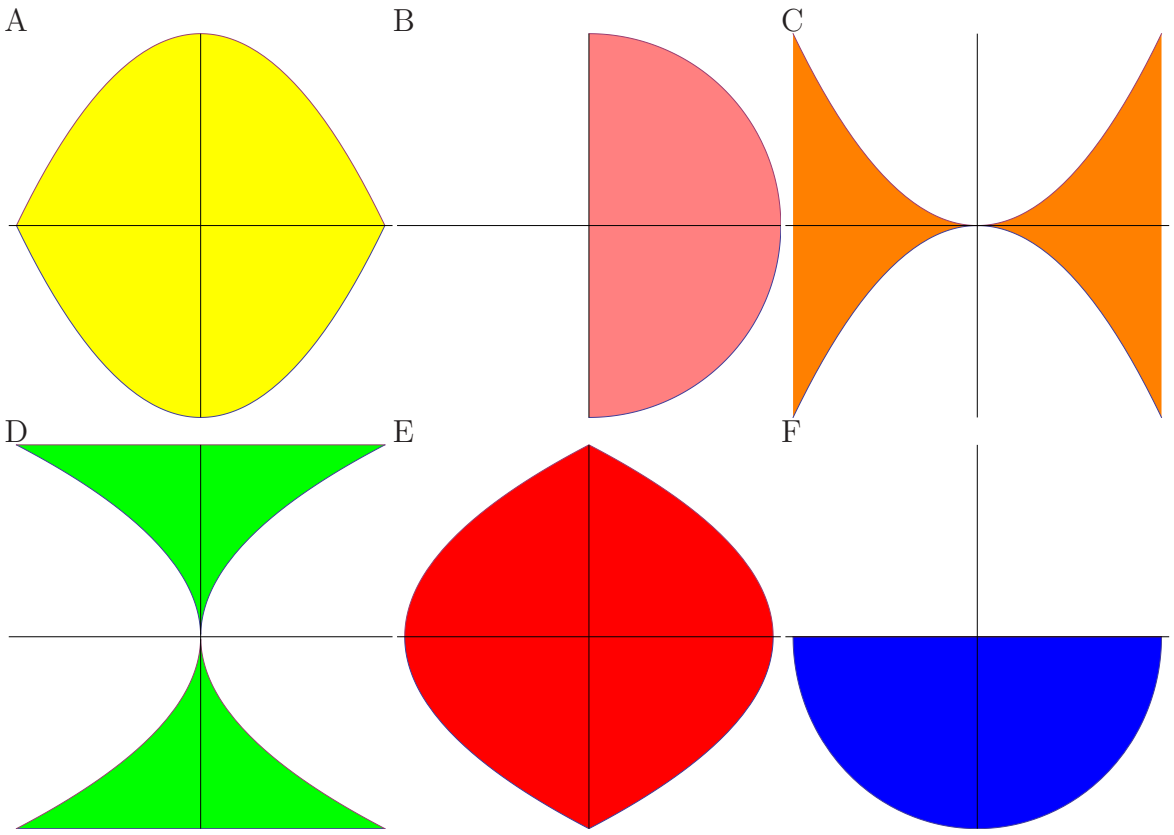
Solution:
The integrals have not been switched.

20) [T] [F] $\int\int_R \sin(x+y)\, dxdy = 0$ for $R = \{-\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$.

Solution:
Directly integrate. One can see this also by symmetry. The integral has an interpretation as a volume, with exactly the same amount below than above the plane.

Problem 2) (10 points)

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region $A - F$.



Enter A-F	Integral
	$\int_{-1}^{1}\int_{-x^2}^{x^2} f(x,y)\,dydx.$
	$\int_{-1}^{1}\int_{-y^2}^{y^2} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{y^2-1}^{1-y^2} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} f(x,y)\,dxdy.$
	$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} f(x,y)\,dydx.$
	$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dydx.$

b) (4 points) Fill in one word names (like "Heat","Wave" etc) for the partial differential equations:

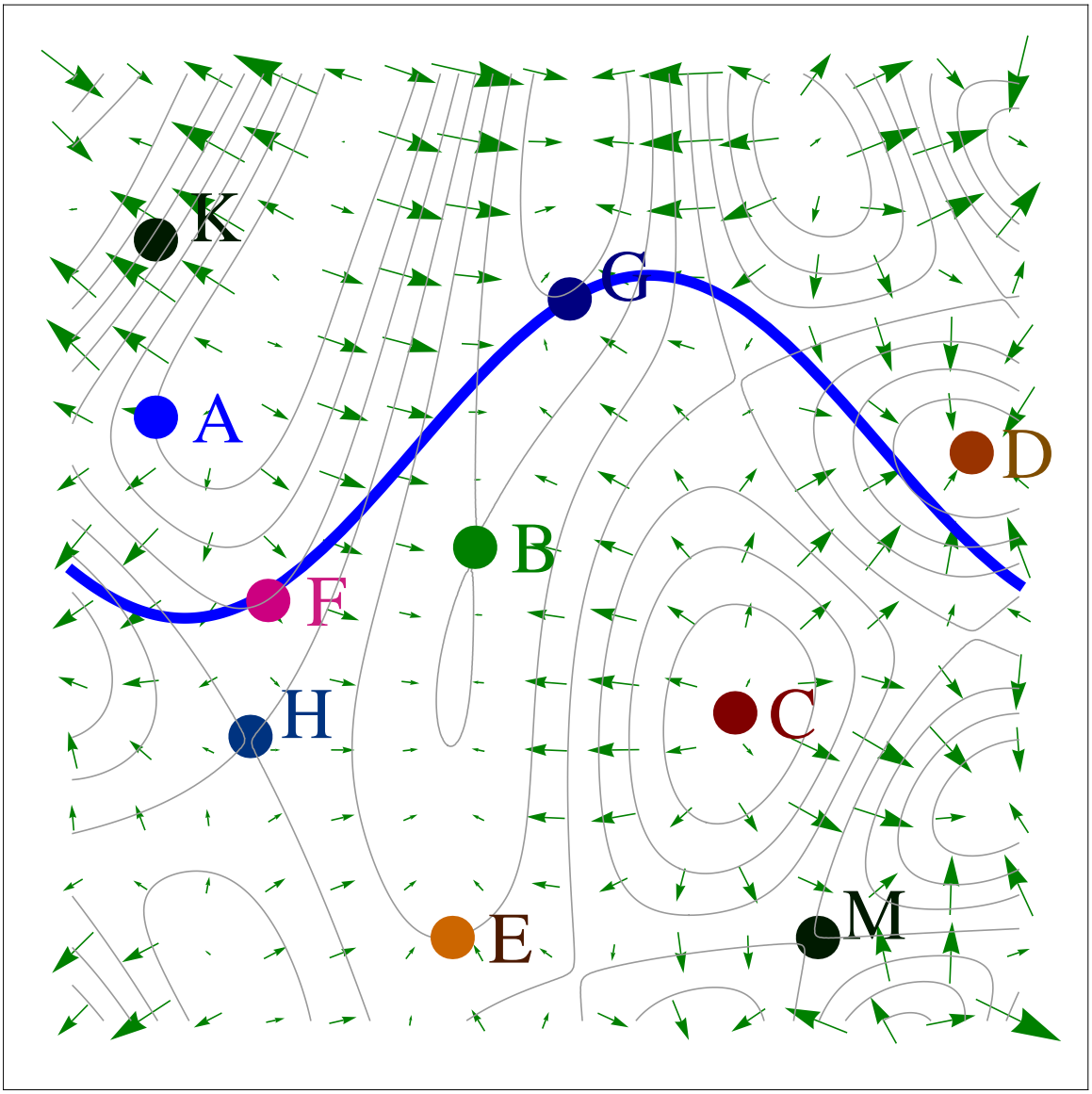
Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = g_{yy}$
	$g_{xx} = -g_{yy}$
	$g_x = g_{yy}$

Solution:
a) C,D,E,B,A,F
b) Transport, Wave, Laplace, Heat.

Problem 3) (10 points)

(10 points) A function $f(x, y)$ of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve $g(x, y) = 0$ which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters $A, B, C, D, E, F, G, H, K, M$ appears exactly once.

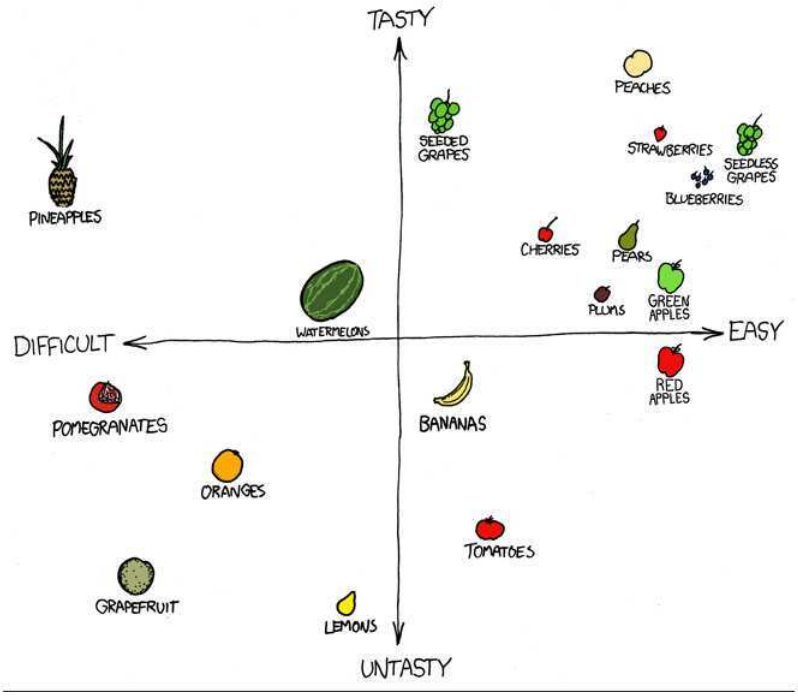
Enter A-P	Description
	a local maximum of $f(x, y)$.
	a local minimum of $f(x, y)$.
	a saddle point of $f(x, y)$ where $f_{xx} < 0$.
	a saddle point of $f(x, y)$ where $f_{xx} > 0$.
	a saddle point of $f(x, y)$ where f_{xx} is close to zero
	a point, where $f_x = 0$ and $f_y \neq 0$
	a point, where $f_y = 0$ and $f_x \neq 0$
	the point, where $\|\nabla f\|$ is largest
	a local maximum of $f(x, y)$ under the constraint $g(x, y) = 0$.
	a local minimum of $f(x, y)$ under the constraint $g(x, y) = 0$.



Solution:
D,C,B,H,M,E,A,K,G,F.

Problem 4) (10 points)

Find and classify all the extrema of the function $f(x,y) = x^5 + y^3 - 5x - 3y$. This function measures "eat temptation" in the x=Easy-y=Tasty plane. Is there a global minimum or global maximum?



The "Easy-Tasty plane" was introduced in the XKCD cartoon titled "F&#% Grapefruits".

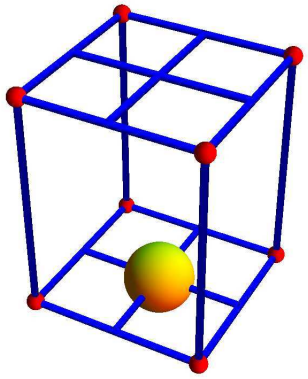
Solution:
Find the critical points $\nabla f(x,y) = \langle 0, 0\rangle$, then find the discriminant D and f_{xx}.

ne]hline point	D	f_{xx}	type
$(-1,-1)$	120	-20	maximum
$(-1,1)$	-120	-20	saddle
$(1,-1)$	-120	20	saddle
$(1,1)$	120	20	minimum

Problem 5) (10 points)

After having watched the latest Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x, y, z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x, y, z) = xyz$.





Solution:
The gradients are $\nabla f(x, y, z) = \langle yz, xz, xy \rangle$ and $\nabla g(x, y, z) = \langle 6, 6, 4 \rangle$. The Lagrange equations are

$$yz = \lambda \cdot 6$$
$$xz = \lambda \cdot 6$$
$$xy = \lambda \cdot 4$$

Getting rid of λ and using that x, y, z and so λ must all be positive for having a positive volume gives $x = y = 2z/3$. Plugging this into the constraints gives $\boxed{x = 16/9, y = 16/9, z = 8/3}$.

Problem 6) (10 points)

a) (8 points) Find the tangent plane to the surface $f(x, y, z) = x^2 - y^2 + z = 6$ at the point $(2, 1, 3)$.

b) (2 points) A curve $\vec{r}(t)$ on that tangent plane of the function $f(x, y, z)$ in a) has constant speed $|\vec{r}\,'| = 1$ and passes through the point $(2, 1, 3)$ at $t = 0$. What is $\frac{d}{dt} f(\vec{r}(t))$ at $t = 0$?

Solution:
a) The gradient is $\nabla f(x,y,z) = \langle 2x, -2y, 1\rangle$. At the point $(2,1,3)$ this is $\langle 4, -2, 3\rangle$. The plane has the form $4x-2y+z = d$ where d can now be obtained by plugging in the point, which is $8-2+3=9$. The plane is $\boxed{4x-2y+z=9.}$
b) Since the curve is on the surface and f does not change, we have $d/dt f(\vec{r}(t)) = 0$. We can also see it from the fact that $\vec{r}'(t)$ is perpendicular to the gradient. The answer is $\boxed{0}$.

Problem 7) (10 points)

a) (5 points) Estimate $\sqrt{\sin(0.0004) + 1.001^2}$ using linear approximation.

b) (5 points) We know $f(0,0) = 1$, $D_{\langle \frac{3}{5}, \frac{4}{5}\rangle} f(0,0) = 2$ and $D_{\langle -\frac{4}{5}, \frac{3}{5}\rangle} f(0,0) = -1$. If $L(x,y)$ is the linear approximation to $f(x,y)$ at the point $(0,0)$, find $L(0.06, 0.08)$.

Solution:
a) $f_x = \cos(x)/(2\sqrt{\sin(x)+y^2})$ and $f_y = 2y/(2\sqrt{\sin(x)+y^2})$ at the point $(0,1)$ we get $\nabla f(0,1) = \langle 1/2, 1\rangle$. The linearization is

$$L(x,y) = 1 + \frac{1}{2}\cdot 0.0004 + 1\cdot 0.001 = 1 + 0.0002 + 0.001 = 1.0012\ .$$

The answer is $\boxed{1.0012}$.
b) Write $\nabla f = \langle a, b\rangle$. We get $a\cdot(3/5) + b\cdot(4/5) = 2, (-4/5)\cdot a + (3/5)\cdot b = -1$. We can solve for a, b to get $a = 2, b = 1$. We have therefore the linear approximation

$$L(x,y) = f(0,0) + 2\cdot 0.06 + 1\cdot 0.08 = 1.2\ .$$

The answer is $\boxed{1.2}$.

Problem 8) (10 points)

a) (5 points) Find the following double integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\pi \sin(\pi y)}{y^2 - \sqrt{y}}\, dydx\ .$$

b) (5 points) Evaluate the following double integral

$$\int\int_R \frac{\sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\, dxdy$$

over the region

$$R = \{x^2 + y^2 \leq 1, x > 0 \} .$$

Solution:
a) Make a picture and change the order of integration.

$$\int_0^1 \int_{y^2}^{\sqrt{y}} \frac{\pi \sin(\pi y)}{(y^2 - \sqrt{y})} \, dxdy = \int_0^1 -\sin(\pi y) \, dy = -2 .$$

The answer is $\boxed{-2}$.
b) We use polar coordinates:

$$\int_0^1 \int_{-\pi/2}^{\pi/2} \frac{\sin(\pi r)}{r} r \, d\theta \, dr = 2\pi/\pi = 2 .$$

The answer is $\boxed{2}$.
Remark. In a), Mathematica 8 can not compute the integral. We have to help it to change the order of integration. This example shows that Mathematica does not have the change of order of integration trick implemented.

Problem 9) (10 points)

a) (8 points) Find the surface area of the surface parametrized as

$$\vec{r}(u, v) = \langle u - v, u + v, (u^2 - v^2)/2 \rangle ,$$

where (u, v) is in the unit disc $R = \{u^2 + v^2 \leq 1 \}$.

b) (2 points) Give a nonzero vector $\vec{n}$ normal to the surface at $\vec{r}(4, 2) = \langle 2, 6, 6 \rangle$.

Solution:
a) We have $r_u = \langle 1, 1, u\rangle$ and $r_v = \langle -1, 1, -v\rangle$ so that $r_u \times r_v = \langle -u - v, v - u, 2\rangle$. Its length is $|r_u \times r_v| = \sqrt{2}\sqrt{u^2 + v^2 + 2}$. Use polar coordinates to integrate this over R:

$$\int_R |r_u \times r_v| \, dudv = \int_0^{2\pi} \int_0^1 \sqrt{2(r^2+2)}r \, drd\theta \, .$$

Use substitution $u = r^2 + 2$ to solve the inner integral:

$$\int_0^{2\pi} \frac{\sqrt{2}}{3}(r^2+2)^{3/2}|_0^1 \, d\theta \, .$$

Evaluate this to get the final answer $\boxed{(\pi/3)(6\sqrt{6} - 8)}$.
b) We have computed $\vec{r}_u \times \vec{r}_v$ already in a). Just take $(u, v) = (4, 2)$ to get $\boxed{\langle -6, -2, 2\rangle}$. Of course any vector parallel to this is also correct.

Problem 10) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y} \, dydx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2 + y^2) \, dydx \, ,$$

where R is the ring $1 \leq x^2 + y^2 \leq 9$.

Solution:
a) To change the order of integration, make a figure. The integration region is the upper left triangle in the square $[0, \pi/2] \times [0, \pi/2]$. We get

$$\int_0^{\pi/2} \int_0^y \cos(y)/y \, dxdy = \int_0^{\pi/2} \cos(y) \, dy = \sin(y)_0^{\pi/2} = 1 \, .$$

The answer is $\boxed{1}$.
b) This is a polar integration problem

$$\int_0 0^{2\pi} \int_1^3 r^2 r \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4}|_1^3 \, d\theta = \int_0^{2\pi} 20 \, d\theta = 40\pi \, .$$

The answer is $\boxed{40\pi}$.

11/1/2017 Second Hourly Practice 7 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] If $f(x,y) = 1$ is a curve, and near $(2,3)$ one can write y as a function of x, then $y' = -f_y(2,3)/f_x(2,3)$.

Solution:
The order is wrong: the f_y is in the denominiator.

2) [T] [F] If $\iint_R f(x,y)\, dA = 0$, then the function $f(x,y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.

Solution:
If $f = xy$ and R is the disc, then the integral is zero but f is nonzero.

3) [T] [F] The directional derivative in the direction of the gradient is $|\nabla f|$.

Solution:
Indeed, $D_{\nabla f/|\nabla f|} = \nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$.

4) [T] [F] The linearization of $f(x,y) = x^3 + y^3$ at $(1,1)$ is the quadratic function $L(x,y) = 3x^2 + 3y^2$.

Solution:
The linearization is a linear (affine) function and not quadratic.

5) [T] [F] The function $f(x,y) = x^2 + y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.

Solution:
Almost, a factor 4 is missing.

6) [T] [F] The function x^2y^2 has no local minimum at $(0,0)$ because the discriminant function D is zero there.

Solution:
There can be a local minimum with $D = 0$.

7) [T] [F] The double integral $\int_0^{\pi/4} \int_0^2 r^3 \, dr d\theta$ is the volume of the part of a solid cylinder $x^2 + y^2 \leq 4$ which is below the paraboloid $z = x^2 + y^2$ and above the xy plane.

Solution:
We do not integrate from 0 to 2π.

8) [T] [F] The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is perpendicular to the level surface of f through (x_0, y_0, z_0).

Solution:
It indeed is! This is an important fact.

9) [T] [F] If $f(x, y, z) = 3x - 4z$, then the minimal possible directional derivative $D_{\vec{u}} f$ at any point in space is -5.

Solution:
The gradient has length 5. The directional derivative into the direction of the gradient is the length of the gradient.

10) [T] [F] If (x, y) is not a critical point, then the directional derivative $D_{\vec{v}} f$ can take both positive and negative values for different choices of $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

Solution:
Because of the symmetry $D_{-v} f = -D_v f$ the integral is zero.

11) [T] [F] Using linearization of $f(x, y) = x/y$ we can estimate $1.01/1.001 = f(1.01, 1.001) \sim 1 + 0.01 - 0.001 = 1.009$.

Solution:
$L(x, y) = 1 + 1 \cdot 0.01 - 1 \cdot 0.001$.

12) [T] [F] If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.

Solution:
Local max or min, but not necessary global max or min.

13) [T] [F] For a rectangular region R, Fubini tells that $\int_0^2 \int_0^3 f(x,y)\ dxdy = \int_0^2 \int_0^3 f(x,y)\ dydx$ for any continuous function $f(x,y)$.

Solution:
We also have to switch the integration bounds.

14) [T] [F] If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y)\ dxdy > 0$.

Solution:
The critical point can be surrounded by a small region only, where f is positive.

15) [T] [F] If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

Solution:
Yes, this is the chain rule.

16) [T] [F] $\int_0^1 \int_0^1 f_{xy}(x,y)\ dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.

Solution:
This is a consequence of the fundamental theorem of calculus

17) [T] [F] If $f_{yy}(x,y) > 0$ everywhere, then f can not have any local maximum.

Solution:
We would have $f_{yy} < 0$ at a local maximum.

18) [T] [F] The double integral $\int_0^1 \int_0^1 x^2 - y^2 \, dxdy$ is the volume of the solid below the graph of $f(x, y) = x^2 - y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy-plane.

Solution:
It is a signed volume. There can be part below.

19) [T] [F] For any unit vector $\vec{v}$ and any differentiable function f, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

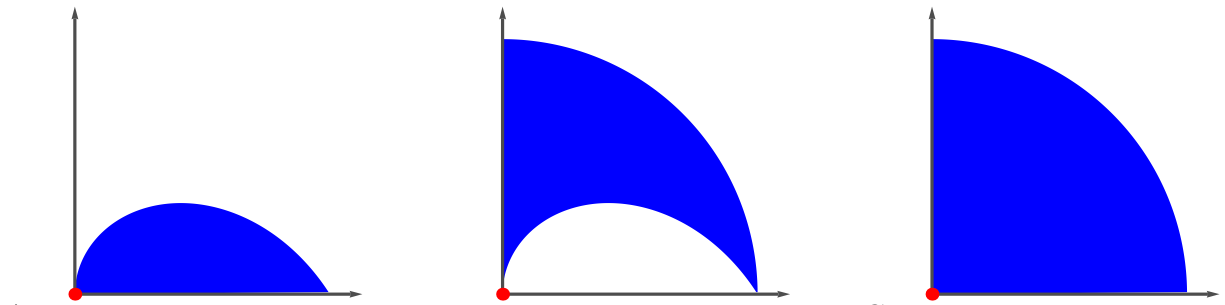
Solution:
Write down the definition. The sum is $\nabla f \cdot (v - v) = 0$.

20) [T] [F] The surfaces $x + y + z = 0$ and $x^2 + y^2 + z^2 + x + y + z = 0$ have the same tangent plane at $(0, 0, 0)$.

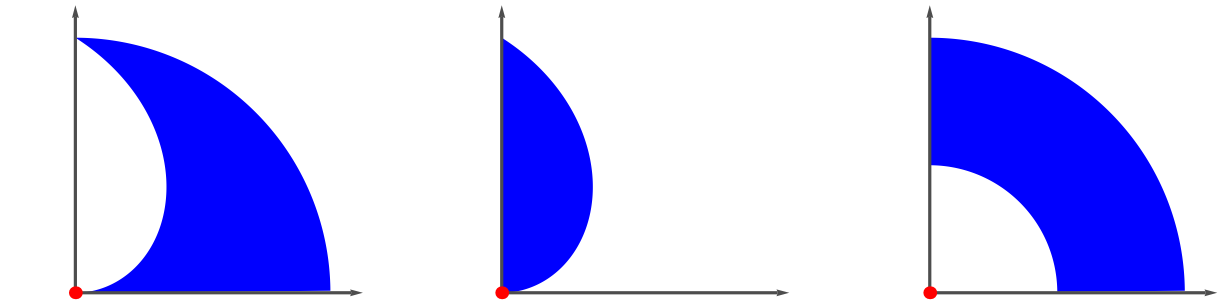
Solution:
They have the same gradient at $(0, 0, 0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



A B C



Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta)r\ drd\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_\theta^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta)r\ drd\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_t + uu_s - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$

Enter A,B,C,D here	PDE
	$u_{tt} - u_{ss} = 0$
	$u_t - u_{ss} = 0$

A) Wave equation	B) Heat equation	C) Burgers equation	D) Laplace equation

Solution:

a)
C D
E B
A F

b)
C A
D B

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x, y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x, y)$?

Solution:
The gradient of f is

$$\langle 6x + 3x^2, 6y + 3y^2 \rangle = \langle 3x(2 + x), 3y(2 + y) \rangle \ .$$

There are 5 critical points:

x	y	D	f_{xx}	Type	f value
-2	-2	36	-6	maximum	13
-2	0	-36	-6	saddle	9
0	-2	-36	6	saddle	9
0	0	36	6	minimum	5

b) There is no global maximum (nor global minimum). For $y = 0$, have $5+3x^2+x^3$ which grows like x^3 for $x \to \infty$.

Problem 4) (10 points)

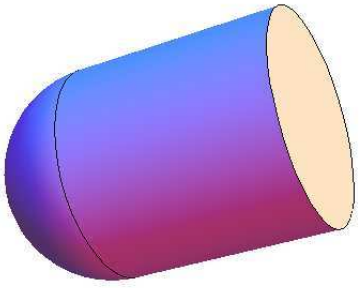
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

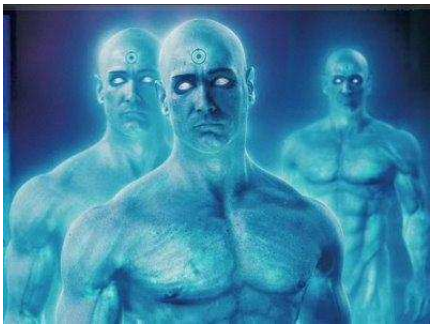
$$f(h, r) = 3r^2 + 2rh$$

under the constraint

$$g(h, r) = 2r^3 + 3r^2 h = 1 \ .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

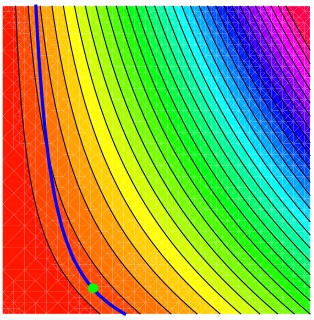




Solution:
The Lagrange equations are

$$\begin{aligned} 2h + 6r &= \lambda(6hr + 6r^2) \\ 2r &= 3\lambda r^2 \\ 3hr^2 + 2r^3 &= 1 \, . \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide the second equation by r. This allows to eliminate λ and $2h+6r = 4h+4r$ which is $h = r$. The third equation gives us $h = r = 1/5^{1/3}$. The point where the minimum occurs is $(1/5^{1/3}, 1/5^{1/3})$. The minimal value of f is $5/5^{2/3}$.

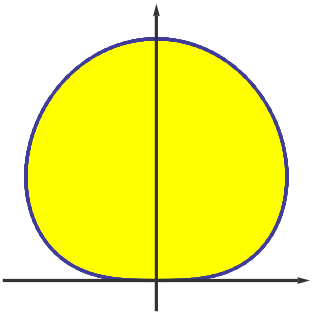


Problem 5) (10 points)

A region R in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int\int_R 1 \, dxdy$$

of this nihilistic object.



Solution:

$$\int_0^{\pi} \int_0^{\theta(\pi-\theta)} r \; dr d\theta = \int_0^{\pi} \theta^2(\pi-\theta)^2/2 \; d\theta = \pi^5/60 \; .$$



Problem 6) (10 points)

a) (4 points) If

$$f(x,y) = y\cos(x-y) \; ,$$

find equation of plane tangent to $z = f(x,y)$ at the point $(2,2,2)$.

b) (3 points) Find the equation of the tangent line to $f(x,y) = 2$ at $(2,2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Solution:
a) Define $g(x,y,z) = f(x,y) - z$. It is important to deal with a function of three variables when looking at planes. The graph of g is equal to the level surface $z - f(x,y) = 0$. Then $\nabla g(2,2,2) = \langle 0, 1, -1 \rangle$. The tangent plane is of the form $y - z = d$ where d is a constant. Plugging in $(2,2,2)$ gives $y - z = 0$.
b) $\nabla f(x,y) = \langle -y\sin(x-y), \cos(x-y) + \sin(x-y)$ and $\nabla f(2,2) = \langle 0, 2 \rangle$ so that the equation of the tangent line is $y = d$ for a constant d. Plugging in the point $(2,2)$ gives $y = 2$.
c) $L(2.1, 1.9) = 2 + 0(0.1) + 1(-0.1) = 1.9$.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.



Image source: Harvard Press release on robobees.seas.harvard.edu

a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b?

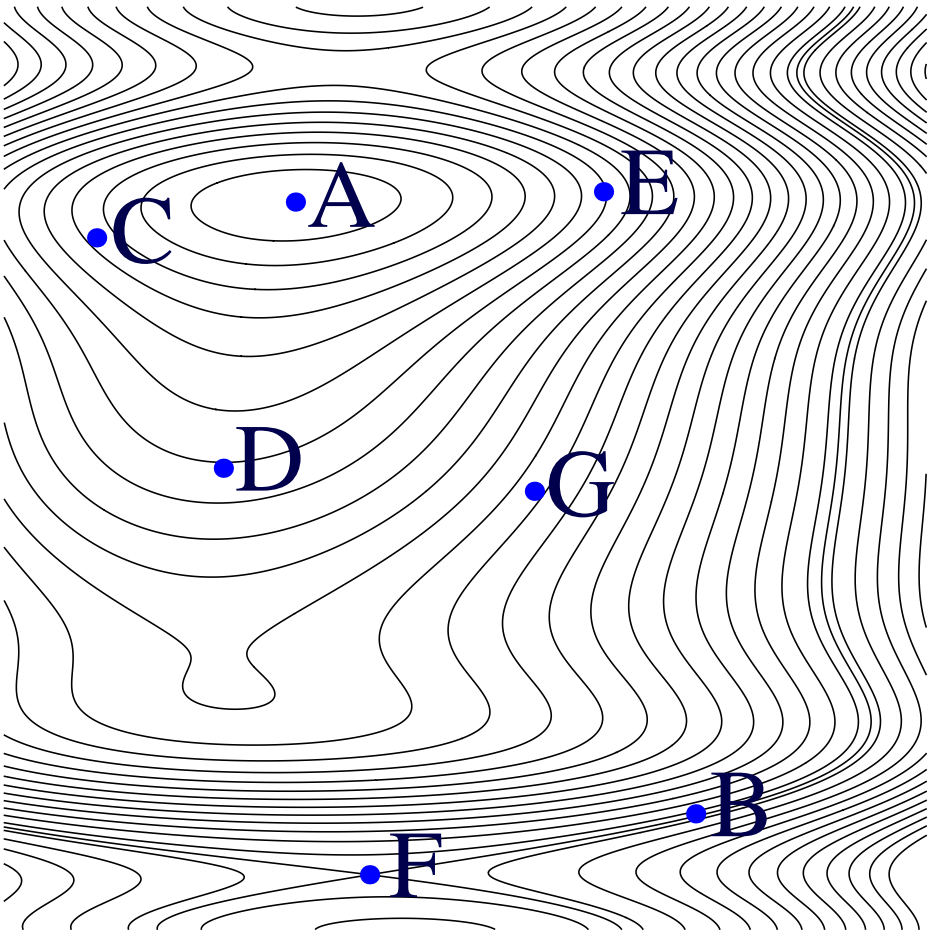
b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

Solution:
a) $\vec{r}\,'(t) = \langle 1 - 3t^2, 6t - 3 \rangle$. So that $\vec{r}\,'(0) = \langle 1, -3 \rangle, \vec{r}\,'(1) = \langle -2, 3 \rangle$. If the gradient of f at $(0, 0)$ is $\langle a, b \rangle$ we get by the chain rule $d/dt f(\vec{r}(t)) = \langle a, b \rangle \cdot \vec{r}\,'(t)$ which is either $\langle a, b \rangle \cdot \langle 1, -3 \rangle = a - 3b$ or $\langle a, b \rangle \cdot \langle -2, 3 \rangle = -2a + 3b$.
b) We know that $a - 3b = 3$ and $-2a + 3b = 3$. This is a system of linear equations which has the solution $\langle a, b \rangle = \langle -6, -3 \rangle$.

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x, y) = 0$ and $f_y(x, y) \neq 0$.
	$f_y(x, y) = 0$ and $f_x(x, y) \neq 0$.
	$f(x, y)$ has either a max or a min.
	$f(x, y)$ has a saddle point.
	$f(x, y)$ has no max nor min but is extremal under a constraint $y = c$ for some c.
	$f(x, y)$ has no max nor min but is extremal under a constraint $x = c$ for some c.
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	$D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d.



Solution:
D,E,A,F,(D or F), (E or F), B,G,C,C.
To the tangent line (last choice): The equation $x+y=d$ means that the gradient vector is $\langle 1,1\rangle$ at the point. Use now that the gradient vector is perpendicular to the level curve.

Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4}\, dydx\ .$$

Solution:
We have to change the order of integration

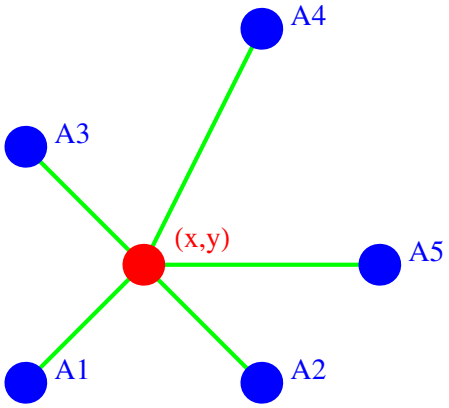
$$\int_0^1 \int_0^{1-\sqrt{y}} x^3/(1-\sqrt{y})^4 dxdy = \int_0^1 (1/4)\, dy = 1/4\ .$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.



Solution:
The gradient of f is

$$\nabla f(x, y) = \langle -14 + 10x, -12 + 10y \rangle\ .$$

It leads to the solution $(x, y) = (7, 6)/5 = (1.4, 1.2)$.
(Side remark: In general the average $\sum_{i=1}^n A_i/n$ is the only critical point because the function $f(X) = \sum_{i=1}^n |x - A_i|^2$ has the gradient $\sum_{i=1}^n 2(X - A_i) = 0$ showing $nX = \sum_i A_i$. This is true in any dimension and any number of mass points.)
The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\ .$$

From this we can read $D = 100$ and $f_{xx} = 10$. The second derivative test shows that the point is a minimum. We have $f(1.4, 1.2) = 14$.

11/1/2017 Second Hourly Practice 8 **Math 21a, Fall 2017**

Name:

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MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
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8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] Every function $f(x,y)$ of two variables has either a global minimum or a global maximum.

Solution:
Take for example $f(x,y) = x + y$. This function has a constant nonzero gradient and so no critical point. It is unbounded above and below.

2) [T] [F] The linearization of the function $f(x,y) = e^{x+3y}$ at $(0,0)$ is $L(x,y) = 1 + x + 3y$.

Solution:
Use the definition of linearization. The gradient of f is $\nabla f = \langle e^{x+3y}, 3e^{x+3y}\rangle$. At $(0,0)$this is $\langle 1, 3\rangle$. We have $f(0,0) = 1$ so that $L(x,y) = 1 + x + 3y$.

3) [T] [F] The function $f(x,y,z) = x^2\cos(z) + x^3y^2z + (y-2)^3y^5$ satisfies the partial differential equation $f_{xyxzxy} = 12$.

Solution:
Use Clairot.

4) [T] [F] If $xe^z = y^2z$, then $\partial z/\partial x = e^z/(y^2 - xe^z)$.

Solution:
This is a direct application of implicit differentiation $z_x = -f_x/f_z$.

5) [T] [F] The function $\cos(x^2)\cos(y^2)$ has a local maximum at $(0,0)$.

Solution:
The value at $(0,0)$ is equal to 1. The functions and so the product take values between -1 and 1.

6) [T] [F] The value of the double integral $\int_0^{\pi/4}\int_0^2 x^3\cos(y)\ dxdy$ is the same as $(\int_0^2 x^3\ dx)(\int_0^{\pi/4}\cos(y)\ dy)$.

Solution:
The function $\cos(y)$ is a constant for the inner integral so that we can pull it out of the inner integral.

7) [T] [F] The gradient of $f(x, y)$ is always tangent to the level curves of f.

Solution:
It is perpendicular

8) [T] [F] If $f(x, y, z) = x - 2y + z$, then the largest possible directional derivative $D_{\vec{u}}f$ at any point in space is $\sqrt{6}$.

Solution:
The gradient has length $\sqrt{6}$. The directional derivative into the direction of the gradient is the length of the gradient.

9) [T] [F] $\int_0^1 \int_0^1 (x^2 + y^2)\ dxdy = \int_0^1 \int_0^1 r^3\ drd\theta$.

Solution:
While the substitution of the function and the r factor have been done correctly, the region changes. The right integral defines a sector, while the left integral is an integral over the unit square.

10) [T] [F] It is possible that the directional derivative $D_{\vec{v}}f$ is positive for all unit vectors $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

11) [T] [F] Using linearization of $f(x, y) = xy$ we can estimate $f(0.999, 1.01) \sim 1 - 0.001 + 0.01 = 1.009$.

Solution:
$L(x, y) = 1 - 1 \cdot 0.001 + 1 \cdot 0.01$.

12) [T] [F] Given a curve $\vec{r}(t)$ on a surface $g(x,y,z) = -1$, then $\frac{d}{dt}g(\vec{r}(t)) < 0$.

Solution:
It is zero.

13) [T] [F] If $f(x,y)$ has a local minimum at $(0,0)$ then it is possible that $f_{xy}(0,0) > 0$.

Solution:
$D = f_{xx}f_{yy} - f_{xy}^2 > 0$ is still possible, if f_{xx} and f_{yy} are large. For example $x^2+y^2+xy/10$ has a local minimum at $(0,0)$ even so $f_{xy} > 0$.

14) [T] [F] The function $f(x,y) = -x^8 - 2x^6 - y^8$ has a local minimum at $(0,0)$.

Solution:
One can not use the second derivative test because the discriminant is zero. But the function is zero at $(0,0)$ and strictly negative everywhere else. Therefore, $(0,0)$ is a global maximum. It is definitly not a minimum.

15) [T] [F] If $\vec{r}(t)$ is a curve in space and f is a function of three variables, then $\frac{d}{dt}f(\vec{r}(t)) = 0$ for $t = 0$ implies that $\vec{r}(0)$ is a critical point of $f(x,y,z)$.

Solution:
We can have $r(t) = (t,0,0)$ and $f(x,y,z) = x^2 + (y-1)^2$.

16) [T] [F] Let a, b, c be the number of saddle points, maxima and minima of a function $f(x,y)$. Then $a \leq b + c$.

Solution:
Already $x^2 - y^2$ is a counter example.

17) [T] [F] If $f(x,y)$ is a nonzero function of two variables and R is a region, then $\int\int_R f(x,y)\,dxdy$ is the volume under the graph of f and therefore a positive value.

Solution:
if f is replaced by $-f$, then the sign of the integral changes too.

18) T [F] We extremize $f(x,y)$ under the constraint $g(x,y)=c$ and obtain a solution (x_0,y_0). If the Lagrange multiplier λ is positive, then the solution is a minimum.

Solution:
There is no relation between the sign of λ and minima and maxima. Change $g=c$ to $-g=-c$ and the sign of λ changes.

19) T [F] The tangent plane to a surface $f(x,y,z)=1$ intersects the surface in exactly one point.

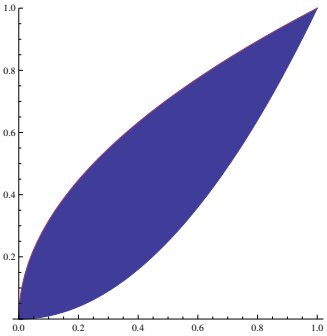
Solution:
Take a one sheeted hyperboloid.

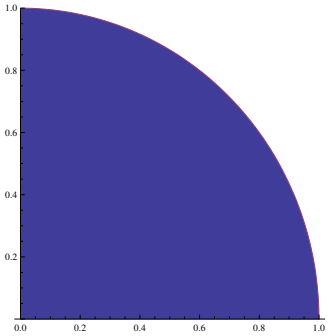
20) T [F] Let $\vec{v}$ be a vector of length 1 in space. Given a function $f(x,y,z)$ of three variables. If (x_0,y_0,z_0) is a critical point of f, then it is a critical point of $g(x,y,z)=D_v f(x,y,z)$.

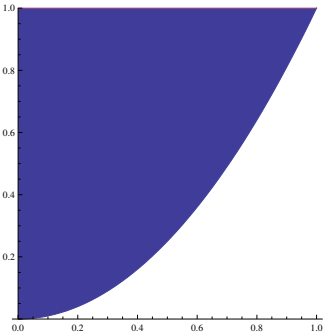
Solution:
Let $\vec{v}=\langle 1,0,0\rangle$. Now $g(x,y,z)=f_x(x,y,z)$ and $\nabla g=\langle f_{xx},f_{xy},f_{xz}\rangle$.

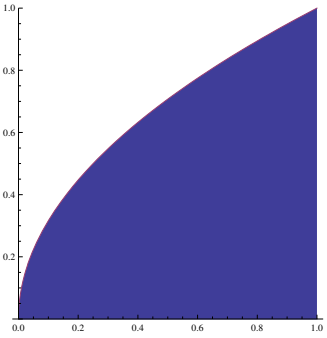
Problem 2) (10 points)

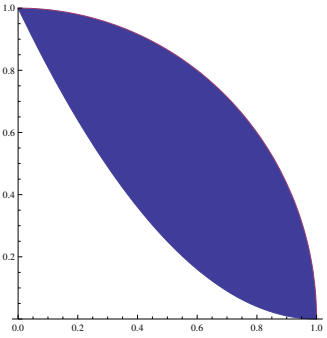
a) (6 points) Match the regions with the corresponding double integrals

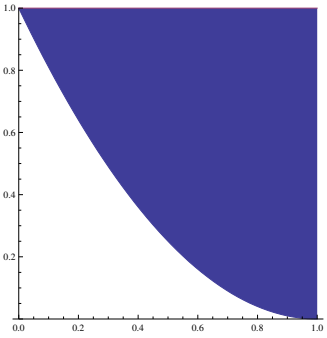
a 

b 

c 

d 

e 

f 

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\, dydx$
	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\, dxdy$
	$\int_0^1 \int_{y^2}^1 f(x,y)\, dxdy$

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dydx$
	$\int_0^1 \int_{(1-x)^2}^1 f(x,y)\, dydx$
	$\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x,y)\, dydx$

b) (4 points) Match the PDE's with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$f_{xx} = -f_{yy}$
	$f_x = f_y$

Enter A,B,C,D here	PDE
	$f_{xx} = f_{yy}$
	$f_x = f_{yy}$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Solution:

a) a b / c f / d e

b) D A / C B

Problem 3) (10 points)

a) (3 points) Find and classify all the critical points of $f(x,y) = xy - x$ on the plane.

b) (2 points) Decide whether an absolute maximum or an absolute minimum of f exists on the plane R^2.

c) (3 points) Use the method of Lagrange multipliers to find the maximum and minimum of f on the boundary $x^2 + 4y^2 = 12$ of the elliptical region $G : x^2 + 4y^2 \leq 12$.

d) (2 points) Find the absolute maximum and absolute minimum of f on the region G given in c).

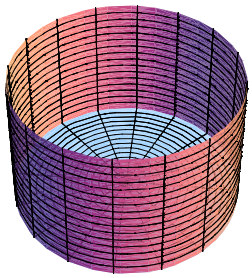
Solution:
a) $\nabla f = \langle y-1, x\rangle = \vec{0}$ for $(x,y) = (0,1)$. Since $f_{xx} = f_{yy} = 0$ and $f_{xy} = 1$, the discriminant is $D = 0^2 - 2^2 < 0$ and $\boxed{(0,1) \text{ is a saddle point}}$.
b) There is $\boxed{\text{no global maximum, nor any global minimum}}$ on the plane. On the x-axes $y = 0$ for example, we have $f(x,0) = -x$ which is unbounded both from above and from below.
c) The Lagrange equations are

$$\begin{aligned} y - 1 &= \lambda\, 2x \\ x &= \lambda\, 8y \\ x^2 + 4y^2 &= 12\,. \end{aligned}$$

$y \neq 0$, because otherwise the second equation would give $x = 0$, contradicting the constraint. Also $x \neq 0$, because otherwise, the first equation would give $y = 1$, again contradicting the constraint. Dividing the first by the second gives $(y-1)/x = (1/4)x/y$ or $4y(y-1) = x^2$. Plugging this into the constraint gives $4y(y-1)+4y^2 = 12$. The solutions of this quadratic equation are $y = 3/2$ or $y = -1$. The extrema are $\boxed{(\pm 2\sqrt{2}, -1) \text{ and } (\pm\sqrt{3}, 1.5)}$. Since $f(2\sqrt{2},-1) = -4\sqrt{2}, f(-2\sqrt{2},-1) = 4\sqrt{2}, f(\sqrt{3},1.5) = \frac{\sqrt{3}}{2}$ and $f(-\sqrt{3},1.5) = -\frac{\sqrt{3}}{2}$, $\boxed{\text{the maximum is } (x,y) = (-2\sqrt{2},-1)}$ and $\boxed{\text{the minimum is } (x,y) = (2\sqrt{2},-1)}$.
d) From parts (a) and (c) we have a list of all candidates for global extrema. The global maximum value of f on G is $f(-2\sqrt{2},-1) = \boxed{4\sqrt{2}}$, the global minimal value on G is $f(2\sqrt{2},-1) = \boxed{-4\sqrt{2}}$.

Problem 4) (10 points)

Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.



Solution:
The Lagrange equations are

$$\begin{aligned} 2xy\pi &= (2x\pi + 2y\pi)\lambda \\ \pi x^2 &= 2\pi x\lambda \\ \pi x^2 + 2\pi xy &= \pi \end{aligned}$$

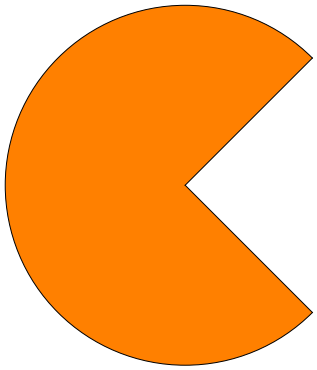
Since $x = 0$ is not possible (it would violate the constraint), we can divide the second equations by x and divide the first by the second equation. This gives $\boxed{x = y = 1/\sqrt{3}}$. The maximum value is $\boxed{\pi\sqrt{3}/9}$.

Problem 5) (10 points)

The Pac-Man region R is bounded by the lines $y = x, y = -x$ and the unit circle. The number

$$a = \frac{\int\int_R x \, dxdy}{\int\int_R 1 \, dxdy}$$

defines the point $C = (a, 0)$ called center of mass of the region. Find it.



Solution:

$$\int_{\pi/4}^{7\pi/4}\int_0^1 r\cos(\theta)\, r dr d\theta = (1/3)\sin(\theta)|_{\pi/4}^{7\pi/4} = -\sqrt{2}/3\ .$$

$$\int_{\pi/4}^{7\pi/4}\int_0^1 r dr d\theta = (1/2)(7\pi/4 - \pi/4) = 6\pi/8 = 3\pi/4\ .$$

The second integral is the area of the Pac-Man, which is 3/4 of the area of the full disc. Dividing the first by the second integral gives the result $\boxed{a = -4\sqrt{2}/(9\pi)}$. The center of mass is $\boxed{(-4\sqrt{2}/(9\pi), 0)}$.

Problem 6) (10 points)

a) (5 points) Find the tangent plane to the surface $\sqrt{xyz} = 60$ at $(x, y, z) = (100, 36, 1)$.

b) (5 points) Estimate $\sqrt{100.1 * 36.1 * 0.999}$ using linear approximation. Here, for clarity reasons, we use $*$ for the usual multiplication for numbers.

Solution:
a) We have

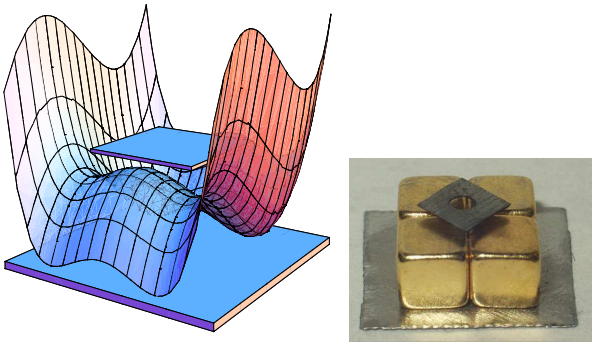
$$\nabla f(x, y, z) = \langle \sqrt{\frac{yz}{x}}, \sqrt{\frac{xz}{y}}, \sqrt{\frac{xy}{z}} \rangle /2$$

$$\nabla f(100, 36, 1) = \langle \frac{6}{10}, \frac{10}{6}, 60 \rangle /2$$

The tangent plane is $\boxed{(3/10)x + (5/6)y + 30z = 90}$. We have obtained the constant on the right by plugging in the point $(x, y, z) = (100, 36, 1)$.
b) Since $f(100, 36, 1) = 60$, we have $L(x, y, z) = 60 + (3/10)(x-100) + (5/6)(y-36) + 30(z-1)$. We have $L(100.1, 36.1, 0.999) = 60 + 0.03 + 0.08333.. - 0.03 = \boxed{60.08333... = 60 + 1/12}$. This is very close to the actual value 60.0832455.... You have in this problem computed the square root of a real number by hand with an accuracy of 4 digits after the comma.

Problem 7) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form $f(x, y) = x^4 + y^3 - 2x^2 - 3y$. Find all critical points of this function and classify them. Is there a global minimum?



Right picture credit: Wikipedia.

Solution:
To find the critical points, we have to solve the system of equations $f_x = 4x^3 - 4x = 0, f_y = 3y^2 - 3 = 0$. The first equation gives $x = 0$ or $x = \pm 1$. The second equation $f_y = 3y^2 - 3 = 0$ gives $y = \pm 1$. There are $3 \cdot 2 = 6$ critical points. We compute the discriminant $D = 6y(12x^2 - 4)$ and $f_{xx} = 12x^2 - 4$ at each of the 6 points and use the second derivative test to determine the nature of the critical point.

point	D	f_xx	nature	value
(-1, -1)	-48	8	saddle	1
(-1, 1)	48	8	min	-3
(0, -1)	24	-4	max	2
(0, 1)	-24	-4	saddle	-2
(1, -1)	-48	8	saddle	1
(1, 1)	48	8	min	-3

There is no global minimum, nor any global maximum since for $x = 0$, the function is $f(0, y) = y^3 - 3y$ which is unbounded from above and from below (it goes to $\pm\infty$ for $y \to \pm\infty$).

Problem 8) (10 points)

Let $f(x, y) = xy$.

a) (2 points) Find the direction of maximal increase at the point $(1, 1)$.

b) (3 points) Find the directional derivative at $(1, 1)$ in the direction $\langle 3/5, 4/5 \rangle$.

c) (2 points) The curve $\vec{r}(t) = \langle \sqrt{2}\sin(t), \sqrt{2}\cos(t) \rangle$ passes through the point $(1, 1)$ at some time t_0. Find $\frac{d}{dt} f(\vec{r}(t))$ at time t_0 directly.

d) (3 points) Find $\frac{d}{dt} f(\vec{r}(t))$ at time t_0 using the multivariable chain rule.

Solution:
a) $\nabla f(x,y) = \langle y, x\rangle$, $\nabla f(1,1) = \langle 1,1\rangle$. The direction of maximal increase is $\boxed{\langle 1,1\rangle/\sqrt{2}}$.
b) $D_v f(1,1) = \langle 1,1\rangle \cdot \langle 3/5, 4/5\rangle = \boxed{7/5}$.
c) It is at the time $t_0 = \pi/4$, where the curve passes through thte point $(1,1)$. We have $f(\vec{r}(t)) = 2\cos(t)\sin(t) = \sin(2t)$ and $d/dt f(\vec{r}(t)) = 2\cos(2t)$ which is $\boxed{0}$ at time $t = \pi/4$.
d) By the multi variable chain rule, $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(1,1) \cdot \langle -\sin(\pi/4), \cos(\pi/4)\rangle = \boxed{0}$.

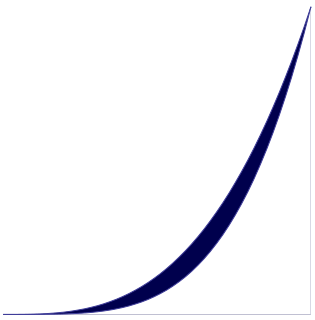
Problem 9) (10 points)

Integrate the function

$$f(x,y) = \frac{y^5 - 1}{y^{1/3} - y^{1/4}}$$

on the finite region bounded by the curves $y = x^3$ and $y = x^4$.

Solution:
Make a picture! The two graphs intersect at 0 and 1 forming a grass shaped region.



The type I integral

$$\int_0^1 \int_{x^4}^{x^3} \frac{y^5 - 1}{y^{1/3} - y^{1/4}}\, dydx$$

can not be evaluated (at least not without going through difficult substitution/partial fraction procedures which can fill pages).
We decide therefore, to change the order or integration and write a type II integral:

$$\int_0^1 \int_{y^{1/3}}^{y^{1/4}} \frac{y^5 - 1}{y^{1/3} - y^{1/4}}\, dxdy$$

Now the inner integral can be solved and give $(1 - y^5)$. We end up with $\int_0^1 (1 - y^5)\, dy = \boxed{5/6}$.

Problem 10) (10 points)

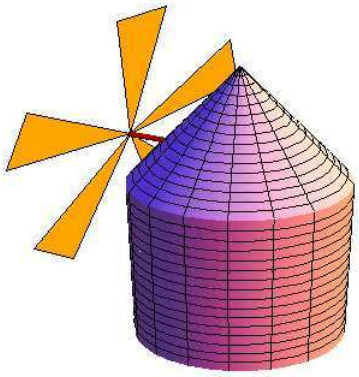
The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius r, the height of the side wall is h and the height of the roof is h, then the volume is

$$V(h,r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and assume the cost of the building is

$$A(h,r) = \pi r^2 + 2\pi rh + \pi 2r^2 = \pi(3r^2 + 2rh)$$

which is the area of the ground plus the area of the wall plus $2\pi rh$, the cost for the roof. For fixed volume $V(h,r) = 4\pi/3$, minimize the cost $A(h,r)$ using the Lagrange multiplier method.



Solution:
After dividing out some constants and taking $g = hr^2 = 1$, the Lagrange equations become

$$\begin{aligned} 6r + 2h &= \lambda 2hr \\ 2r &= \lambda r^2 \\ r^2 h &= 1 \end{aligned}$$

The second equation can be divided by r since $r = 0$ is incompatible with the third equation. The first can be divided by 2. We get

$$\begin{aligned} 3 * r + h &= \lambda hr \\ 2 &= \lambda r \\ r^2 h &= 1 \end{aligned}$$

You can plug in λr from the second equation into the first to get

$$\begin{aligned} 3r + h &= 2h \\ r^2 h &= 1 \end{aligned}$$

The first equation shows $h = 3r$ and plugging this into the third equation gives $\boxed{r = 1/3^{1/3}}$ and $\boxed{h = 3r = 3^{2/3}}$.

11/1/2017 Second Hourly Practice 9 **Math 21a, Fall 2017**

Name:

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- Start by printing your name in the above box and **check your section** in the box to the left.
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- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
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1		20
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4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] The directional derivative $D_{\vec{v}}f$ is a vector perpendicular to $\vec{v}$.

Solution:
The directional derivative is a scalar, not a vector.

2) [T] [F] Using linearization of $f(x,y) = xy$ we can estimate $f(0.9, 1.2) \sim 1 - 0.1 + 0.2 = 1.1$.

Solution:
$L(x,y) = 1 - 1 \cdot 0.1 + 1 \cdot 0.2$.

3) [T] [F] Given a curve $\vec{r}(t)$ on a surface $g(x,y,z) = 1$, then $\frac{d}{dt}g(\vec{r}(t)) = 0$.

Solution:
This fact is used in the proof that level surfaces are perpendicular to gradients.

4) [T] [F] Given a function $f(x,y)$ such that $\nabla f(0,0) = \langle 2, -1\rangle$. Then $D_{\langle 0,-1\rangle}f(0,0) = 0$.

Solution:
The directional derivative in the direction $\langle 0, -1\rangle$ is equal to 1 which is nonzero.

5) [T] [F] $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ is a surface of revolution.

Solution:
The parametrization given is a helicoid and not rotationally symmetric. It ressembles the parametrization $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$ of the cone, which is rotationally symmetric.

6) [T] [F] If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$.

7) [T] [F] If $f(x,y)$ has a local maximum at $(0,0)$ then it is possible that $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) < 0$.

Solution:
The conditions imply that $D = f_{xx}f_{yy} - f_{xy}^2 < 0$.

8) [T] [F] The integral $\int_0^x \int_0^y 1 \, dxdy$ computes the area of a region in the plane.

Solution:
This is not a valid double integral. The outer integral should not contain variables.

9) [T] [F] The function $f(x,y) = x^2 + y^4$ has a local minimum at $(0,0)$.

Solution:
One can not use the second derivative test but the function is zero at $(0,0)$ but positive everywhere else.

10) [T] [F] The integral $\int_0^1 \int_0^1 x^2 + y^2 \, dxdy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.

Solution:
In general $\int\int_R f(x,y) \, dydx$ is the volume under the graph of f

11) [T] [F] There exists a region in the plane, which is neither a type I integral, nor a type II integral.

Solution:
Take for example an S shaped region.

12) [T] [F] Fubini's theorem assures that $\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_0^y f(x,y)\, dxdy$.

Solution:
Fubini only applies to rectangular regions.

13) [T] [F] The function $f(x,y) = \sin(x)\cos(y)$ satisfies the partial differential equation $f_{xx} + f_{yy} = 0$.

Solution:
Just differentiate. It is a solution to the wave wequation, not the Laplace equation.

14) [T] [F] Let $L(x,y)$ be the linearization of $f(x,y) = \sin(x(y+1))$ at $(0,0)$. Then, the level curves of $L(x,y)$ consist of lines.

Solution:
The function $L(x,y)$ is a linear function of the form $ax + by + c$ it has lines.

15) [T] [F] For any smooth function $f(x,y)$, the inequality $|\nabla f| \geq |f_x + f_y|$ is true.

Solution:
If $\nabla f = \langle a, b\rangle$, we square the claim, we get $a^2 + b^2 \geq (a+b)^2$. This is wrong for $(a,b) = (1,1)$.

16) [T] [F] Any differentiable function $f(x,y)$ which satisfies the partial differential equation $||\nabla f||^2 = 0$ is constant.

Solution:
The condition $||\nabla f||^2 = 0$ implies that that the gradient is zero and so that all directional derivatives are zero.

17) [T] [F] If $x + \sin(xy) = 1$, $dy/dx = \frac{-(1+y\cos(yx))}{(x\cos(xy))}$.

Solution:
This is implicit differentiation.

18) [T] [F] The directional derivative $D_v f(1,1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1,1)$.

Solution:
The level curve is perpendicular to the gradient.

19) [T] [F] If (a,b) is a maximum of $f(x,y)$ under the constraint $g(x,y)=0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a,b).

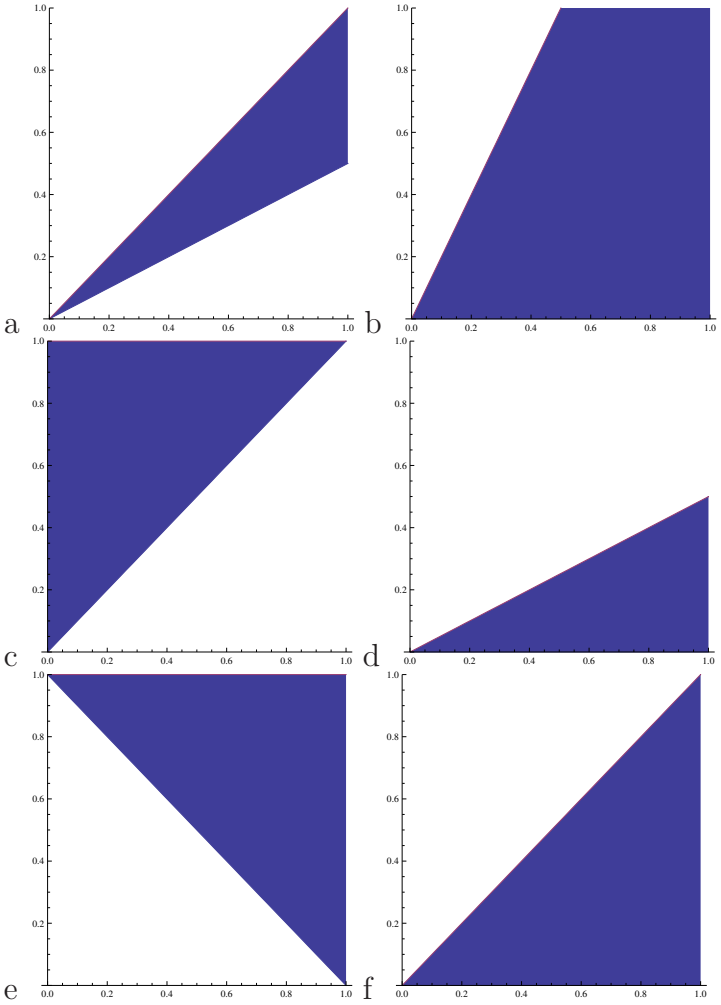
Solution:
False, by changing g to $-g$, we can change the Lagrange multiplier, but the discriminant stays the same.

20) [T] [F] If $D_{\langle 1/\sqrt{2},1/\sqrt{2}\rangle} f(1,2) = 0$ and $D_{\langle -1/\sqrt{2},1/\sqrt{2}\rangle} f(1,2) = 0$, then $(1,2)$ is a critical point.

Solution:
Indeed, if $\nabla f = \langle a,b\rangle$, then $\langle a,b\rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = 0$ and $\langle a,b\rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle = 0$ which imples $a = b = 0$.

Problem 2) (10 points)

Match the regions with the corresponding double integrals



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x/2}^{x} f(x,y)\, dydx$
	$\int_0^1 \int_0^{y} f(x,y)\, dxdy$
	$\int_0^1 \int_0^{x/2} f(x,y)\, dydx$
	$\int_0^1 \int_{y/2}^{1} f(x,y)\, dxdy$
	$\int_0^1 \int_0^{x} f(x,y)\, dydx$
	$\int_0^1 \int_{1-x}^{1} f(x,y)\, dydx$

Solution:
a,c,d,b,f,e.

Problem 3) (10 points)

Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.
a) (5 points) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.
b) (5 points) The surface in a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.

Solution:
a) $\nabla g(x, y, z) = \langle 2x, 4y, -1 \rangle$.
$\nabla g(2, 0, 1) = \langle 4, 0, -1 \rangle$ The equation of the tangent plane is

$$4x + 0y - z = 7 \ .$$

where the value of 7 has been obtained by plugging in the point $(2, 0, 1)$. The final answer is $\boxed{4x - z = 7}$.
b) $z = x^2 + 2y^2 - 3 = f(x, y)$.

$$\nabla f(x, y) = \langle 2x, 4y \rangle \ .$$

and

$$\nabla f(2, 0) = \langle 4, 0 \rangle \ .$$

The equation of the tangent line is

$$4x = 8$$

or $\boxed{x = 2}$.

Problem 4) (10 points)

a) (5 points) Use the technique of linear approximation to estimate $f(\pi/2 + 0.1, 2.9)$ for

$$f(x, y) = (10 \sin(x) - 5y^2 + 8)^{1/3} \ .$$

b) (5 points) Find the unit vector at $(\pi/2, 3)$, in the direction where the function increases fastest.

Solution:
a) $f(\pi/2, 3) = -3$ and

$$\nabla f(x, y) = \frac{1}{3}(10\sin(x) - 5y^2 + 8)^{-2/3}\langle 10\cos(x), -10y\rangle \ .$$

so that

$$\nabla f(\pi/2, 3) = \frac{1}{27}\langle 0, -30\rangle = \langle 0, -10/9\rangle \ .$$

The estimation is $-3 + 0.1 \cdot 0 - 0.1 \cdot (-10/9) = -3 + 1/9 = -26/9$.
b) It is $(0, -1)$.

Problem 5) (10 points)

The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t\rangle$.

a) (2 points) State the chain rule which applies in this situation.
b) (4 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.
c) (4 points) At which time t does the observer go in the direction, in which the pressure decreases most?

Solution:
a) The multivariable chain rule is

$$\frac{d}{dt}p(\vec{r}(t)) = \nabla p(\vec{r}(t)) \cdot \vec{r}\,'(t) \ .$$

b) $\nabla p(x, y, z) = \langle 2x, 2y, -3z^2\rangle$, $\vec{r}\,'(t) = \langle 1, 1, -1/t^2\rangle$. We have $\vec{r}(2) = \langle 2, 2, 1/2\rangle$ and $\vec{r}\,'(2) = \langle 1, 1, -1/4\rangle$. By the chain rule in a), we have

$$\nabla p(2, 2, 1/2) \cdot \vec{r}\,'(2) = \langle 4, 4, -3/4\rangle \cdot \langle 1, 1, -1/4\rangle = 8 + 3/16 \ .$$

c) The direction in which the pressure decreases most at the observers position $\vec{r}(t)$ is $-\nabla p(\vec{r}(t)) = \langle -2t, -2t, 3/t^2\rangle$. The question is, when this vector is parallel to the velocity vector $\langle 1, 1, -1/t^2\rangle$. If we set

$$\langle -2t, -2t, 3/t^2\rangle = c\langle 1, 1, -1/t^2\rangle \ ,$$

we get by comparing the first coordinate $c = -2t$. The third component equation reads $3/t^2 = 2t/t^2$ and gives $3 = 2t$ leading to $\boxed{t = 3/2}$.

Problem 6) (10 points)

The coffee chain **Astrbucks** [1] has branches at $(0,0), (0,3)$ and $(3,3)$ (JFK street, Church street, and Broadway) near Harvard square. A caffeine addicted [politically correct: loving] mathematician wants to rent an apartment at a location, where the sum of the squared distances $f(x,y)$ to all those shops is a local minimum. The function is

$$f(x,y) = (x-0)^2+(y-0)^2+(x-0)^2+(y-3)^2+(x-3)^2+(y-3)^2 = 27-6x+3x^2-12y+3y^2 \ .$$

a) (5 points) Where does the mathematician have to live to locally minimize $f(x,y)$?
b) (3 points) For every local minimum answer: Is this local minimum a **global** minimum?
c) (2 points) Is there a global maximum to this problem? If yes, give it. If no, why not?

Solution:
a) $\nabla f(x,y) = \langle 6x-6, 6y-12\rangle$ is the zero vector for $(x,y) = (1,2)$. By the second derivative test, this is a local minimum.
b) Yes, this is a global minimum. The function can be written with a completion of squares as

$$27-3+(3-6x+3x^2)-12+(12-12y+3y^2) = 12+3(1-x)^2+3(2-y)^2$$

which has as a graph an elliptic paraboloid with global minimum at $(1,2)$. Many different attempts have been used here to justify the fact that we have a global minimum. Note it is not true that if a function $f(x,y)$ has one local minimum and no other critical point, then this local minimum has to be a global minimum. An example (provided by Chen-Yu Chi) is $f(x,y) = x^3 + e^{(3y)} - 3xe^y$. It has a local minimum at $(1,0)$ but not other critical point and no global minimum nor maximum.
c) It is possible to argue that the function increases monotonically if x^2+y^2 is large enough and goes to ∞. This prevents the existence of a global maximum.





Problem 7) (10 points)

[1] This problem was sponsored by *Astrbucks*©.

Find all the critical points of $f(x,y) = 3xy + x^2y + xy^2$ and classify them as saddle points, local maxima or local minima.

Solution:
The gradient is

$$\nabla f(x,y) = \langle 3y + 2xy + y^2, 3x + x^2 + 2xy \rangle \ .$$

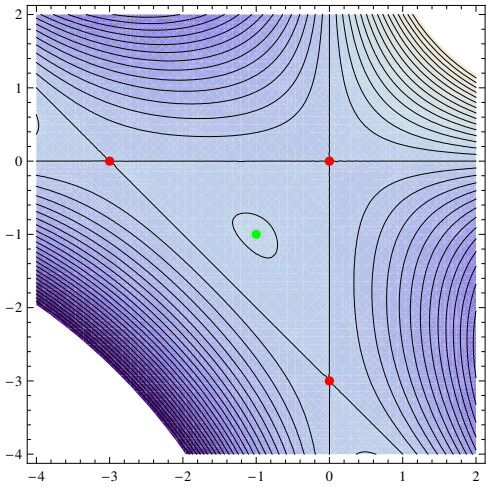
Factor out y in the first component and x in the second component.

$$\nabla f(x,y) = \langle y(3 + 2x + y), x(3 + x + 2y) \rangle = \langle 0, 0 \rangle \ .$$

If $y = 0$, then the first equation holds and the second equation needs either $x = 0$ or $x = -3$. If $x = 0$ then the second equation holds and the first equation needs either $y = 0$ or $y = -3$. If both x and y are not zero, then $3 + 2x + y = 0, 3 + x + 2y = 0$ which has the solution $x = y = -1$. We have therefore 4 solutions. We evaluate the discriminant $D = 4xy - (3 + 2x + 2y)^2$ and the second derivative $f_{xx} = 2y$ at each point:

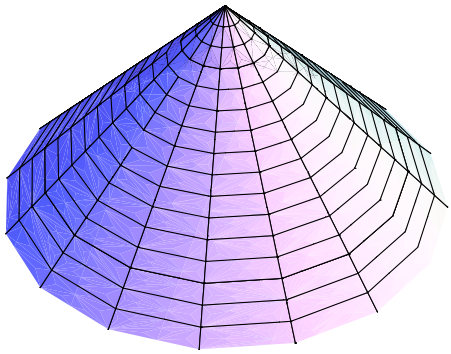
point	D	f_{xx}	nature of the critical point
(-3,0)	-9	0	saddle point
(-1,-1)	3	-2	local max
(0,-3)	-9	-6	saddle point
(0,0)	-9	0	saddle point

The picture shows some level curves of the function $f(x,y)$:



Problem 8) (10 points)

A solid cone of height h and with base radius r has the volume $f(h,r) = h\pi r^2/3$ and the surface area $g(h,r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h,r) = \pi$ use the Lagrange method to find the cone with maximal volume.



Solution:
The Lagrange equations are after dividing out π factors in all 3 equations

$$\begin{aligned} f_r = 2rh/3 &= \lambda(2r + \sqrt{r^2+h^2} + r^2/\sqrt{r^2+h^2} = g_r \\ f_h = r^2/3 &= \lambda rh/\sqrt{r^2+h^2} = g_h \\ r\sqrt{r^2+h^2} + r^2 &= 1 \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide by r in the second and third equation equation:

$$\begin{aligned} 2rh/3 &= \lambda(2r + \sqrt{r^2+h^2} + r^2/\sqrt{r^2+h^2} \\ r/3 &= \lambda h/\sqrt{r^2+h^2} \\ \sqrt{r^2+h^2} &= \frac{1-r^2}{r} \end{aligned}$$

Now plug the third equation into the first two to have a simpler system and also square the third equation

$$\begin{aligned} 2rh/3 &= \lambda(2r + (1-r^2)/r + r^3/(1-r^2)) \\ r/3 &= \lambda hr/(1-r^2) \\ r^2+h^2 &= \frac{(1-r^2)^2}{r^2} \end{aligned}$$

We get rid of λ by dividing the first by the second equation

$$\begin{aligned} 2h &= 1/(hr^2) \\ r^2+h^2 &= \frac{(1-r^2)^2}{r^2} \end{aligned}$$

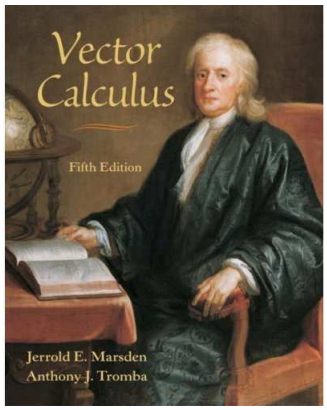
The first equation gives $h^2 = 1/(2r^2)$. Plugging this into the third gives $r = 1/2$. The solution is $\boxed{r = 1/2, h = \sqrt{2}}$.

Problem 9) (10 points)

Marsden and Tromba pose in their textbook the following riddle: Suppose $w = f(x, y)$ and $y = x^2$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + 2x\frac{\partial w}{\partial y}$$

so that $0 = 2x\frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$.



a) Find an explicit example of a function $f(x, y)$, where you see the argument is false.

b) What is flawed in the above application of the chain rule?

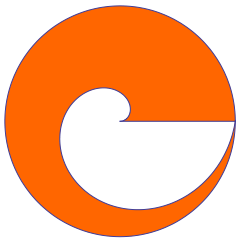
Solution:
a) Take $w = f(x, y) = x^2 + y^2$ for example. Then $w_x = 2x, w_y = 2y$. b) On the left hand side of the chain rule, we have not a partial derivative any more. Lets write this more clearly. We have a function of two variables x, y and both variables x, y are functions of a third variable t. In our case $x(t) = t, y(t) = t^2$. The chain rule gives the derivative $\frac{d}{dt}f(x(t), y(t))$ as $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = f_x(t, t^2) + f_y(t, t^2)2t$. The false argument had written the left hand side of the chain rule as f_x again. Because the variable t was the variable x itself, this confusion was possible.

Problem 10) (10 points)

Evaluate the double integral

$$\int\int_R \sqrt{x^2 + y^2}\, dxdy$$

where R is the region bounded by the positive x-axes, the spiral curve $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle, 0 \le t \le 2\pi$ and the circle with radius 2π.



Solution:
Use polar coordinates, where $\sqrt{x^2 + y^2} = r$ and where $dxdy$ is replaced by $rdrd\theta$.

$$\int_0^{2\pi}\int_\theta^{2\pi} r^2\, drd\theta = \int_0^{2\pi} (8\pi^3 - \theta^3)/3\, d\theta\ .$$

This integral is $\boxed{4\pi^4}$.

11/1/2017 Second Hourly Practice 10 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | | F | $f(x$

Solution:
The function g has always $(0,0)$ as a critical point, even if f has not.

2) | T | | F | If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

Solution:
At a critical point the gradient is $(a,b) = (0,0)$, which implies $f = 0$.

3) | T | | F | Given 2 arbitrary points in the plane, there is a function $f(x,y)$ which has these points as critical points and no other critical points.

Solution:
Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates (-1,0),(1,0). The function $f(x,y) = -y^2(x^3 - 1)$ has the two points as critical points. One is a local max, the other is a saddle point.

4) | T | | F | If (x_0, y_0) is the maximum of $f(x,y)$ on the disc $x^2+y^2 \leq 1$ then $x_0^2+y_0^2 < 1$.

Solution:
The maximum could be on the boundary.

5) | T | | F | There are no functions $f(x,y)$ for which every point on the unit circle is a critical point.

Solution:
There are many rotationally symmetric functions with this property.

6) | T | | F | An absolute maximum (x_0, y_0) of $f(x,y)$ is also an absolute maximum of $f(x,y)$ constrained to a curve $g(x,y) = c$ that goes through the point (x_0, y_0).

Solution:
The Lagrange multiplier vanishes in this case.

7) [T] [F] If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution:
Look at a camel type surface. It has a saddle between the local maxima.

8) [T] [F] There exists a function $f(x, y)$ of two variables which has no critical points at all.

Solution:
True. Every non-constant linear function for example.

9) [T] [F] If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:
False, f could be constant.

10) [T] [F] $(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

Solution:
$(0, 0)$ is a saddle point.

11) [T] [F] If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

Solution:
Adding $x^4 + y^3$ does not change the first and second derivatives.

12) [T] [F] Every critical point (x, y) of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

Solution:
The second derivative test give for negative D that we have a saddle point.

13) [T] [F] If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

Solution:
If $f_{xx}(0,0) < 0$ then on the x-axis the function $g(x) = f(x,0)$ has a local maximum. This means that there are points close to $(0,0)$ where the value of f is smaller.

14) [T] [F] In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

15) [T] [F] The function $f(x,y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0,0)$.

Solution:
The function is both smaller and bigger than $f(0,0)$ for points near $(0,0)$.

16) [T] [F] It is possible to find a function of two variables which has no maximum and no minimum.

Solution:
There are many linear functions like that.

17) [T] [F] The value of the function $f(x,y) = \sqrt{1+3x+5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use formula for $L(x,y)$.

18) [T] [F] The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:
By Clairots theorem, we can have all three x derivatives at the beginning.

19) [T] [F] If $\vec{r}(t)$ is a curve with unit speed in the plane with $\vec{r}(0) = (0,0)$ and $D_{\vec{r}\,'(0)} f(0,0) = 0$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at the time $t = 0$.

Solution:
By the chain rule.

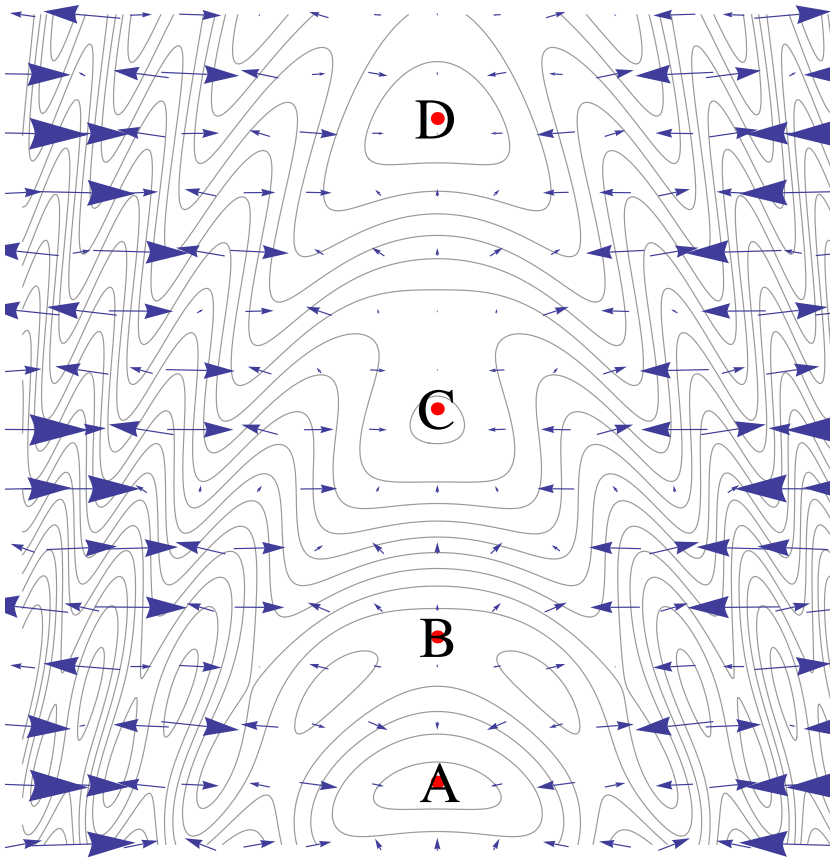
20) [T] [F] If a function $f(x,y)$ satisfies the partial differential equation $f_x^2 - f_y^2 = 0$, then f is the constant function.

Solution:
We can have $f = x + y$ for example.

Problem 2) (10 points)

a) (5 points) The picture below shows the contour map of a function $f(x, y)$ which has many critical points. Four of them are outlined for you on the y axes and are labeled A, B, C, D and ordered in increasing y value. The picture shows also the gradient vectors. Determine from each of the 4 points whether it is a local maximum, a local minimum or a saddle point. No justification is necessary in this problem.

Point	Max	Min	Saddle
D			
C			
B			
A			



b) (5 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x, y)\, dxdy$
	$\int_0^1 \int_y^1 f(x, y)\, dxdy$
	$\int_0^1 \int_0^{1-y} f(x, y)\, dxdy$
	$\int_0^1 \int_0^y f(x, y)\, dxdy$

I) $\int_0^1 \int_0^x f(x, y)\, dydx$
II) $\int_0^1 \int_0^{1-x} f(x, y)\, dydx$

III) $\int_0^1 \int_x^1 f(x,y)\, dydx$
IV) $\int_0^1 \int_0^{x-1} f(x,y)\, dydx$
V) $\int_0^1 \int_{1-x}^1 f(x,y)\, dydx$

Solution:
a) D is a local maximum because the gradient arrows point towards it and gradient arrows point into the direction of maximal increase. Similarly, C and A are local maxima. The point B is a saddle point.
b) Make a picture of the region. Each of the regions is a triangle in the unit square. The first is the upper right half, the second the lower right, the third the lower left, the forth the upper left half. The solution is $\boxed{\text{V,I,II,III}}$.

Problem 3) (10 points)

When Ramanujan, the amazing India born mathematician was sick in the hospital in england and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered: "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.



a) (5 points) Find the linearization $L(x,y,z)$ of the function $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$.

b) (5 points) Use the technique of linear approximation to estimate $9.001^3 + 10.02^3 - 12.001^3$. Since we are not all Ramanujans, you can leave the end result as a product and sum of numbers. For example, $234 \cdot 0.001 - 100 \cdot 0.002$ would be an acceptable end result.

Solution:
We make a linear approximation of $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$. We have $\nabla f(x,y,z) = (3x^2, 3y^2, -3z^2)$ which is $(243, 300, -432)$. a) $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$.
b) $L(9.001, 10.02, 12.001) = 1 + 243 \cdot 0.001 + 300 \cdot 0.02 - 432 \cdot 0.001$.

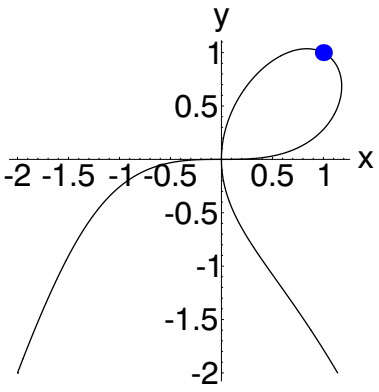
Solution:
$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.

Problem 4) (10 points)

Consider the equation

$$f(x,y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x = 1, y = 1$, the function can be written as a graph $y = y(x)$. Find the slope of that graph at the point $(1,1)$.



Solution:
Use the formula for implicit differentiation which is derived from the chain rule $f_x(x,y(x))\cdot 1 + f_y(x,y(x)) \cdot y'(x) = 0$. The slope is $y'(x) = -f_x(x,y)/f_y(x,y)_{(x,y)} = (1,1) = -1/2$.
An other possibility to solve this problem is to find the equation of the tangent line which is $f_x(1,1)(x-1) + f_y(1,1)(y-1) = 0$ and find the slope m by writing this equation as $y = mx + b$. It gives of course the same result.

Problem 5) (10 points)

a) Find a point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ which is locally closest to the origin.
b) Is this a global minimum? Hint: look at points $(x,y,z) = (1,-1/n,8/n)$ where n is an integer.

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \frac{-1}{x^2} \\ 2y &= \lambda \frac{-1}{y^2} \\ 2z &= \lambda \frac{-8}{z^2} \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
b) consider the points $(x,y,z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points lye on the surface $g(x,y,z) = 1$. Their distance to the origin however decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 6) (10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum among them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2 - 3, 3y^2 - 12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

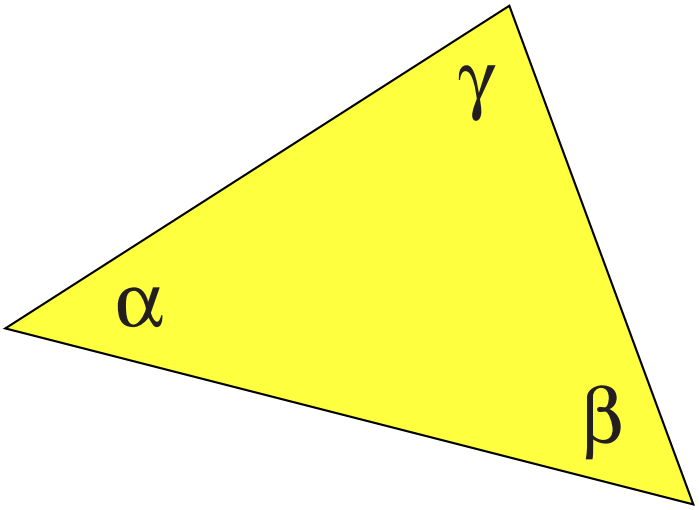
Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

Problem 7) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log(\sin(\alpha)\sin(\beta)\sin(\gamma))$$

is maximal?



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.

Problem 8) (10 points)

Let $g(x, y)$ be the distance from a point (x, y) to the curve $x^2 + 2y^2 + y^4/10 = 1$. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve.

Hint: no computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f?

Remark: This problem only needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Solution:
The level curves of g are curves, for which the distance to the curve is constant. Lets look at the level curves of f, if f is the solution to the PDE. The PDE $|\nabla f|^2 = 1$ tells us that the gradient of f is a unit vector everywhere. This means that the directional derivative in the gradient direction is 1 everywhere on a level curve. This implies that the level curves of f are equidistributed too. The level curves of f and g are the same. Because f and g are both zero at the curve, the two functions are the same.

Solution:
More explanation: if you move perpendicular to a level curve, then your function changes according to the length of the gradient. If the gradient is large, then the function changes a lot, if the gradient is small, then the function changes little. The given partial differential equation tells that that the gradient of f has length 1. This means that if go along the steepest decent directions all the time, then you descend by the exactly same amount with which you go away from the level curve. So, the level curve of the height function $f(x, y)$ is equal to the level curve of the distance function $g(x, y)$.
While the solution to the problem is difficult to explain, it essentially just asks you to see what are the consequences of having a gradient of length 1 everywhere. If you have a mountain of height f(x,y) for which the gradient is 1 everywhere, it has the property that if you decent always perpendicular to the level curve (the fastest decent), then the height decreases in the same way as the distance you go away (the slope is 1 in that direction, the directional derivative is 1 in the gradient direction).
If you are still confused, look at the one dimensional problem: the function $f(x)$ giving the distance to an interval $[a, b]$ satisfies the differential equation $|f'(x)|^2 = 1$. The reasoning is almost the same and if you can answer this, you will understand the 2 dimensional problem better too.

Problem 9) (10 points)

a) (6 points) Find all critical points of $f(x, y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Solution:
a) Lets find the critical points and classify them. Setting the gradient to 0 gives

$$\begin{aligned} f_x &= 3e^y - 3x^2 = 0 \\ f_y &= 3xe^y - 3e^{3y} = 0 \ . \end{aligned}$$

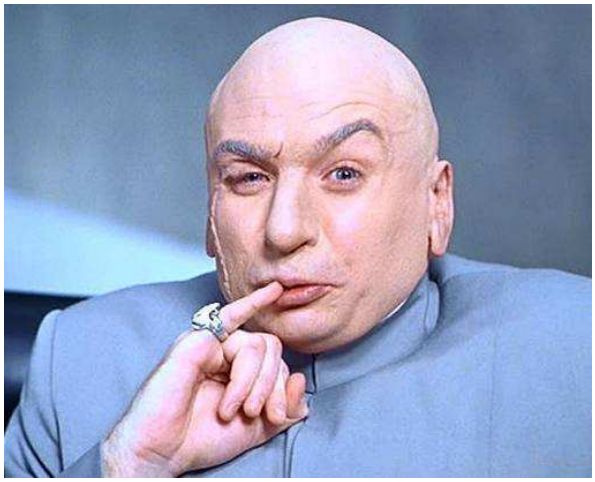
The first equation gives $x = \pm e^{y/2}$. Plugging it into the second gives $y = 0, x = 1$ Applying the second derivative test $f_{xx} = -6$ and $D > 0$ shows that $(1, 0)$ is a local maximum.
b) If we look at the function f restricted to the x axes, we have $g(x) = f(x, 0) = 3x - 1 - x^3$. This goes to $+\infty$ for $x \to -\infty$ and goes to $-\infty$ for $x \to \infty$. We have no global maximum nor a global minimum for f.
Remark: this is a remarkable example. In single variable calculus, sometimes the statement is proven, that if one has a local maximum and no global maximum nor minimum for a function $f(x)$, then there also must exist at least one local minimum. The example here shows that this is not the case for functions for several variables.

Problem 10) (10 points)

a) (5 points) Integrate $f(x, y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \leq 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \ .$$



Solution:
a) Use polar coordinates:

$$\int_0^1 \int_0^{2\pi} (r^2\cos^2(\theta) - r^2\sin^2(\theta))r \; d\theta dr = \int_0^1 r^3 \; dr(\int_0^{2\pi} \cos(2\theta) \; d\theta) = (1/4)\cdot 0 = 0 \; .$$

The final answer is $\boxed{\text{zero}}$.
b) Write it in more convenient coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dydx \; .$$

This is a quarter disc in the x, y plane. Now use polar coordinates. The integral is evil because now, the θ, r have a different meaning. The integral in polar coordinates is

$$\int_0^1 \int_0^{\pi/2} r^2\sin^2(\theta)r \; d\theta dr$$

which is $\int_0^{\pi/2} \sin^2(\theta) \; d\theta \int_0^1 r^3 \; dr = (\pi/4)(1/4) = \pi/16$. (to compute the first integral, use the double angle formula $(1-\cos(2\theta))/2 = \sin^2(\theta)$.)

11/1/2017 Second Hourly Practice 11 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F] The

Solution:
The directional derivative is a scalar

2) [T] [F] At a critical point of a function f, the gradient vector has length 1.

Solution:
The gradient vector is the zero vector there.

3) [T] [F] At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

Solution:
The tangent plane is horizontal there.

4) [T] [F] For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}}f(x, y)$ is nonzero.

Solution:
Take a vector vector is perpendicular to the gradient, the directional derivative $D_{\vec{u}}f = \nabla f \cdot \vec{u}$ is zero.

5) [T] [F] If $f_{xx}(0, 0) = 0, D = f_{xx}f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0, 0) = \langle 0, 0\rangle$, then $(0, 0)$ is a saddle point.

Solution:
Because $f_{xx} = 0$, we have $D = f_{xx}f_{yy} - f_{xy}^2 = -f_{xy}^2$ which can not be positive. Because $D \neq 0$, we must have $D < 0$. By the second derivative test, the critical point is a saddle point.

6) [T] [F] A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

Solution:
The maximum can be either in the interior or at the boundary.

7) [T] [F] The function $f(x,y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0,0)$.

Solution:
It is a saddle point.

8) [T] [F] If (x,y) is a maximum of $f(x,y)$ under the constraint $g(x,y) = 5$ then it is also a maximum of $f(x,y) + g(x,y)$ under the constraint $g(x,y) = 5$.

Solution:
Indeed, on the constraint curve, the function $f + g$ is just $f + 5$, which has the same maxima and minima as f on that curve.

9) [T] [F] The functions $f(x,y)$ and $g(x,y) = (f(x,y))^6$ always have the same critical points.

Solution:
The gradient of g is $6f^5(x,y)\nabla f$. So, the second function has critical points, where f vanishes.

10) [T] [F] For $f(x,y,z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1,1,1)$ is perpendicular to the surface $f(x,y,z) = 4$ at the point $(1,1,1)$.

Solution:
This is a basic property of gradients.

11) [T] [F] $f(x,y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

Solution:
The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

12) [T] [F] If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

Solution:
The discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ can be positive. An example is $f(x, y) = 100x^2 + 100y^2 - xy$.

13) [T] [F] If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c, h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1, 1, 1)$ is a critical point of f.

Solution:
Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0, 0, 0)$.

14) [T] [F] Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

Solution:
The Lagrange equations tell that $\nabla f(x, y) = (0, 0)$.

15) [T] [F] At a saddle point, all directional derivatives are zero.

Solution:
Because $\nabla f(x, y) = (0, 0)$ at a saddle point, all directional derivatives $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ are zero.

16) [T] [F] The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

Solution:
This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

17) [T] [F] At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

Solution:
Indeed, at a local maximum, $f_{yy} \leq 0$.

18) [T] [F] It is possible that $f(x,y)$ attains a maximum under the constraint $g(x,y) = 0$ at a point, where $\nabla f \neq \lambda \nabla g$.

Solution:
If $\nabla g = 0$.

19) [T] [F] Any Lagrange problem which asks for an extremum of $f(x,y)$ under a constraint $g(x,y) = 0$ has either a maximum or a minimum.

Solution:
Take $f(x,y) = x^3 + y^3$ and $g(x,y) = x - y$.

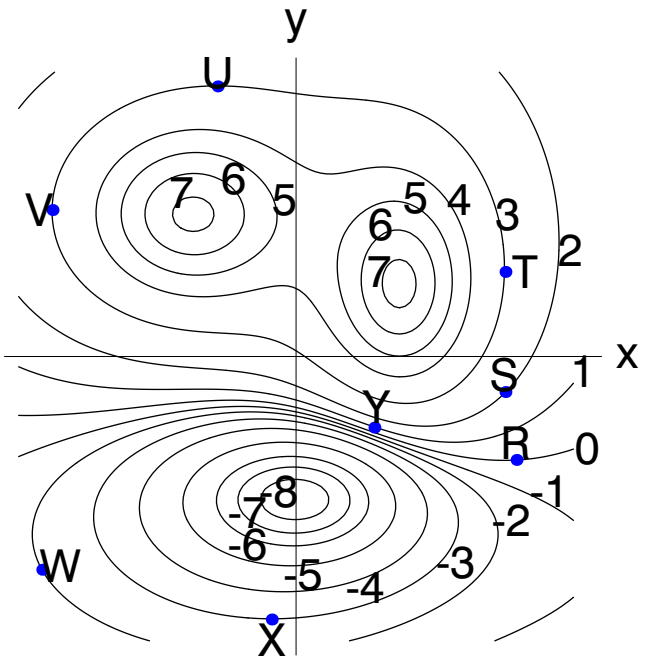
20) [T] [F] The function $u(x,y) = \sin(x+y)$ satisfies the PDE $u_{xx} + u_{yy} - 2u_{xy} = 0$.

Solution:
Actually $(\partial_x - \partial_y)f = 0$ and so $(\partial_x - \partial_y)^2 f = 0$.

Problem 2) (10 points) No justifications needed.

a) (4 points) Fill in the boxes. You do not need to give additional explanations.

Chain rule:	$\frac{d}{dt} f(\vec{r}(t)) =$ ☐ $\cdot \vec{r}\,'(t)$
Directional derivative D_v	$D_{\langle 2,3\rangle/\sqrt{13}} f(1,1) = \nabla f(1,1) \cdot$ ☐
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) =$ ☐ $+ \nabla f(1,1) \cdot (x-1, y-1)$
Equation of tangent line at $(1,1)$	$\nabla f(1,1) \cdot \langle x-1, y-1\rangle =$ ☐
Critical point $(1,1)$ of f	$\nabla f(1,1) =$ ☐
Lagrange equations	$\nabla f(x,y) =$ ☐ $\nabla g(x,y),\ g(x,y) = c.$
Type I integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y)$ ☐ .
Type II integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y)$ ☐ .
Integration in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)}$ ☐ $f(r\cos(\theta), r\sin(\theta))\ dr d\theta.$
Area	$\int\int_R$ ☐ $dxdy$



b) (2 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

c) (2 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

d) (2 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Solution:
a) $\nabla f(r(t))$, $\langle 2,3\rangle/\sqrt{13}$, $f(1,1)$, 0, $\langle 0,0\rangle$, λ, $dydx$, $dxdy$,r,1.
b) At the point Y the level curves are closest to each other indicating the steepest place and so the largest gradient.
c) At the points V and Y, the function increases, if we go into the x direction. In the other points, the function decreases, if we go into the x direction.
d) In order to have a zero directional derivative, we need the gradient to be zero or perpendicular into the direction $\vec{v}$. This is the case at the point S.

Problem 3) (10 points)

a) Locate and classify all the critical points of

$$f(x, y) = 3y - y^3 - 3x^2y \ .$$

Solution:
This is a routine problem. Find the gradient, put it to zero to find the critical points and apply the second derivative test.

x	y	D	f_{xx}	Nature	f
-1	0	-36	0	saddle	0
1	0	-36	0	saddle	0
0	-1	36	6	minimum	-2
0	1	36	-6	maximum * 2	

b) Where on the parameterized surface

$$\vec{r}(x, y) = \langle u, v, w \rangle = \langle xy^3, x^2/2, 3y^2/2 \rangle$$

is the function $g(u, v, w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x, y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Solution:
For $f(x, y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$ the gradient is $\nabla f(x, y) = \langle y^3 - x, 3xy^2 - 3y \rangle$. It is zero if $3y - 3y^5 = 0$ or $y(1 - y^4) = 0$ which means $y = 0$ or $y = \pm 1$. In the case $y = 0$, we have $x = 0$. In the case $y = 1$, we have $x = 1$, in the case $y = -1$, we have $x = -1$. The critical points are $(0, 0), (1, 1), (-1, -1)$.
The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 3 - 6xy - 9y^4$. The entry f_{xx} is -1 everywhere. Applying the second derivative test gives

Critical point	(0,0)	(1,1)	(-1,-1)
Discriminant	3	-12	-12
f_{xx}	-1	-1	-1
Analysis	max	saddle	saddle

Problem 4) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4 - \sqrt{x}} \, dx \, dy \ .$$

Solution:
Change the order of integration:

$$\int_0^{16} \int_{\sqrt{x}}^4 \frac{x^4}{4-\sqrt{x}}\, dydx$$

The inner integral $\int_{\sqrt{x}}^4 \frac{x^4}{4-\sqrt{x}}\, dx$ can now be computed and gives x^4 so that we end up with

$$\int_0^{16} x^4\, dx = 16^5/5$$

Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x,y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have an absolute maximum or absolute minimum? Make sure to justify also this answer.

Solution:
a) Lets find the critical points and classify them. Setting the gradient to 0 gives

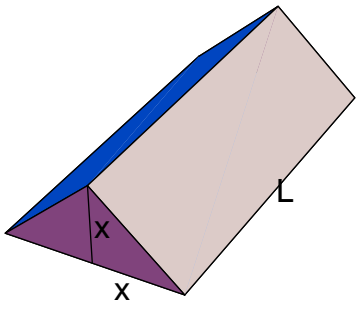
$$\begin{aligned} f_x &= 3e^y - 3x^2 = 0 \\ f_y &= 3xe^y - 3e^{3y} = 0\ . \end{aligned}$$

The first equation gives $x = \pm e^{y/2}$. Plugging this into the second gives $y = 0, x = 1$. The case $x = -1$ is not possible. Applying the second derivative test $f_{xx} = -6$ and $D > 0$ shows that $(1,0)$ is a local maximum.
b) If we look at the function f restricted to the x axes, we have $g(x) = f(x,0) = 3x-1-x^3$. This goes to $+\infty$ for $x \to -\infty$ and goes to $-\infty$ for $x \to \infty$. We have no global maximum nor a global minimum for f.
Remark: this is a remarkable example. In single variable calculus, sometimes the statement is proven, that if one has a local maximum and no global maximum nor minimum for a function $f(x)$, then there also must exist at least one local minimum. The example here shows that this is not the case for functions for several variables.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x,y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to

minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.



Solution:
The Lagrange equations

$$\nabla f = \lambda \nabla g, g = 1$$

are

$$\begin{aligned} 4x + 4y &= \lambda 2xy \\ 4x &= \lambda x^2 \\ x^2 y &= 1 \end{aligned}$$

Eliminating λ gives $(4x + 4y)/4y = \lambda 2xy/\lambda x^2$ or $y/x + 1 = 2y/x$ so that $1 = y/x$. The only critical point with positive x, y is $(1,1)$. The minimum of f is $f(1,1) = 6$. The minimal surface area is 6.

Problem 7) (10 points)

Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x,y) = (x^4 - x, (y^2 - 1)) = (0,0)$ so that the critical points are $(0,1), (0,-1), (1,1), (1,-1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = 4x^3 - 1$.

Point	D	f_{xx}	type
$(0,1)$	$D = -2$	-	saddle
$(0,-1)$	$D = 2$	-1	local max
$(1,1)$	$D = 6$	3	local min
$(1,-1)$	$D = -6$	-	saddle

Problem 8) (10 points)

The temperature distribution in a room is $T(x, y, z) = x + y + z$. On which point of the parametrized surface

$$\vec{r}(s, t) = \langle x, y, z \rangle = \langle s^2 + t^2, st, 2s - t \rangle$$

is the temperature extremal? Is it a maximum or a minimum?

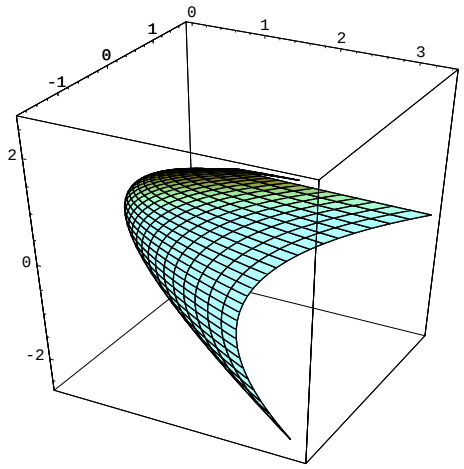
Solution:
We have to extremize

$$f(s, t) = s^2 + t^2 + st + 2s - t \ .$$

The gradient

$$\nabla f(s, t) = \langle 2s + t + 2, 2t + s - 1 \rangle$$

The gradient is zero at $(s, t) = (-5/3, 4/3)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 3$. Since $f_{xx} = 0$, we have a local minimum.



Solution:
The function to extremize is $f(s, t) = s^2 + t^2 + st + 2s - t$. The gradient is $\nabla f(s, t) = \langle 2s + t + 2, 2t + s - 1 \rangle$. The system of equations

$$\begin{aligned} 2s + t + 2 &= 0 \\ 2t + s - 1 &= 0 \end{aligned}$$

has the solution $s = -5/3, t = 4/3$. The discriminant is $D = 2 \cdot 2 - 1 = 3 > 0$ and $f_{ss} = 2 > 0$. So the point $(-5/3, 4/3)$ is a $\boxed{\text{minimum}}$.

Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta^2$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi - (x^2+y^2)^{1/4})}\, dx\, dy\ .$$



Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = drd\theta$ does not work. The integral $\int_0^\pi \int_0^{\theta^2} \frac{\cos(r)}{r(\pi-\sqrt{r})} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^{\pi^2} \int_{\sqrt{r}}^{\pi} \frac{\cos(r)}{r(\pi - \sqrt{r})} r\, d\theta\, dr$$

Evaluating the inner integral leads to $\int_0^{\pi^2} \cos(r)\, dr = \boxed{\sin(\pi^2)}$.

Problem 10) (10 points)

Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$.

a) Find the linear approximation of $f(1.01, 0.98)$.
b) What is the gradient ∇f at $(1, 1)$?
c) What is the equation $ax + by = d$ of the tangent line at $(1, 1)$?

Solution:
a) The approximation must lie on the tangent plane. Plug in $x = 1.01, y = 0.98$. We get $z = (9 - 2 * 1.01 - 3 * 0.98)/2 = 2.02$.
b) The normal vector to the graph of $f(x, y)$ is the gradient of $g(x, y, z) = z - f(x, y)$ at the point $(1, 1, 2)$ which is $\langle -f_x(1, 1), -f_y(1, 1), 1\rangle$. We know this is parallel to $\langle 2, 3, 2\rangle$. Because of the last equation, we know that $\langle -f_x(1, 1), -f_y(1, 1), 1\rangle = \langle 1, 3/2, 1\rangle$. Therefore $f_x = -1, f_y = -3/2$.
c) We have $a = f_x, b = f_y$ so that the equation is $-x - 3/2y = d$, Because the point $(1, 1)$ is on the line, we have $-x - 3/2y = -5/2$ or $2x + 3y = 5$.

11/1/2017 Second Hourly Practice 12 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) [T] [F] Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.

Solution:
On every line through the critical point, we have a local minumum. So, it is a local minimum, not a local max.

2) [T] [F] Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$.

Solution:
Use Clairot's theorem tells $g_x = f_{yx} = f_{xy} = g_y$.

3) [T] [F] The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface.

Solution:
The first equation is a single cone, the second equation is a double cone.

4) [T] [F] Even with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x, y)$ at (a, b) is non-zero.

Solution:
We have $D_v f(x, y) = f_x v_1 + f_y v_2$.

5) [T] [F] There exists a pair of different points on a sphere, for which the tangent planes are parallel.

Solution:
Take the antipodes.

6) [T] [F] If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

7) [T] [F] Assume we have a smooth function $f(x, y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x, y) = 0$. Then $(0, 0)$ is a critical point with $D < 0$.

Solution:
It can not be a saddle point, nor can it be a local maximum or local minimum. If it would be a saddle point, the level curves through the point consist of two lines. A concrete example, where three level curves pass through the same point is a monkey saddle. This is a case, where $D = 0$.

8) [T] [F] The gradient of $f(x, y)$ is perpendicular to the graph of f.

Solution:
This is a misconception. The gradient of f is a two dimensional vector. It is perpendicular to level curves.

9) [T] [F] The level curves of a linearization $L(x, y)$ of a function $f(x, y) = \sin(x + y)$ at $(0, 0)$ consist of lines.

Solution:
The function $L(x, y)$ is a linear function of the form $ax + by + c$.

10) [T] [F] If $x^4y + \sin(y) = 0$ then $y' = 4x^3y/(x^4 + \cos(y))$.

Solution:
The sign is wrong.

11) [T] [F] The linearization $L(x, y)$ at a critical point (x_0, y_0) of a function $f(x, y)$ is a constant function.

Solution:
If the gradient is $(a, b) = \nabla f$, then the linearization has the form $L(x, y) = ax + by + c$.

12) [T] [F] The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s,t), y(s,t), z(s,t)\rangle = \langle s, t, f(s,t)\rangle$ for some function $f(s,t)$ for which the parametrization covers the entire surface.

Solution:
The surface is not a graph.

13) [T] [F] The tangent plane to the graph of $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x, y, z)$ of $z - f(x, y)$.

Solution:
It is the level surface $L(x, y, z) = f(x_0, y_0, f(x_0, z_0)$.

14) [T] [F] The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

Solution:
The critical points of F are points where $f_x = \lambda g_x, f_y = \lambda g_y, g = 0$ which is exactly the Lagrange equations.

15) [T] [F] The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle.

Solution:
It is a half line, the intersection of a plane with a half plane.

16) [T] [F] If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axes the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

17) [T] [F] If $f(x,y,z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$.

Solution:
The gradient is a vector, not a scalar.

18) [T] [F] A function $f(x,y)$ in the plane always has a local minimum or a local maximum.

Solution:
Take the example $f(x,y) = xy$.

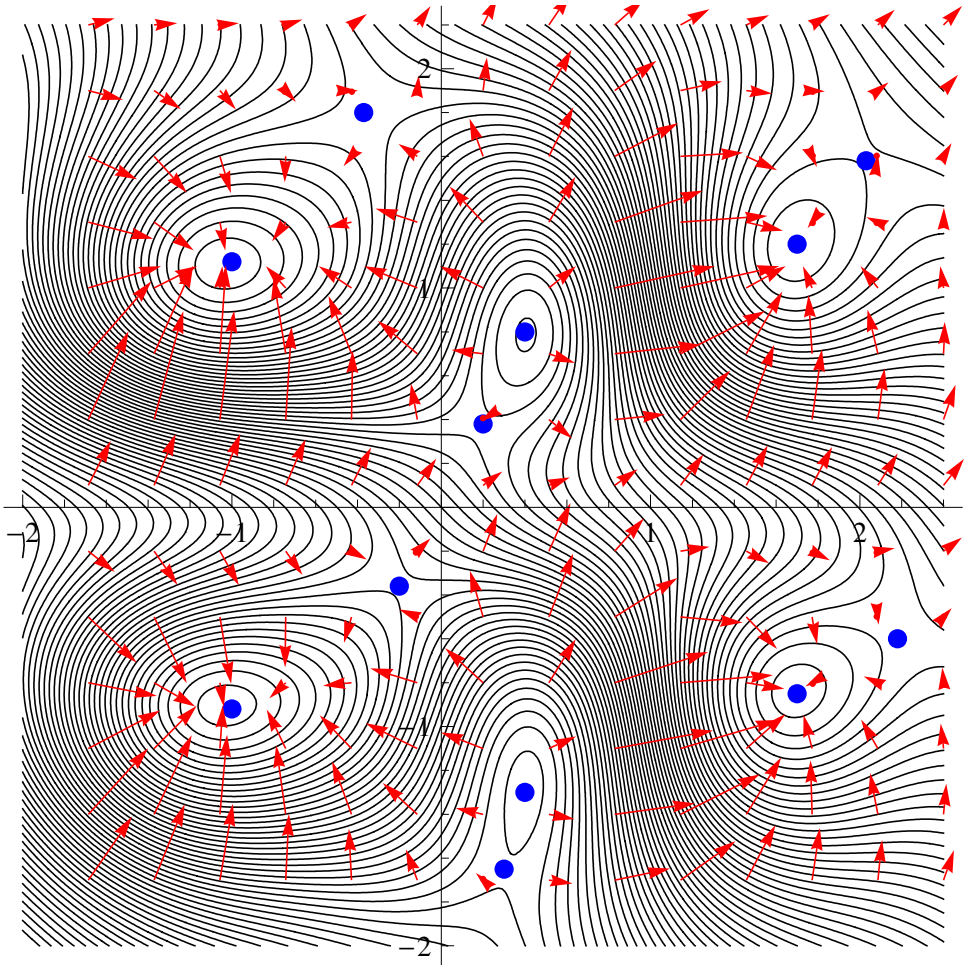
19) [T] [F] For any smooth function $f(x,y)$, the inequality $\|\nabla f\| \geq |f_x + f_y|$ is true.

Solution:
If $\nabla f = \langle a, b \rangle$, we square the claim, we get $a^2 + b^2 \geq (a+b)^2$. This is wrong for $(a,b) = (1,1)$.

20) [T] [F] If a function $f(x,y)$ satisfies $|\nabla f(x,y)| = 1$ everywhere in the plane, then f is constant.

Solution:
Counter examples are $f(x,y) = \sqrt{x^2+y^2}$, or $f(x,y) = x$.

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x, y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x, y)$ has		local maxima
The function $f(x, y)$ has		local minima
The function $f(x, y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Solution:
a) 4 maxima, 2 minima and 6 saddle points (look at the direction of the arrows.
b) Left: Laplace and wave, Right: heat and transport.

Problem 3) (10 points)

Find the cos of the angle between the sphere

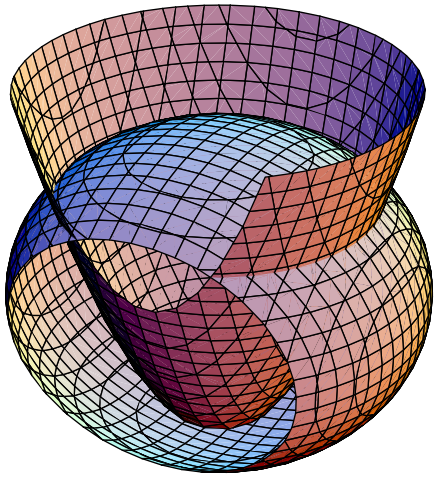
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P.



Solution:
The angle between the surfaces is the angle between the normal vectors. The gradients are $(4, -2, 4)$ and $(-4, 2, 1)$. The angle beween them satisfies $\cos(\theta) = -16/\sqrt{36 \cdot 21} = -8/3\sqrt{21}$. Both signs are ok.

Problem 4) (10 points)

a) You know that

$$-2x + 5y + 10z = 2$$

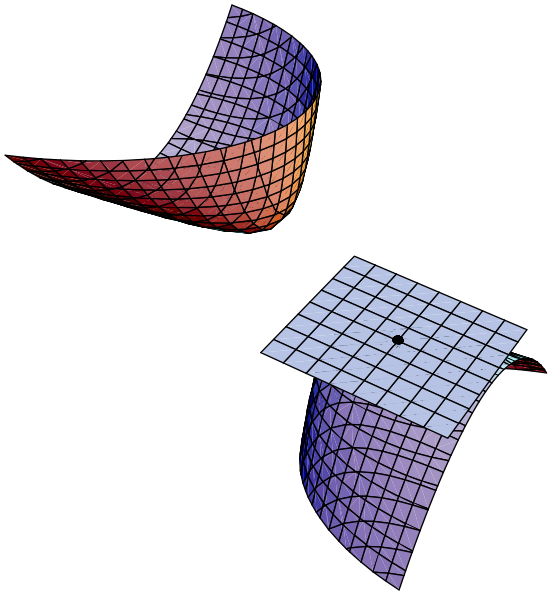
is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$. Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.

Solution:
a) The gradient is parallel to $\langle -2, 5, 10\rangle$. Because the gradient of the graph f is the gradient to the level surface $z - f(x, y) = 0$ which is $\nabla g(x, y) = \langle -f_x, -f_y, 1\rangle$, the gradient is $\langle f_x, f_y\rangle = \langle 2/10, -5/10\rangle = \langle 1/5, -1/2\rangle$.
The linearization is

$$L(x, y) = f(-1, 2) + (1/5)(x + 1) - (1/2)(y - 2) = -1 + (x + 1)/5 - (y - 2)/2$$

We get $L(-0.998, 2.0001) = -1 + 0.0004 - 0.0005 = -0.99965$.
b) The gradient is $\nabla f(x, y, z) = \langle 2x + 3z, 4y, 3x\rangle$. At the point $(2, 0, -1)$, we get $\nabla f(2, 0, -1) = \langle 1, 0, 6\rangle$. The equation of the tangent plane is $x + 6z = d$, where d is obtained by plugging in the point $(2, 0, -1)$. We have $x + 6z = -4$.
The linearization is $L(x, y, z) = 0 + (x - 2) + 6(z + 1)$. Plugging in the point $(x, y, z) = (2.001, 0.01, -1.0001)$ gives $L(2.001, 0.01, -1.0001) = 0.001 + 6(-0.0001) = 0.0004$.



Problem 5) (10 points)

a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the

elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 \ .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 \ .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Solution:
a) $\nabla f(x,y) = \langle y, x \rangle = \langle 0, 0 \rangle$ implies $x = y = 0$. The only critical point in the interior is $\boxed{(0,0)}$. The discriminant is $D = -1^2 = -1$. The point is a $\boxed{\text{saddle point}}$.
b) With $g(x,y) = x^2 + y^2/4$, we have the Lagrange equations

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 2y/4 \\ x^2 + y^2/4 &= 1 \end{aligned}$$

Dividing the first equation by the second to get $y/x = 4x/y$ which means $y^2 = 4x^2$ or $y = \pm 2x$. The third equation gives $2x^2 = 1$ or $x = \pm 1/\sqrt{2}$. The third equation gives $y = 2\sqrt{1 - x^2} = \pm\sqrt{2}$. The critical points are

$$\{(1/\sqrt{2}), \sqrt{2}\}, \{(-1/\sqrt{2}), \sqrt{2}\}, \{(1/\sqrt{2}), -\sqrt{2}\}, \{-(1/\sqrt{2}), -\sqrt{2}\} \ .$$

The value of the function at these points are $1, -1, -1, 1$. The first and last are maxima, the second and third are minima.
c) There is no global maximum, nor a global minimum in the interior of the disc because there is no local maximum in the interior of the disc. The global maxima as well as the minima are on the boundary. The global maximum is at $\pm(1, \sqrt{2})/\sqrt{2}$. The global minimum is $\pm(-1, \sqrt{2})/\sqrt{2}$.

Problem 6) (10 points)

a) Assume $f(x,y) = e^{2x-y-2} + y + \sin(x-1)$ and $x(t) = \cos(5t), y(t) = \sin(5t)$. What is

$$\frac{d}{dt} f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x, y, z) = (1, 1, 1)$. Find the gradient

$$\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \rangle$$

of $z(x, y)$ at the point $(1, 1)$.

Solution:
a) At time $t = 0$, the curve passes through the point $(1, 0)$. The gradient of f is $\nabla f(x, y) = \langle 2e^{2x+y-2} + \cos(x-1), -e^{2x-y-2} + 1 \rangle$. At the point $(1, 0)$, we have $\nabla f(1, 0) = \langle 3, 0 \rangle$. The velocity vector at time $t = 0$ is $\vec{r}\,'(t) = \langle -5\sin(t), 5\cos(t) \rangle$. At time $t = 0$, the velocity vector is $\langle 0, 5 \rangle$. By the chain rule, the rate of change

$$\frac{d}{dt} f(x(t), y(t)) = \nabla f(x(t), y(t)) \cdot \vec{r}\,'(t) = \langle 2, 0 \rangle \langle 0, 5 \rangle = 0 \, .$$

b) We have $z_x = -f_x/f_z, z_y = -f_y/f_z$. Because $\nabla f = \langle yz+y, xz+x+z^2, xy+3z^2+2yz \rangle = \langle f_x, f_y, f_z \rangle$. and $\nabla f(1, 1, 1) = \langle 2, 3, 6 \rangle$, we have $z_x = -1/3, z_y = -1/2$. In summary $\boxed{\nabla z(1, 1) = \langle -1/3, -1/2 \rangle}$.

Problem 7) (10 points)

The temperature in a room is given by $T(x, y, z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1, 0, 0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3, 2, 1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.



Solution:
$\nabla T(x,y,z) = \langle 2x, 4y, -3\rangle$. $\nabla T(1,0,0) = \langle 2,0,-3\rangle$. To increase the heat, we have to go into the direction $\boxed{\langle 2,0,-3\rangle/\sqrt{13}}$.
b) We have to find a direction $\vec{v} = \langle x,y,0\rangle = \langle \cos(\theta), \sin(\theta), 0\rangle$ such that $D_{\vec{v}} f(3,2,1) = 0$. We have $\nabla T(3,2,1) = \langle 6,8,-3\rangle$ and so $D_{\vec{v}} f(3,2,1) = 6x + 8y = 0$. This means $x = -(4/3)y$. Normalized, the vector is $\vec{v} = \langle -4/5, 3/5, 0\rangle$.

Problem 8) (10 points)

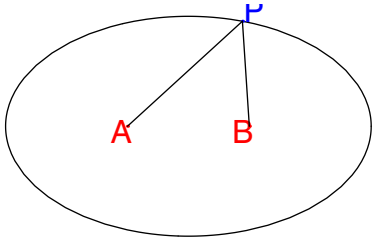
Let $g(x,y)$ denote the distance of a point $P = (x,y)$ to a point A and $h(x,y)$ the distance from P to a point B. The set of points (x,y) for which $f(x,y) = g(x,y) + h(x,y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot r' = -\nabla h \cdot r'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P. (Hint: check that $|\nabla f| = |\nabla g| = 1$).

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.



Solution:
a) The crucial point is that ∇f is perpendicular to the curve. That implies $\nabla g + \nabla h$ is perpendicular to the curve.
b) That is a direct consequence of a): the velocity vector $\vec{r}'(t)$ is tangent to the level curve. The vector $\nabla g(r(t)) + \nabla h(r(t))$ is perpendicular to the level curve and so to $\vec{r}\,'(t)$.
c) The cosine of the angle α between ∇g and $\vec{r}\ '$ is $|\nabla g \cdot r'|/(|\nabla g||r'|)$, the cosine of the angle β between ∇h and $\vec{r}\ '$ is $|\nabla h \cdot r'|/(|\nabla h||r'|)$. Note that $|\nabla h| = |\nabla g| = 1$. [Take $g(x,y) = (x^2+y^2)^{1/2}$ and compute the gradient. $\nabla g(x,y) = (2x,2y)/2(x^2+y^2)^{1/2} = (x,y)/(x^2+y^2)^{1/2}$. This is a special case of the Eiconal problem which appeared in other practice problems, where we take the distance to a curve.]

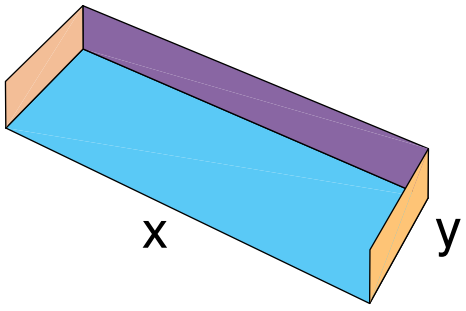
$$\cos(\alpha) = \frac{|\nabla g \cdot r'|}{|\nabla g|\ |r'|} = \frac{|\nabla g \cdot r'|}{|r'|} = \frac{|\nabla h \cdot r'|}{|r'|} = \frac{|\nabla h \cdot r'|}{|\nabla h|\ |r'|} = \cos(\beta)\ .$$

Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x,y) = xy + 2x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is constant and equal to 4.



Solution:
The Lagrange equations for the function given are

$$y + 2 = \lambda y$$
$$x + 2 = \lambda x$$
$$xy = 4$$

Dividing the first by the second, gives

$$(y + 2)/(x + 2) = y/x$$

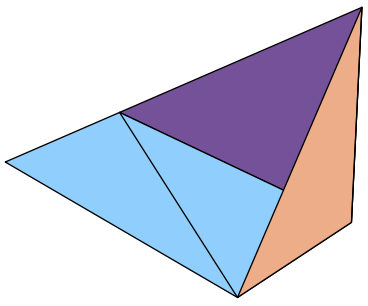
or $x(y + 2) = y(x + 2)$. This gives $x = y = 2$.
The picture was inconsistent with the function. We actually had intended to take

$$f(x, y) = xy + x + 2y$$

which leads to the minimum $(x, y) = (2\sqrt{2}, \sqrt{2})$. If somebody would solve this problem, then it would give full credit also.

Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?



Solution:
The Lagrange equations are

$$\begin{aligned} bc &= \lambda(b + c/2) \\ ac &= \lambda(a + c/2) \\ ab &= \lambda(a + b)/2 \\ ab + bc/2 + ac/2 &= 12 \, . \end{aligned}$$

Dividing the first to the second equation leads to $a = b$. Dividing the second to the third equation gives $c = 2b$. Substituting a and c gives $b^2 + b^2 + b^2 = 12$ or $b = 2$. Therefore $a = 2, b = 2, c = 4$ is the optimal configuration. The maximal volume is $f(2, 2, 4) = 8/3$.

12/15/2017 FINAL EXAM **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3 and 6, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The length of the vector $\vec{v} = \langle 3, 4, 5 \rangle$ is equal to the distance from the point $(1, 1, 1)$ to the point $(-2, -3, -4)$.

Solution:
By definition, $d(P, Q) = |\vec{PQ}|$ for any two points P, Q.

2) [T] [F] There is a non-constant vector field $\vec{F}(x, y, z)$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.

Solution:
This can not be true as the curl is a vector field while div is a scalar field.

3) [T] [F] For every $\vec{v}$ and $\vec{w}$, the projection of $\vec{v}$ onto $\vec{w}$ always has the same length as the projection of $\vec{w}$ onto $\vec{v}$.

Solution:
The projection of v onto w does not depend on the length of w while the projection of w onto v does not depend on the length of v. A simple counter example is $\vec{v} = \langle 1, 0, 0 \rangle, \vec{w} = \langle 2, 0, 0 \rangle$.

4) [T] [F] If $\vec{F}(x, y, z) = \langle \sin(z), \cos(z), 0 \rangle$, then $\text{curl}(\text{curl}(\text{curl}(\vec{F}))) = \vec{F}$.

Solution:
Just compute the curl once and see that already then $\text{curl}(\vec{F}) = \vec{F}$.

5) [T] [F] If S is the graph $z = x^6 + y^6$ above $x^2 + y^2 \leq 1$ oriented upwards and $\vec{F}(x, y, z) = \langle 0, 0, z \rangle$, then the flux of $\vec{F}$ through S is positive.

Solution:
The field always has an acute or right angle with the normal vector to the surface.

6) [T] [F] If S is the unit sphere oriented outwards and $\vec{F}(x, y, z) = \langle 0, 0, z^2 \rangle$, then the flux of $\vec{F}$ through the upper hemisphere of S is the same as the flux through the lower hemisphere.

Solution:
The flux through the lower part is the negative of the flux through the upper part.

7) [T] [F] If the divergence of $\vec{F}$ is zero, then $\vec{F}$ is a gradient vector field.

Solution:
Take for example $\langle -y, x, 0\rangle$. It has zero divergence but it is not a gradient field.

8) [T] [F] If E is the unit ball $x^2+y^2+z^2 \leq 1$ and $\vec{F}$ is the curl of some other vector field, then $\int\int\int_E \text{div}(\vec{F})dV = 4\pi/3$.

Solution:
The result is zero because div(curl(F))=0.

9) [T] [F] The curve $\vec{r}(t) = \langle t, t\rangle$ is a flow line of $\vec{F}(x, y) = \langle x, 2y\rangle$.

Solution:
The flow lines are parabola for the given field. We can also just compute the velocity $\langle 1, 1\rangle$ of the curve. It is not always parallel to the field $\vec{F}(x, y)$. Already at $(x, y) = (1, 1)$, it is false.

10) [T] [F] The integral $\iint_R \sqrt{1+f(u,v)^2}\ dudv$ is the surface area of the surface parametrized by $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$ for $(u, v) \in R$.

Solution:
The integration factor is $|\vec{r}_{\vec{u}} \times \vec{r}_{\vec{v}}| = \sqrt{1+f_u^2+f_v^2}$ and not $\sqrt{1+f(u,v)^2}$.

11) [T] [F] The volume of a parallelepiped with corners $A, B = A+\vec{v}, C = A+\vec{w}, D = A+\vec{v}+\vec{w}$ and $A+\vec{u}, B+\vec{u}, C+\vec{u}, D+\vec{u}$ is $|\vec{u}\cdot(\vec{v}\times\vec{w})|$.

Solution:
This triple scalar product indeed is the volume.

12) [T] [F] The curvature of $y = x^2$ at $(0, 0)$ is larger than the curvature of $y = 3x^2$ at $(0, 0)$.

Solution:
The second parabola is bent more, meaning the curvature is larger. It is also possible to compute the curvature with the formula as you have done in a homework for the parabola but that would take more time.

13) [T] [F] If $\vec{F}$ and $\vec{G}$ are two vector fields for which the divergence is the same, then $\vec{F} - \vec{G}$ is a constant vector field.

Solution:
Take for example $\vec{F} = \langle x, 0, 0\rangle$ and $\vec{G} = \langle 0, y, 0\rangle$. They both have divergence 1 but their difference is not constant.

14) [T] [F] If $\vec{F}, \vec{G}$ are two vector fields which have the same curl, then $\vec{F} - \vec{G}$ is irrotational.

Solution:
Yes, as the curl is then constant 0.

15) [T] [F] The parametrization $\vec{r}(u, v) = \langle 1 + u, v, u + v\rangle$ describes a plane.

Solution:
It is indeed a plane. No catch.

16) [T] [F] Any function $u(x, y)$ that obeys the partial differential equation $u_x + u_y - u_{xx} = 1$ has no local minima.

Solution:
At a local minimum, we have $\nabla u = \langle u_x, u_y\rangle = \langle 0, 0\rangle$, so that $u_{xx} = -1$ which is incompatible with a local minimum.

17) [T] [F] If $\vec{F} = \langle P, Q, R\rangle$ is a vector field so that $\langle P_x, Q_y, R_z\rangle = \langle 0, 0, 0\rangle$, then it is incompressible meaning that the divergence is zero everywhere.

Solution:
Indeed, $P_x + Q_y + R_z$ is then zero.

18) [T] [F] If $f(x, g(x)) = 0$, then $g'(x) = -f_x/f_y$ provided f_y is not zero.

Solution:
This is implicit differentiation. The formula follows from the chain rule.e

19) [T] [F] The equation $x^2 - (y-1)^2 + z^2 + 2z = -1$ represents a two-sheeted hyperboloid.

Solution:
It is a cone.

20) [T] [F] If $\vec{F}(\vec{r}(u,v)) = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$, then the absolute value of the flux of $\vec{F}$ through a closed bounded surface S parametrized by $\vec{r}(u,v)$ is the surface area of S.

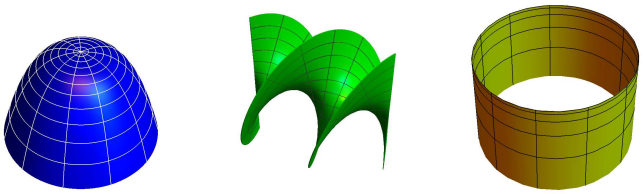
Solution:
Write down the flux integral. A factor $|\vec{r}_u \times \vec{r}_v|$ cancels out and gets to $\int\int |\vec{r}_u \times \vec{r}_v|\, dudv$.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(u,v)$	A-C
$\langle u\sin(v), u\cos(v), -u^3\rangle$	
$\langle u, v\sin(u), v\cos(u)\rangle$	
$\langle \sin(u), \cos(u), -v^3\rangle$	

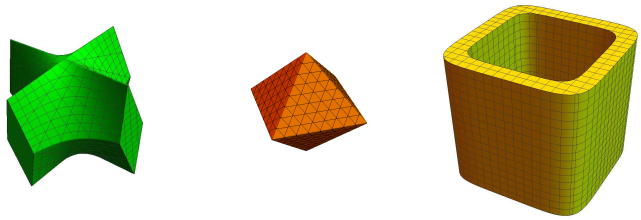
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$1 < x^6 + y^6 < 2$	
$\|x\|\|y\| + \|z\| < 1$	
$\|x\| + \|y\| + \|z\| < 3$	

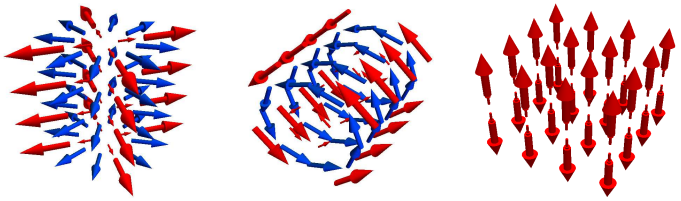
A B C



c) (2 points) The figures display vector fields $\vec{F}$. Match them.

Field	A-C
$\vec{F}(x,y,z) = \langle 0, 0, z\rangle$	
$\vec{F}(x,y,z) = \langle -z, 0, x\rangle$	
$\vec{F}(x,y,z) = \langle x, y, 0\rangle$	

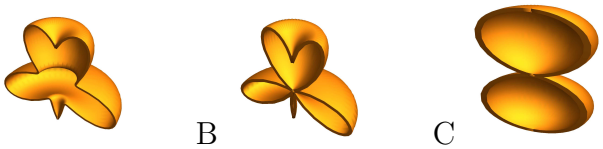
A B C



d) (2 points) Match the spherical plots.

Surface	A-C
$\rho(\theta,\phi) = 1 + \sin(4\phi)$	
$\rho(\theta,\phi) = 2 + \sin(4\phi)$	
$\rho(\theta,\phi) = 1 + \cos(2\phi)$	

A B C

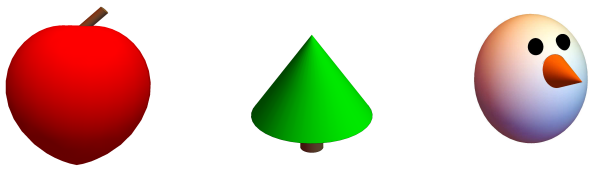


e) (1 point) Name a partial differential equation (PDE) for a function $u(t,x)$ discussed in this course which involves a term uu_x.

f) (1 point) Match each surface S to a graphic that contains S.

Surface S	A-C
$r^2 - (1-z)^2 = 0$	
$x^2 + y^2 + z^2 = 1$	
$\rho(\theta,\phi) = \sin^2(\phi/2)\phi$	

A B C



Solution:
a) ABC
b) CAB
c) CBA
d) BAC
e) Burgers equation
f) BCA

Problem 3) (10 points)

a) (3 points) Ed Sheeran's **"Shape of You"**, released in January this year, has been a critical success: it peaked at number-one on the singles charts of 44 countries and is currently the most streamed song on **Spotify.** But what is the shape of you? Which of the letters are not simply connected (SC)?

	Check if not SC
S	
H	
A	
P	
E	

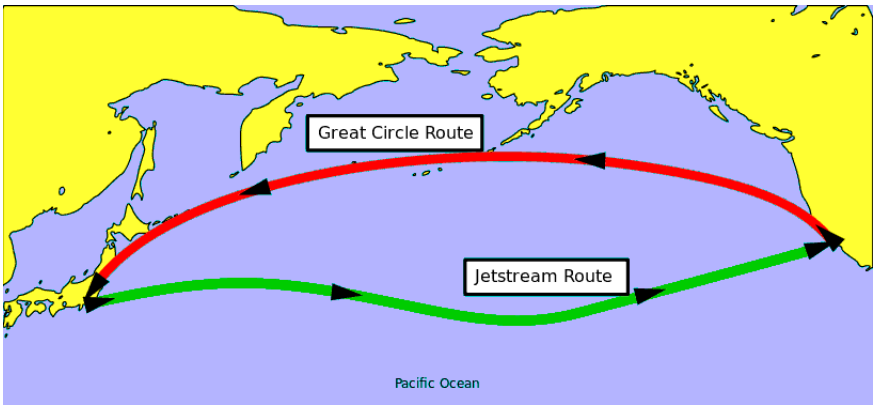
	Check if not SC
O	
F	
Y	
O	
U	



b) (4 points)

A plane flies from **Los Angeles** to **Tokyo** along the great circle route A and comes back via the jet stream route B. There is a force field $\vec{F}$ acting on the plane so that the work along A is $\int_A \vec{F} \cdot \vec{dr}$ and the work along B is $\int_B \vec{F} \cdot \vec{dr}$. You know that there is a potential function f such that $\vec{F} = \nabla f$. Check the statements that must be true.

$\int_A \vec{F} \cdot \vec{dr} = 0$	
$\int_B \vec{F} \cdot \vec{dr} = 0$	
$\int_A \vec{F} \cdot \vec{dr} + \int_B \vec{F} \cdot \vec{dr} = 0$	
$\int_A \vec{F} \cdot \vec{dr} - \int_B \vec{F} \cdot \vec{dr} = 0$	



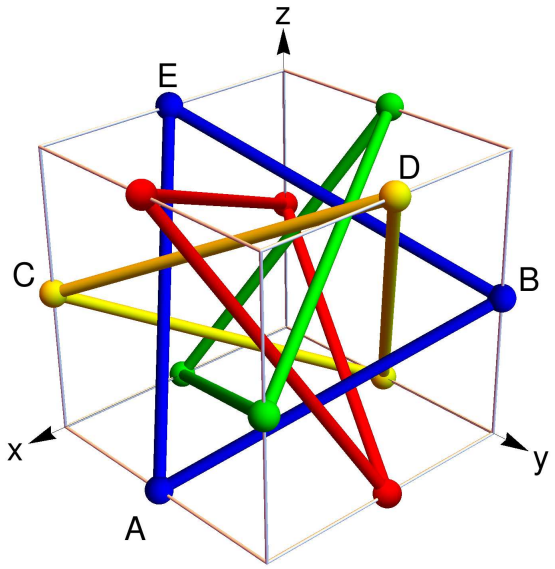
c) (3 points) Let E be the solid given by $x^8 + y^8 + z^8 \leq 4, y \geq 0$. Let S be the boundary of E with outward orientation. Consider the vector fields $\vec{F} = \langle x, y, z \rangle$, $\vec{G} = \langle x, y, -z \rangle$ and $\vec{H} = \langle x + y, y^2 + z^2, yz \rangle$. Check the correct box in each line:

Flux integral	$< \text{Vol}(E)$	$= \text{Vol}(E)$	$> \text{Vol}(E)$
$\iint_S \vec{F} \cdot d\vec{S}$			
$\iint_S \vec{G} \cdot d\vec{S}$			
$\iint_S \vec{H} \cdot d\vec{S}$			

Solution:
a) A, P, O, O are not simply connected.
b) The third is the only true identity as this is a closed loop.
c) Larger, Equal, Larger.
Use the divergence theorem. Larger (actually three times as large), then equal. For the third we note that the divergence is $1 + 3y$ which is positive in the part $y > 0$ where the solid is.

Problem 4) (10 points)

In topology one knows the **Danzer cube**. It is an example of what one calls a "non-shellable triangulation" of the cube. The picture shows four of the triangles.



a) (5 points) Find the distance between the line joining

$$A = (2, 1, 0) \text{ and } B = (0, 2, 1)$$

and the line joining

$$C = (2, 0, 1) \text{ and } D = (1, 2, 2) \, .$$

b) (5 points) Find the area of the triangle ABE, where

$$E = (1, 0, 2) \, .$$

Solution:
a) We use the distance formula $|(\vec{AB} \times \vec{CD}) \cdot \vec{AC}|/|(\vec{AB} \times \vec{CD})|$ which is with $\vec{AB} = \langle -2, 1, 1\rangle$ and $\vec{CD} = \langle -1, 2, 1\rangle$ and $\vec{AB} \times \vec{CD} = \langle -1, 1, -3\rangle$. The anser is $\boxed{4/\sqrt{11}}$.
b) We have to compute $|\langle \vec{AB} \times \vec{AE}|/2$ which is $\boxed{3\sqrt{3}/2}$.

Problem 5) (10 points)

a) (6 points) Find the **surface area** of

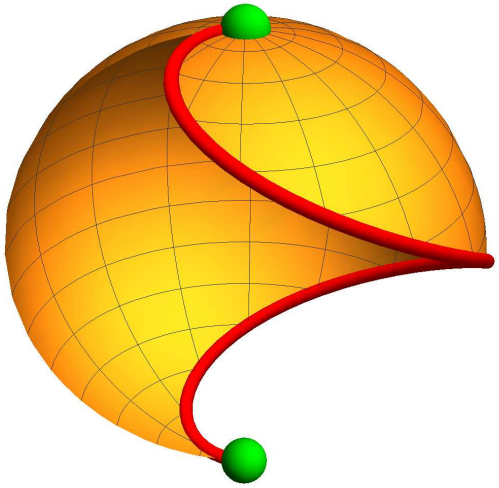
$$\vec{r}(t, s) = \langle \cos(t)\sin(s), \sin(t)\sin(s), \cos(s)\rangle$$

$0 \leq t \leq 2\pi, 0 \leq s \leq t/2$.

b) (4 points) The part of the boundary curve when $s = t/2$ is defined as

$$\vec{r}(t) = \langle \cos(t)\sin(t/2), \sin(t)\sin(t/2), \cos(t/2)\rangle \ .$$

Compute the number $|\int_0^{2\pi} \vec{r}\,'(t)\ dt|$.



Solution:
a) We know the value of $|r_t \times r_s| = \sin(s)$ already as this is just a sphere of radius 1. We now integrate

$$\int_0^{2\pi} \int_0^{t/2} \sin(s) ds dt = \int_0^{2\pi} 1\ dt = 2\pi \ .$$

The answer $\boxed{2\pi}$ could also have been by symmetry as the surface is half of the sphere. This was of course accepted, but there had to be an explanation.
b) By the fundamental theorem of calculus this is $|\vec{r}(2\pi) - \vec{r}(0)|$ which is $|\langle 0, 0, 1\rangle - \langle 0, 0, -1\rangle| = \boxed{2}$. There was no need to integrate here. It is just the distance between the north and south pole.

Problem 6) (10 points)

The **Longy School of Music** at 27 Garden Street shows an abstract art work featuring a sphere, a cone and a cylinder. You build a model. Your parametrization should use the variables provided. No further justifications are needed in this problem.

a) (2 points) Parametrize the sphere $x^2 + y^2 + (z-5)^2 = 9$.

$$\vec{r}(\theta, \phi) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

b) (3 points) Parametrize the cylinder $x^2 + y^2 = 4$.

$$\vec{r}(\theta, z) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

c) (3 points) Parametrize the cone $4y^2 + 4(z-5)^2 = x^2$.

$$\vec{r}(\theta, x) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$

d) (2 points) Parametrize the grass floor $z = \sin(99x + 99y)$.

$$\vec{r}(x, y) = \langle \boxed{}, \boxed{}, \boxed{} \rangle$$



Photo: O. Knill, December 2017

Solution:

a) $x^2 + y^2 + (z-5)^2 = 9$ is a sphere of radius 3 translated up a bit.

$$\vec{r}(\theta, \phi) = \langle 3\cos(\theta)\sin(\phi), 3\sin(\theta)\sin(\phi), 5 + 3\cos(\phi)\rangle$$

b) $x^2 + y^2 = 4$ is a cylinder of radius 2.

$$\vec{r}(\theta, z) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle$$

c) $4y^2 + 4(z-5)^2 = x^2$ is a cone translated up

$$\vec{r}(\theta, x) = \langle x, (x/2)\cos(\theta), 5 + (x/2)\sin(\theta)\rangle$$

d) $z = \sin(99x + 99y)$ is a graph.

$$\vec{r}(x, y) = \langle x, y, \sin(99x + 99y)\rangle$$

Problem 7) (10 points)

a) (6 points) Find the **linearization** $L(x, y)$ of

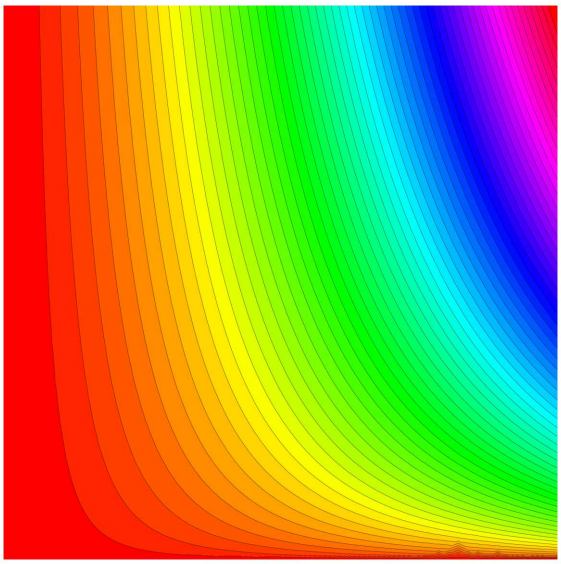
$$f(x, y) = \sqrt{x^3 y}$$

at $(x_0, y_0) = (10, 1000)$.

b) (4 points) Estimate the value

$$\sqrt{11^3 \cdot 999}$$

using the linearization in a).



Solution:
a) We compute the gradient of f as

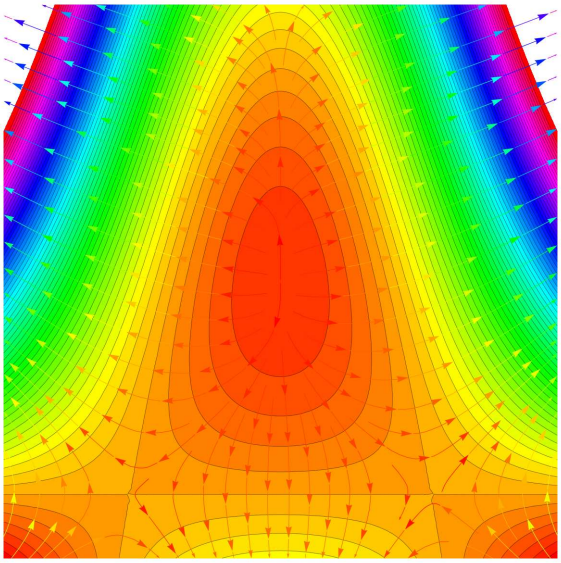
$$\nabla f(x,y) = \langle 3x^2y/(2\sqrt{x^3y}), x^3/(2\sqrt{x^3y})\rangle$$

which is at $(10, 1000)$ equal to $\langle a, b\rangle = \langle 150, 1/2\rangle$. The linearization is $\boxed{L(x,y) = 1000 + 150(x-10) + (1/2)(y-10)}$. It could be simplified to $-505+150x+y/2$ but the first expression is actually better to continue in b)
b) We can estimate $L(11, 999) = 1000 + 150 \cdot 1 + (1/2)(-1)$ which is $\boxed{1150 - 1/2}$.
As a comparison, the numberical result is 1153.11. The linearization is by about 3 promille off.

Problem 8) (10 points)

The vector field $\vec{F}$ is a **gradient field** with potential function

$$f(x,y) = y^2 + 4yx^2 + 4x^2 \ .$$



a) (8 points) Find and classify all the critical points of f.

b) (2 points) Does f have a global maximum or global minimum on the whole xy-plane? Only a brief explanation is needed. Some context: the critical points of f are the equilibrium points of $\vec{F}$. A critical point is called a **sink** if all vectors nearby point towards it. **Sinks** correspond to maxima of f. The minima of f are also called **source** as all vectors points nearby point away of it. An equilibrium is called **hyperbolic** if there are vectors pointing both away and towards it. These are the saddle points of f.

Solution:
a) We look for critical points, points where the gradient

$$\nabla f = \langle 8x + 8xy, 4x^2 + 2y \rangle$$

is zero. We see that either $x = 0$, implying $y = 0$, or then $y = -1$ implying $x = \pm 1/\sqrt{2}$. We have $f_{xx} = 8 + 8y$ and $D = f_{xx} f_{yy} - f_{xy}^2 = 2(8 + 8y) - (8x)^2$.

- At the point $\boxed{(0,0)}$ we have $D = 16$ and $f_{xx} = 8 > 0$ so that we have there a $\boxed{\textbf{minimum}}$.
- At the points $\boxed{(\pm 1/\sqrt{2}, -1)}$ we have $D = -32$ so that these are both $\boxed{\textbf{saddle points}}$.

b) If we put $x = y$, we get the function $5x^2 + 4x^3$ of one variable which has is unbounded in both directions: it is dominated by the x^3 term. The function gets arbitrary large and arbitrary small. There is $\boxed{\text{neither a global maximum, nor a global minimum}}$.
P.S. one can not argue with Bolzano since we do not have a closed bounded region. The fact that we have not a closed bounded region does also not imply automatically that we don't have a global max or min. The function $\sin(x) + \sin(y)$ for example is defined everywhere but has many global maxima and minima.

Problem 9) (10 points)

The top tower of the Harvard **Memorial Hall** is a **square frustum** of height $h = 9$. On the **Moscow Papyrus** written in 1850 BC, the volume of such a truncated square pyramid with side lengths x, y of the top and bottom faces, has already been given with the formula $h(x^2 + xy + y^2)/3$. Using this almost four-millennia year old formula and Lagrange, find the minimal volume

$$f(x, y) = 3x^2 + 3xy + 3y^2$$

under the constraint

$$g(x, y) = 3x + 2y = 14 \ .$$

You don't have to justify whether the solution is a minimum.



Photo: O. Knill, November 2017

Solution:
The challenge was here only not to fall in awe over the daring cultural bridge between the awesome ancient Egyptian mathematics and the awesome Memorial hall, whose "frustum" tower was built in 1718, destroyed in a fire of 1956 and rebuilt in 1996. The Lagrange equations are

$$\begin{aligned} 6x + 3y &= \lambda 3 \\ 3x + 6y &= \lambda 2 \\ 3x + 2y &= 14 \end{aligned}$$

Elliminating λ gives $x = 4y$. Plugging this into the third equation gives $y = 1$ and so $x = 4$. The solution is $\boxed{(4, 1)}$.

Problem 10) (10 points)

When properly aired, sand becomes **liquid sand** and you can take a bath in a sand tank. Assume the force field acting on a body floating in it is

$$\vec{F} = \langle -y, x, z \rangle \ .$$

The flux $\int\int_S \vec{F} \cdot d\vec{S}$ of this vector field through the surface of the body is the uplift. What is the uplift of the football

$$x^2 + y^2 \leq \cos^2(z)$$

with $-\pi/2 \leq z \leq \pi/2$, with the outward orientation?



Source: Youtube, Mark Rober, November 28, 2017

Solution:
We use the divergence theorem. The vector field has divegence 1. The flux integral is therefore the volume of the solid of revolution given by rotating the cos graph around the z-axes in the interval from $-\pi/2$ to $\pi/2$. This volume is

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{\cos(z)} r dr dz d\theta$$

which is $2\pi \int_{-\pi/2}^{\pi/2} \cos^2(z)/2 \, dz = 2\pi \int_{-\pi/2}^{\pi/2} (1 + \cos(2z))/4 dz = 2\pi\pi/4$. The result is $\boxed{\pi^2/2}$.

Problem 11) (10 points)

Find the **flux** of the curl of the vector field

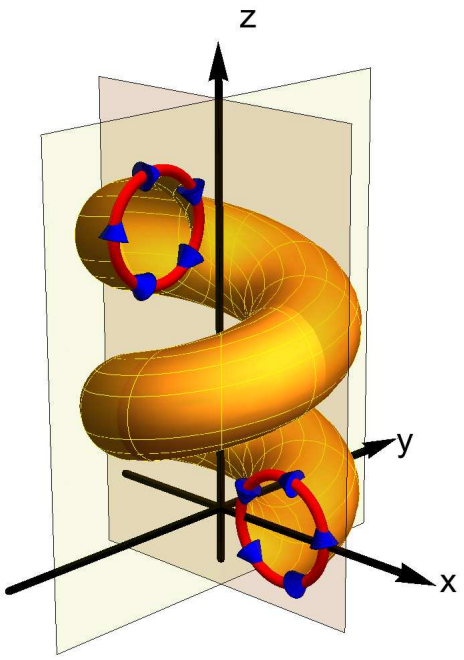
$$\vec{F}(x,y,z) = \langle -z, z+\sin(xyz), x-3\rangle + \langle x^5, y^7, z^4\rangle$$

through the **twisted surface** oriented inwards and parametrized by

$$\vec{r}(t,s) = \langle (3+2\cos(t))\cos(s), (3+2\cos(t))\sin(s), s+2\sin(t)\rangle$$

where $0 \le s \le 7\pi/2$ and $0 \le t \le 2\pi$.

Hint: This parametrization leads correctly already to a vector $\vec{r}_t \times \vec{r}_s$ pointing inwards. The boundary of the surface is made of two circles $\vec{r}(t,0)$ and $\vec{r}(t,7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).



Solution:
We use Stokes theorem. Instead of computing the flux integral we compute the line integral along the two circles. The vector field was already split so that the second part is a gradient field. Both line integrals with that vector field are zero by the fundamental theorem of line integrals. The lower circle is already oriented correctly, the second one not. The first circle is obtained by putting $s = 0$, the second one is obtained by putting $s = 7\pi/2$:

$$\vec{r}(t) = \langle 3+2\cos(t)), 0, 2\sin(t)\rangle \ ,$$

$$\vec{r}(t) = \langle 0, -3-2\cos(t)4\pi, 7\pi/2+2\sin(t)\rangle \ .$$

The first line integral $\int_0^{2\pi} \langle -2\sin(t), 2\sin(t), 2\cos(t)\rangle dt = 8\pi$. The second line integral $\int_0^{2\pi} \langle 7\pi/2+2\sin(t), 7\pi/2+2\sin(t), -3\rangle dt = 4\pi$ has to be taken negatively. The result is $8\pi - 4\pi = \boxed{4\pi}$.

Problem 12) (10 points)

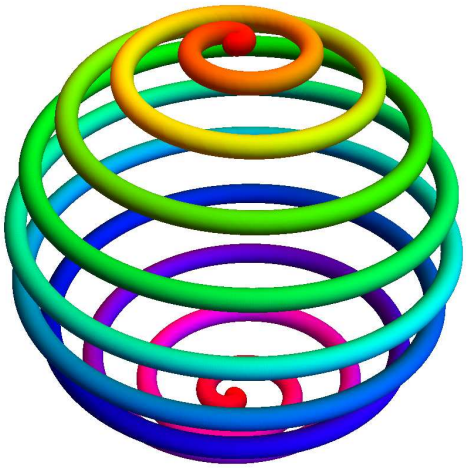
Find the line integral of the vector field

$\vec{F}(x,y,z) = \langle yz + x^2, xz + y^2 + \sin(y), xy + \cos(z)\rangle$

along the **spherical curve**

$$\vec{r}(t) = \langle \cos(20t)\sin(t), \sin(20t)\sin(t), \cos(t)\rangle \ ,$$

where $0 \le t \le \pi$.



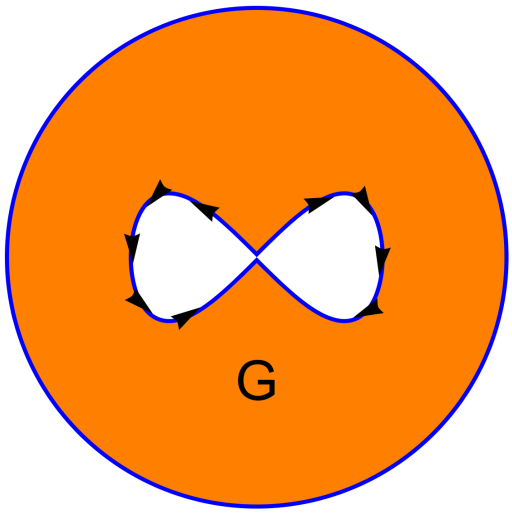
Solution:
The vector field is conservative. The potential f can be obtained by integration. It leads to $f(x,y,z) = x^3/3 + y^3/3 - \cos(y) + \sin(z) + xyz$. Now $\vec{r}(\pi) = \langle 0,0,1\rangle$ and $\vec{r}(0) = \langle 0,0,-1\rangle$. By the fundamental theorem of line integrals, we have $\int_C \vec{F}\cdot d\vec{r} = f(0,0,-1) - f(0,0,1) = \sin(-1) - \sin(1) = \boxed{-2\sin(1)}$.

Problem 13) (10 points)

Look at the shaded region G bounded by a circle of radius 2 and an inner **figure eight lemniscate** with parametric equation

$$\vec{r}(t) = \langle \sin(t), \sin(t)\cos(t)\rangle$$

with $0 \le t \le 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.



Solution:
We use **Greens theorem** wth the vector field $\vec{F} = \langle 0, x \rangle$. Since its curl is constant 1, the area is the line inegral along the boundary. We note however that the curve from $t = 0$ to $t = \pi$ is oriented so that the area. The other part is the same

$$\int_0^\pi \langle 0, \sin(t) \rangle \cdot \langle \cos(t), \cos^2(t) - \sin^2(t) \rangle \, dt$$

is $\int_0^\pi \sin(t)\cos^2(t) - \sin^3(t) \, dt$. Writing in the second integral $\sin^2(t) = 1 - \cos^2(t)$, we get $\int_0^\pi 2\sin(t)\cos^2(t) - \sin(t) \, dt = [-(2/3)\cos^3(t) + \cos(t)]_0^\pi = 4/3 - 2 = -2/3$. The negative area of the two lobes is therefore $-4/3$. The area of the shaded region is $\boxed{4\pi - 4/3}$.

12/15/2017 FINAL PRACTICE I **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The surface $-x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid.

Solution:
Look at the xy-trace.

2) [T] [F] The vector projection $\vec{\text{proj}}_{\vec{w}}(\vec{v})$ of a vector $\vec{v}$ onto a non-zero vector $\vec{w}$ is always non-zero.

Solution:
The two vectors can be perpendicular

3) [T] [F] The linearization of $f(x, y) = 5 + 7x + 3y$ at any point (a, b) is the function $L(x, y) = 5 + 7x + 3y$.

Solution:
A linear function has the function as a linearization.

4) [T] [F] For any function $f(x, y, z)$, for any unit vector $\vec{u}$ and any point (x_0, y_0, z_0) we have $D_{\vec{u}} f(x_0, y_0, z_0) = -D_{-\vec{u}} f(x_0, y_0, z_0)$.

Solution:
Changing the direction of $\vec{u}$ changes the sign by definition $D_u f = \nabla f \cdot \vec{u}$.

5) [T] [F] There is a vector field $\vec{F} = \langle P, Q, R \rangle$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{F})$.

Solution:
The right hand side is a scalar while the left hand side is a vector.

6) [T] [F] The formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ holds if $\vec{v}, \vec{w}$ are vectors in space and α is the angle between them.

Solution:
Yes, this is an important formula for the length of the cross product.

7) [T] [F] If $\vec{F} = \langle -2y, 2x \rangle$ and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise, then $\int_C \vec{F} \cdot d\vec{r} = 16\pi$.

Solution:
Just write out the integrand $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle -4\cos(t), 4\sin(t) \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle = 8\pi$.
When integrated from 0 to 2π, we get 16π.

8) [T] [F] The parametrization $\vec{r}(u, v) = \langle u, \sqrt{1 - u^2 - v^2}, v \rangle, u^2 + v^2 \leq 1$ describes a half sphere.

Solution:
Indeed, the entries x, y, z satisfy $x^2 + y^2 + z^2 = 1$

9) [T] [F] The vectors $\vec{v} = \langle 1, 0, 1 \rangle$ and $\vec{w} = \langle -1, 1, 1 \rangle$ are perpendicular.

Solution:
Their dot product zero.

10) [T] [F] If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) then $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.

Solution:
It would be true if div were replaced by curl.

11) [T] [F] The vector field $\vec{F} = \langle e^x, e^y, e^z \rangle$ satisfies $\text{grad}(\text{div}(\vec{F})) = \vec{F}$.

Solution:
Indeed. Just compute it and use that $(e^x)' = e^x$.

12) [T] [F] If $\vec{F}(x, y, z)$ has zero curl everywhere in space, then $\vec{F}$ is a gradient field.

Solution:
This is a result which follows from the Stokes theorem.

13) [T] [F] If $\vec{r}(u,v)$ is a parametrization of the surface $g(x,y,z) = x^2 + e^{y^3} + z^2 = 5$ then for any u and v we have $\nabla g(\vec{r}(u,v)) \cdot \vec{r}_u(u,v) = 0$.

Solution:
This is true for any surface $g(x,y,z) = c$. The reason is that the gradient of g is perpendicular to the level surface and that $\vec{r}_u$ is tangent to the surface.

14) [T] [F] The equation $\text{grad}(\text{div}(\text{grad}(f))) = \langle 0,0,0\rangle$ always holds.

Solution:
A simple example is $f(x,y,z) = x^3$.

15) [T] [F] There is a non-constant function $f(x,y,z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$ everywhere.

Solution:
The right hand side is 0. So, ∇f has to be constant.

16) [T] [F] If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S oriented outwards is the volume of the enclosed solid.

Solution:
By the divergence theorem.

17) [T] [F] If $f(x,y)$ is maximized at (a,b) under the constraint $g(x,y) = c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel.

Solution:
They indeed are by the Lagrange equations

18) [T] [F] The distance between a point P and the line L through two different points A, B is given by the formula $|\vec{PA} \times \vec{AB}|/|\vec{PA}|$.

Solution:
It would be true if $\vec{PA}$ were replaced by $\vec{AB}$.

19) [T] [F] The unit tangent vector $\vec{T}(t)$ is always perpendicular to the vector $\vec{T}\,'(t)$.

Solution:
We have shown that in class. It follows from differentiating $T \cdot T = 1$.

20) [T] [F] The vector field $\vec{F}(x, y, z) = \langle x^5, x^6, x^7 \rangle$ can not be the curl of another vector field.

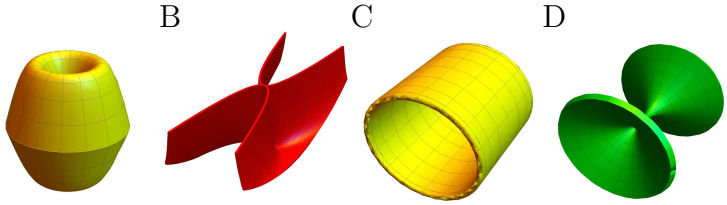
Solution:
Its divergence is not zero. So that it can not happen

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface $\vec{r}(u,v) =$	A-D
$\langle u, u^2v, v^2\rangle$	
$\langle u^2\cos(v), u, u^2\sin(v)\rangle$	
$\langle \cos(v), \sin(u), \sin(v)\rangle$	
$\langle v\cos(u), v\sin(u), \sin(v)\rangle$	

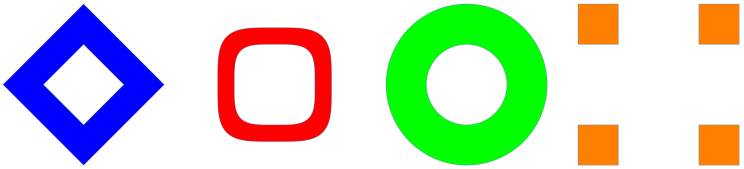
A B C D



b) (2 points) Match the following 2D region plots. There is an exact match.

Region	A-D
$1 \le x^2 \le 4, 1 \le y^2 \le 4$	
$1 \le x^2 + y^2 \le 4$	
$1 \le x^4 + y^4 \le 4$	
$1 \le \lvert x\rvert + \lvert y\rvert \le 2$	

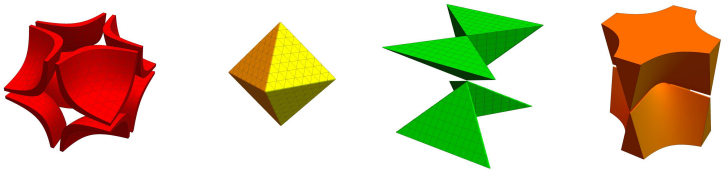
A B C D



c) (2 points) Match the following 3D regions. There is an exact match.

Solid	A-D
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert \le 2$	
$\lvert x\rvert < \lvert y\rvert < \lvert z\rvert$	
$1 < \lvert xyz\rvert < 2$	
$x^2y^2 < z^2$	

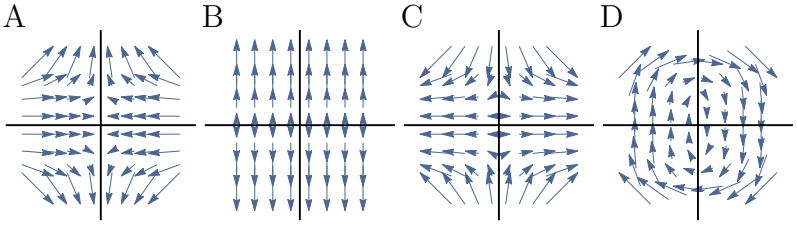
A B C D



d) (2 points) The following figures display vector fields. There is an exact match.

Field	A-D
$\vec{F}(x,y) = \langle 0, y\rangle$	
$\vec{F}(x,y) = \langle -x, y^3\rangle$	
$\vec{F}(x,y) = \langle y^3, -x\rangle$	
$\vec{F}(x,y) = \langle x, -y^3\rangle$	

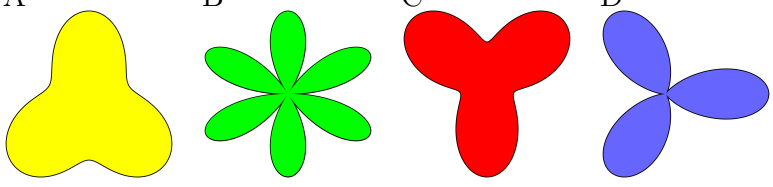
A B C D



e) (2 points) The following figures display polar regions. There is an exact match.

Polar region	A-D
$r \le 1 + \cos(3\theta)$	
$r \le 2 + \sin(3\theta)$	
$r \le 3 - \sin(3\theta)$	
$r \le \lvert\sin(3\theta)\rvert$	

A B C D



Solution:
a) BDCA
b) DCBA
c) BCAD
d) BADC
e) DCAB

Problem 3) (10 points)

a) (6 points)

The concept of **boundary** plays an important role in integral theorems. In each of the following six rows, check exactly one entry which best describes the boundary.

The boundary of	solid	surface	curves	points	empty
$x^2 + y^2 + z^2 = 1$					
$x^2 + y^2 = 1, z = 0$					
$x^2 + y^2 + z^2 \leq 1, x = y = 0$					
$x^2 + y^2 + z^2 \leq 1$					
$x^2 + y^2 \leq 1, z = 0$					
$x^2 + y^2 = 1, z^2 \leq 1$					

b) (4 points) Match the following partial differential equations (PDEs) by picking 4 from the 5 given choices A-E.

PDE	Enter A-E
heat equation	
wave equation	
transport equation	
Burgers equation	

$u_{xx} = u_t$	A
$u_{xx} = u_t + uu_x$	B
$u_x = u_t$	C
$u_{xx} = -u_{tt}$	D
$u_{xx} = u_{tt}$	E

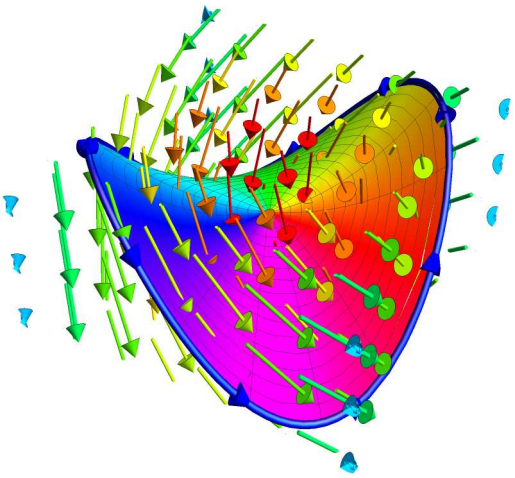
Solution:
a) empty, empty, points, surface, curves, curves.
Here is some more explanation:
The first is a closed surface which has no boundary.
The second is a closed curves which has no boundary.
The third is a line segment which has two points as a boundary.
The fourth is the unit ball which has the unit sphere, a surface, as a boundary.
The fifth is a closed disk, which has a circle, a curve as a boundary.
The sixth is a finite cylinder, which has two circles, two curves, as a boundary.
b) A,E,C,B.

Problem 4) (10 points)

a) (3 points) A surface S is parameterized by

$$\vec{r}(u,v) = \langle u, v, uv \rangle \ ,$$

where $u^2 + v^2 \leq 1$. Find its surface area.



b) (3 points) Parametrize the boundary curve C matching the orientation $\vec{r}_u \times \vec{r}_v$ of S, then compute the line integral $\int_C \vec{F} \cdot \ d\vec{r}$ with $\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$.

c) (2 points) The coordinates of the surface S satisfy $xy - z = 0$. Find the tangent plane to S at $(2,1,2)$.

d) (2 points) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at the point $(2,1)$.

Solution:
a) Since $|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + v^1 + 1}$ we have in polar coordinates $\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} r dr d\theta = 2\pi/3(\sqrt{8} - 1)$.
b) The boundary curve is $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t)\sin(t) \rangle$. The line integral can be computed directly as $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), \cos^2(t) - \sin^2(t) \rangle$. which gives $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2\cos^2(t)$ and the integral is, (use double angle formula) equal to $\boxed{2\pi}$.
P.S. Some have computed it with Stokes using $\text{curl}(\vec{F}) = \langle 0, 0, 2 \rangle$ and $\vec{r}_u \times \vec{r}_v = \langle -v, -u, 1 \rangle$ as computed in a). We have to integrate the function 2 over the disk of radius 1. This is twice the area of the disk, which is again $\boxed{2\pi}$.
c) The gradient vector is $\langle y, x, -1 \rangle$ which is $\langle 1, 2, -1 \rangle$ so that the equation of the plane is $x + 2y - z = d$. The constant can be obtained by plugging in the point. This is $\boxed{x + 2y - z = 2}$.
d) The gradient of f is $\langle y, x \rangle$. At the point $(2,1)$ this is $\langle 1, 2 \rangle$. The linearization is
$\boxed{L(x,y) = f(2,1) + 1(x-2) + 2(y-1) = 2 + (x-2) + 2(y-1)}$

Problem 5) (10 points)

On November 17 2017, the **NASA Eagleworks paper** appeared, making the **EM drive** more probable. It might in future be used for deep space missions. The drive produces a thrust, apparently violating momentum conservation.

a) (5 points) Assume the drive flies in the gravitational field

$$\vec{F}(x,y,z) = \langle x^7 + xy^2z^2, x^2yz^2, x^2y^2z \rangle$$

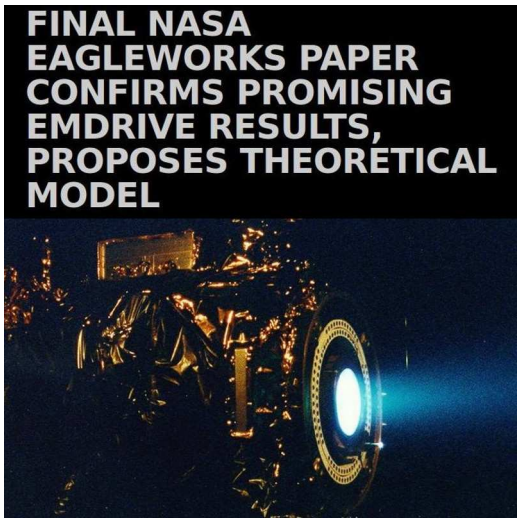
along the path

$$C : \vec{r}(t) = \langle t\cos(t), t\sin(t), t(5\pi - t) \rangle$$

with $0 \leq t \leq 5\pi$. Compute the work

$$\int_0^{5\pi} \vec{F} \cdot d\vec{r}.$$

b) (3 points) Compute $d = |\int_0^{5\pi} \vec{r}\,'(t)\, dt|$.



c) (2 points) If L is the arc length of C, circle the one box below which applies:

$d = L$	$d > L$	$d < L$

Solution:
a) The field is conservative. There is a potential $f(x,y,z) = x^2y^2z^2/2 + x^8/8$. We can by the fundamental theorem of line integral, evaluate $f(-5\pi,0,0) - f(0,0,0) = (5\pi)^8/8$.
b) By the fundamental theorem of calculus, $\int \vec{r}\,'(t)\, dt = \vec{r}(5\pi) - \vec{r}(0) = \langle -5\pi, 0, 0 \rangle$. Its length is $\boxed{5\pi}$.
c) The arc length of the curve is longer than the shortest connection, the line which has been computed in b). Note that the arc length has the length $|\cdot|$ inside the integral. We have

$d = L$	$d > L$	$d < L$

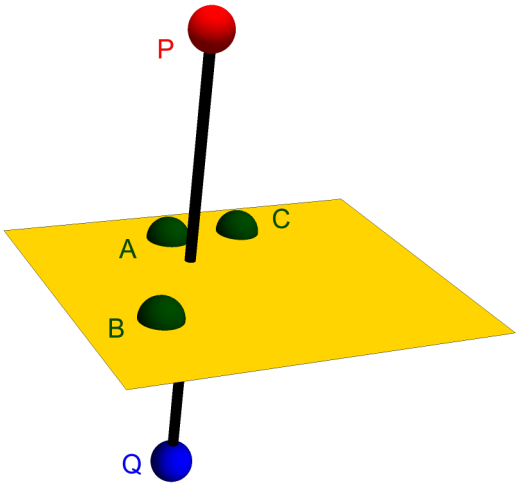
Problem 6) (10 points)

a) (2 points) Find the equation $ax+by+cz=d$ of the plane through $A=(1,1,1), B=(3,4,5), C=(4,4,2)$.

b) (3 points) Compute the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$.

c) (3 points) Determine the volume of the parallelepiped spanned by $\vec{AB}, \vec{AC}, \vec{AP}$ where $P=(1,3,4)$.

d) (2 points) Find the distance $|\vec{PQ}|$, where Q is the mirror image of P opposite of the plane. It is determined by the fact that the middle point $(P+Q)/2$ is on the plane and that $\vec{PQ}$ is perpendicular to the plane.



Solution:

a) The normal vector is $\vec{n} = \vec{AB} \times \vec{AC} = \langle 2,3,4\rangle \cdot \langle 3,3,1\rangle = \langle -9,10,-3\rangle$. The equation is $-9x+10y-3z=d$. The constant is obtained by plugging in a point. It is $\boxed{-9x+10y-3z=-2}$.

b) The area of the parallelogram is the length of the vector $\vec{n} = \langle -9,10,-3\rangle = \boxed{\sqrt{190}}$.

c) The volume of the parallel epiped is $\vec{AP} \cdot \vec{n} = \boxed{11}$.

d) The distance is volume divided by area which is $11/\sqrt{190}$. The distance between P and Q is $\boxed{22/\sqrt{190}}$.

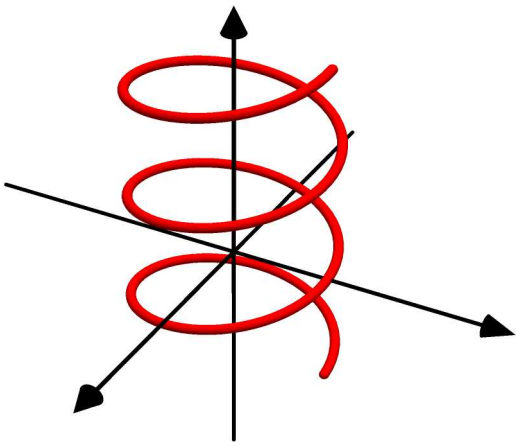
Problem 7) (10 points)

The triple scalar product is also written as

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}).$$

The **torsion** of a space curve is defined as

$$\frac{[\vec{r}\,', \vec{r}\,'', \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2}.$$



a) (3 points) Compute $\vec{r}\,'(0), \vec{r}\,''(0), \vec{r}\,'''(0)$ for

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle.$$

b) (4 points) Compute the torsion of the curve at the point $\vec{r}(0)$.

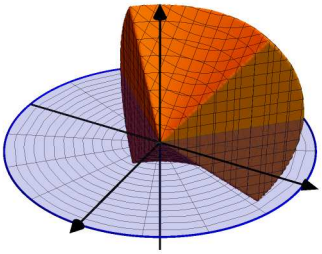
c) (3 points) Assume you have an arbitrary curve $\vec{r}(t)$ which is contained in the xy-plane. What is its torsion?

Solution:
a) $\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$.
$\vec{r}\,''(0) = \langle -1, 0, 0 \rangle$.
$\vec{r}\,'''(0) = \langle 0, -1, 0 \rangle$.
b) The triple scalar product is 1. The cross product $|\vec{r}\,' \times \vec{r}\,''|^2 = |\langle 0, -1, 1 \rangle|^2 = 2$. The torsion is $\boxed{1/2}$.
c) If $\vec{r}(t) = \langle x(t), y(t), 0 \rangle$, then $\vec{r}\,'(t) \times \vec{r}\,'(t) = \langle 0, 0, x'y'' - x''y' \rangle$. On the other hand, $\vec{r}\,'''(t) = \langle x'''(t), y'''(t), 0 \rangle$. The dot product is $\boxed{\text{zero}}$.
P.S. Also more intuitive explanations worked: the three derivatives are all in the plane. They span a parallelepiped of zero volume. Therefore, the torsion, which has the volume in the nominator, is zero.

Problem 8) (10 points)

Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$



a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \, dxdydz.$$

b) (3 points) Let $\vec{F}$ be a vector field

$$\vec{F} = \langle x^3, y^3, z^3 \rangle.$$

Find the flux of $\vec{F}$ through the boundary surface of E, oriented outwards.

Solution:
a) The region is best described in **spherical coordinates**. The ϕ angle goes from $\pi/4$ to $3\pi/4$. The θ angle goes from $\pi/4$ to $3\pi/4$. The radius ρ goes from zero to 3. The integral is

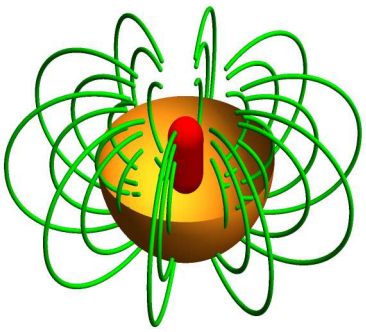
$$\int_{\pi/4}^{3\pi/4} \int_{\pi/4}^{3\pi/4} \int_0^3 \rho^2 \rho^2 \sin(\phi) \, d\rho d\phi d\theta \, .$$

The answer is $(\pi/2)(3^5/5)\sqrt{2} = \boxed{243\pi\sqrt{2}/10}$.
b) Since the divergence is $3x^2 + 3y^2 + 2z^2$ the result is just three times the result found in a). It is $\boxed{729\pi\sqrt{2}/10}$.

Problem 9) (10 points)

The vector field

$$\vec{A}(x, y, z) = \frac{\langle -y, x, 0 \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$



is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards.

Solution:
Since we have an integral of the curl of the vector field $\vec{A}$, we use **Stokes theorem** and integrate $\vec{A}(\vec{r}(t))$ along the boundary curve $\vec{r}(t) = \langle \cos(t), -\sin(t), 0 \rangle$. First of all, we have $\vec{A}(\vec{r}(t)) = \langle \sin(t), \cos(t), 0 \rangle$. The velocity is $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. The integral is $\int_0^{2\pi} -1 \, dt = -2\pi$. The answer is $\boxed{-2\pi}$.

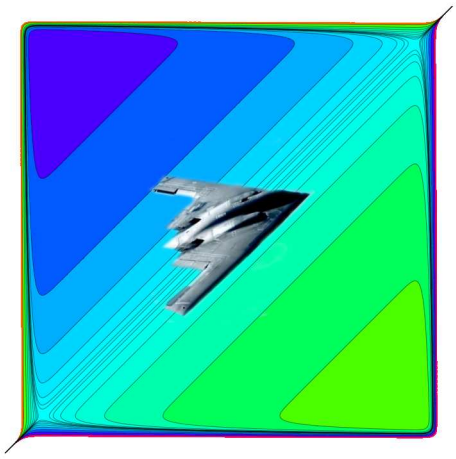
Problem 10) (10 points)

a) (8 points) Classify the critical points of the **area 51** function

$$f(x, y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.

b) (2 points) Does the function have a global maximum or global minimum on the region $x^2 + y^2 \leq 1$ including the boundary? Write "yes" or "no" with a brief explanation. There is no need to find the global extrema.



Solution:
a) Setting the gradient $\nabla f = \langle 51x^{50} - 51, 51y^{50} - 51 \rangle$ to $\langle 0, 0 \rangle$ gives the solutions $x = \pm 1, y = \pm 1$ and so the solutions $(1, 1), (-1, 1), (1, -1), (-1, -1)$. We have $f_{xx} = 50 \cdot 51x^{49}$ and $f_{yy} = -50 \cdot 51y^{49}$ and $f_{xy} = 0$ so that $D = f_{xx}f_{yy} - f_{xy}^2 = -(50 \cdot 51)^2 x^{49} y^{49}$. This is negative if the two coordinates have the same sign and positive else. Furthermore $f_{xx} > 0$ if x is positive. Therefore $\boxed{(1, 1), (-1, -1) \text{ are saddle points}}$ and $\boxed{(-1, 1) \text{ is the maximum}}$ and $\boxed{(1, -1) \text{ is the minimum}}$.
b) Yes by $\boxed{\textbf{Bolzano}}$, the continuous function f has both a maximum and minimum on the closed disc $x^2 + y^2 \leq 1$ as this region is both bounded and closed.

Problem 11) (10 points)

Using the Lagrange method, find the maximum and minimum of the elliptic curve function

$$f(x,y) = y^2 - x^3 - x^2 - x$$

on the circle $g(x,y) = x^2 + y^2 = 1$.



This problem is motivated from a real life application. To encrypt communication in **"WhatsApp"**, the elliptic curve 25519 given by $y^2 = x^3 + 486662x^2 + x$ over the prime $p = 2^{255} - 19 =$

57896044618658097711785492504343953926634992332820282019728792003956564819949

is used.

Solution:
The Lagrange equations are

$$\begin{aligned} -3x^2 - 2x - 1 &= \lambda(2x) \\ 2y &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned}$$

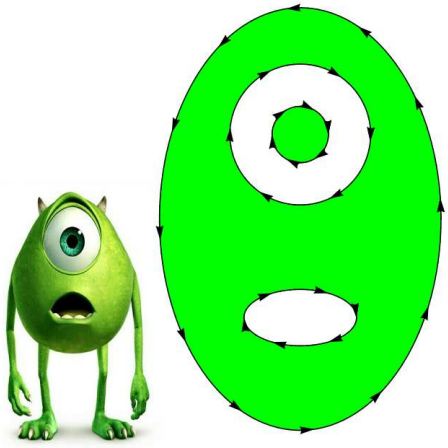
Eliminating λ gives $2y(-3x^2 - 2x - 1) = 2x2y$. The first possibility is that $y = 0$. In that case, $x = \pm 1$. If y is not zero, then we can divide it out and get $-3x^2 - 2x - 1 = 2x$ which is equivalent to $3x^2 + 4x + 1 = (3x + 1)(x + 1)$. We had already $x = -1$ so that the only other solution is $x = -1/3$ which gives $y = \pm\sqrt{8}/3$. Now we have all the critical points. $(1,0), (-1,0), (-1/3, \sqrt{8}/3), (-1/3, -\sqrt{8}/3)$. To find the maximum and minimum, we evaluate the function f at the critical points. The maximum is at $\boxed{(-1/3, \pm\sqrt{8}/3)}$. The minimum is at $\boxed{(1,0)}$.

Problem 12) (10 points)

Given the scalar function $f(x,y) = x^5 + xy^4$, compute the line integral of

$$\vec{F}(x,y) = \langle 5y + 3y^2, 6xy + y^4 \rangle + \nabla f$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. The picture describes the orientations of the boundary curves perfectly and they are as they are! We warn you not to ask about this, or else we will bring in "Mike" from **Monsters, Inc.**



Solution:
The curl is -5. By **Green's theorem** we just have to compute the right areas. If the eye of Mike were oriented differently, we would get the area itself which is $16-(2-1)-3=12$ and the answer would be -60. However, since the inner eye is oriented wrong, that area is counted negatively. So instead of adding 1 we subtract 1 for that line integral contribution. The answer is $(-5)[16-2-3-1] = \boxed{-50}$.

Problem 13) (10 points)

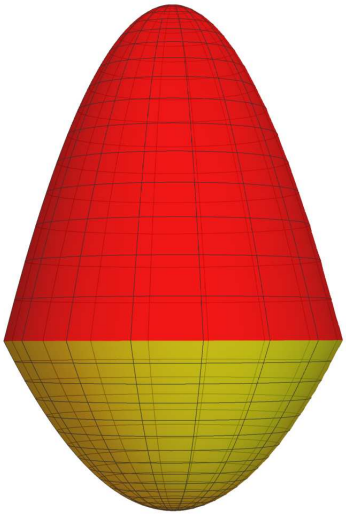
"ProtEgg" is a defense spell. It produces an egg shaped solid E enclosed by the surfaces

$$S : z = 2 - 2x^2 - 2y^2, z \geq 0 \ ,$$

where S is oriented upwards and

$$T : z = x^2 + y^2 - 1, z \leq 0 \ ,$$

where T is oriented downwards.



a) (4 points) Find the volume $\iiint_E 1 \, dxdydz$ of E.

b) (4 points) The spell uses a force field

$$\vec{F}(x, y, z) = \langle 0, 0, 2x^2 + 2y^2 + z \rangle \ .$$

S is parametrized by $\vec{r}(u, v) = \langle u, v, 2 - 2u^2 - 2v^2 \rangle$ with $u^2 + v^2 \leq 1$ oriented upwards. Compute the flux $\iint_S \vec{F} \cdot d\vec{S}$ without an integral theorem.

c) (2 points) The flux $\iint_T \vec{F} \cdot d\vec{S}$ can be determined using an integral theorem. What is the value of the flux? Check all that apply:

A	B	C	D
$\iint_S \vec{F} \cdot d\vec{S}$	$-\iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV + \iint_S \vec{F} \cdot d\vec{S}$	$\iiint_E 1 \, dV - \iint_S \vec{F} \cdot d\vec{S}$

Solution:
a) The region E is described by $x^2+y^2 \le 1, \quad x^2+y^2-1 \le z \le 2-2x^2-2y^2$ which is in polar coordinates:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad r^2-1 \le z =\le 2-2r^2.$$

$$\begin{aligned} \iiint_E 1\,dx\,dy\,dz &= \int_0^{2\pi}\int_0^1\int_{r^2-1}^{2-2r^2} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (3-3r^2)r\,dr\,d\theta = \frac{3\pi}{2}. \end{aligned}$$

The answer is $\boxed{\frac{3\pi}{2}}$.
b) Since $\vec{r}_u = \langle 1, 0, -4u\rangle$ and $\vec{r}_v = \langle 0, 1, -4v\rangle$, we have $\vec{r}_u \times \vec{r}_v = \langle 4u, 4v, 1\rangle$ and

$$\begin{aligned} \iint_S \vec{F}\cdot d\vec{S} &= \iint_{u^2+v^2\le 1} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times = \vec{r}_v)\,du\,dv \\ &= \iint_{u^2+v^2\le 1} 2\,du\,dv = 2\pi. \end{aligned}$$

The answer is $\boxed{2\pi}$.
c) Since S and T bound the solid E, by the **divergence theorem** we have

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_T \vec{F}\cdot d\vec{S} = \iiint_E \mathrm{div}\vec{F}\,dV = \iiint_E 1\,dV.$$

This implies that

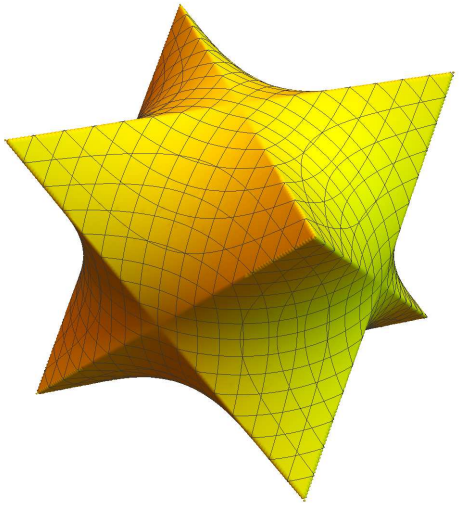
$$\iint_T \vec{F}\cdot d\vec{S} = \iiint_E \mathrm{div}\vec{F}\,dV = \iiint_E 1\,dV - \iint_S = \vec{F}\cdot d\vec{S}.$$

Only answer $\boxed{\text{D}}$ is correct.

Problem 14) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 \ .$$



Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

$$\vec{F}(x,y,z) = \langle 2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z \rangle$$

through the boundary surface S of R, assuming that S is oriented outwards.

Solution:
This is a problem for the divergence theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \mathrm{div}(\vec{V}) \, dV \ .$$

As the divergence is constant 6, the result will be 6 times the volume of the solid. It is $\iiint_E 6dV = 6$ Volume(E) $= \boxed{6\log(256)}$.

12/15/2017 FINAL PRACTICE II **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The parametrization $\vec{r}(u,v) = \langle v^2, v^2\cos(u), v^2\sin(u)\rangle$ describes a cone.

Solution:
Indeed $y^2 + z^2 = x^2$

2) [T] [F] If three vectors $\vec{u}, \vec{v}, \vec{w}$ satisfy $\vec{u}\cdot\vec{v} = 0$ and $\vec{v}\times\vec{w} = \vec{0}$, then $\vec{u}\cdot\vec{w} = 0$.

Solution:
One could think that if the vectors v, w are parallel and $u \cdot v = 0$, then $u \cdot w = 0$. The answer however is false because if $v = 0$ and $u = w = \langle 1, 0, 0\rangle$ then $u\cdot v = 0$ and $v \times w = 0$, but u and w are not orthogonal.

3) [T] [F] Let S be a surface bounding a solid E and $\vec{F}$ is a vector field in space which is incompressible, $\text{div}(\vec{F}) = 0$, then $\iint_S \vec{F}\cdot d\vec{S} = 0$.

Solution:
By the divergence theorem.

4) [T] [F] If $\text{curl}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F}\cdot d\vec{S} = 0$ for any closed surface S.

Solution:
The line integrals would all be zero

5) [T] [F] If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ has zero curl everywhere.

Solution:
Yes, since $\text{curl}(\text{grad}(f)) = 0$

6) [T] [F] If $\vec{F}, \vec{G}$ are two vector fields which have the same divergence then $\vec{F} - \vec{G}$ is constant.

Solution:
Take any two fields which are the curl of two fields.

7) [T] [F] The linearization of the constant function $f(x, y) = 3$ at $(x, y) = (1, 1)$ is the function $L(x, y) = 0$.

Solution:
The linearization is 3.

8) [T] [F] The surface area of a surface S is $\int \int_S \langle x, y, z \rangle \cdot d\vec{S}$.

Solution:
This only would be true for the unit sphere

9) [T] [F] There is a non-constant vector field $\vec{F}(x, y, z)$ such that $\text{curl}(\vec{F}) = \text{curl}(\text{curl}(\vec{F}))$

Solution:
Take a gradient field or the ¡cos(y),0,cos(y)¿

10) [T] [F] If $\vec{F}$ is a vector field and E is the unit ball then $\iiint_E \text{div}(\text{curl}(\vec{F}))\, dV = 0$.

Solution:
Because $\text{div}(\text{curl}(\vec{F}) = \vec{0}$.

11) [T] [F] If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is zero.

Solution:
The flux is the volume.

12) [T] [F] The equation $\text{grad}(\text{div}(\text{grad}(f))) = \vec{0}$ always holds.

Solution:
Take $f = x^3$, then the result is $\langle 1, 0, 0\rangle$.

13) [T] [F] The vector $\vec{k} \times (\vec{j} \times \vec{i})$ is the zero vector, if $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \langle 0, 0, 1\rangle$.

Solution:
$\vec{j} \times \vec{i} = -\vec{k}$ which is parallel to $\vec{k}$.

14) [T] [F] If f is minimized at (a, b) under the constraint $g = c$, then $\nabla f(a, b)$ and $\nabla g(a, b)$ are perpendicular.

Solution:
They are parallel

15) [T] [F] If A, B, C, D are four points in space such that the line through A, B intersects the line through C, D, then A, B, C, D lie on some plane.

Solution:
Yes, it is the plane spanned by the two lines.

16) [T] [F] The chain rule assures that for a vector field $\vec{F} = \langle P, Q, R\rangle$ the formula $\frac{\partial}{\partial u}\vec{F}(\vec{r}(u, v)) = \langle \nabla P \cdot \vec{r}_u, \nabla Q \cdot \vec{r}_u, \nabla R \cdot \vec{r}_u\rangle$ holds.

Solution:
Apply the chain rule to each component.

17) [T] [F] The vector field $\vec{F} = \langle \cos(y), 0, \sin(y)\rangle$ satisfies $\text{curl}(\vec{F}) = \vec{F}$. By the way, it is called the Cheng-Chiang field.

Solution:
Compute the curl and see that the fixed point property holds. Cheng-Chiang is is a Harvard PhD who now is at MIT.

18) [T] [F] The unit tangent vector $\vec{T}(t)$, the normal vector $\vec{N}(t)$ and the binormal vector $\vec{B}(t)$ for a given curve $\vec{r}(t)$ span a cube of volume 1 at $t = 1$.

Solution:
We have to normalize

19) [T] [F] The vector field $\vec{F}(x, y, z) = \langle z, z, z \rangle$ can not be the curl of a vector field.

Solution:
Its divergence is not zero

20) [T] [F] The expression curl(grad(div(grad(div(curl($\vec{F}$)))))) is a well defined vector field in three dimensional space.

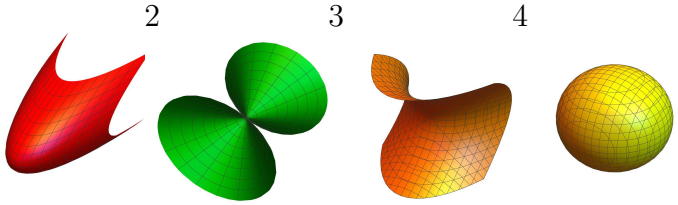
Solution:
This is the implicit equation.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = \langle u, u^2 - v^2, v\rangle$	
$y - x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u, u\sin(v)\rangle$	
$x^2 + y^2 = 1 - z^2$	

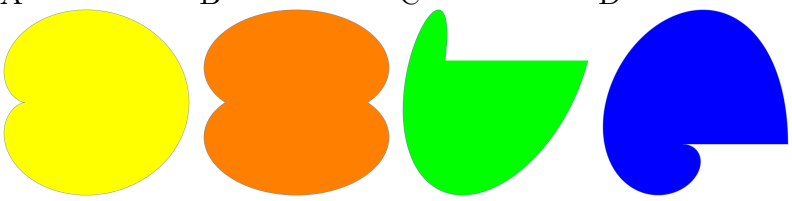
1 2 3 4



b) (2 points) Match the following regions given in polar coordinates (r, θ):

Region	A-D
$r < \theta^2$	
$r < 1 + \cos(\theta)$	
$r < 1 + \lvert\sin(\theta)\rvert$	
$r < (4\pi^2 - \theta^2)$	

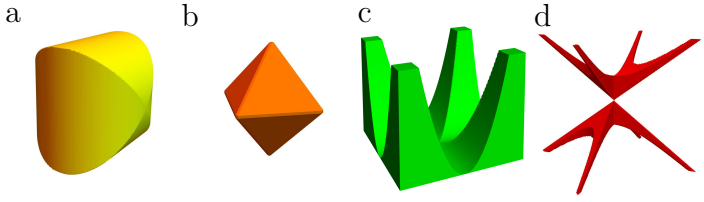
A B C D



c) (2 points) Match the regions. There is an exact match.

Solid	a-d
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert \leq 1$	
$x^2 + y^2 \leq 1, x^2 + z^2 \leq 1$	
$x \leq y^2, x \leq z^2$	
$0 \leq x^2 - y^2 \leq 4, 0 \leq y^2 - z^2 \leq 4$	

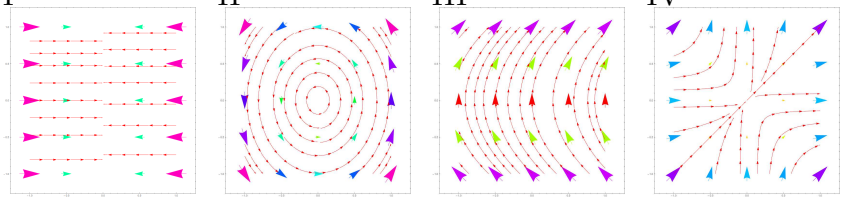
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle y, 1\rangle$	
$\vec{F}(x,y) = \langle -2y, 3x\rangle$	
$\vec{F}(x,y) = \langle -x, 0\rangle$	
$\vec{F}(x,y) = \langle x^2, y^2\rangle$	

I II III IV



e) (1 point) Find the linearization $L(x,y)$ of $f(x,y) = xy$ at $(2,1)$.

f) (1 point) Write down the wave equation for a function $f(t,x)$:

Solution:
a) 3,1,2,4
b) C,A,B,D
c) b,a,c,d
d) III,II,I,IV
e) $2 + (x-2) + 2(y-1)$.
f) $f_{tt} = f_{xx}$.

Problem 3) (10 points)

a) (6 points) In the following, f is a function, $\vec{F}$ a vector field, I is an interval, G is a region in the two dimensional plane, S is a closed surface parametrized by $\vec{r}(u,v)$, C is a closed curve parametrized by $\vec{r}(t)$ and E is a solid. Fill the blanks using "volume", "area", "length" or "0", where a choice can appear a multiple times.

$$\boxed{} = \int\int_G \operatorname{curl}(\langle x, 0\rangle)\, dxdy$$

$$\boxed{} = \int\int_G \operatorname{curl}(\langle -y, 0\rangle)\, dxdy$$

$$\boxed{} = \int\int\int_E \operatorname{div}(\langle x, x, x\rangle)\, dxdydz$$

$$\boxed{} = \int\int_S |\vec{r}_u \times \vec{r}_v|\, dudv$$

$$\boxed{} = \int_C |\vec{r}\,'(t)|\, dt + \int_C \operatorname{grad}(f)\ \cdot d\vec{r}$$

$$\boxed{} = \int\int_S \operatorname{curl}(\vec{F})\ \cdot d\vec{S}$$

b) (4 points)
Complete the formulas of the following boxes

$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}$ [ ] $d\rho d\phi d\theta$ Integral for volume of unit ball $x^2+y^2+z^2 \le 1$.

$\int_0^{2\pi}\int_0^{1}$ [ ] $drd\theta$ Integral for area of unit disc $x^2+y^2 \le 1$

$g_x(x,y) = \frac{\boxed{}}{f_z(x,y,z)}$ Implicit derivative of g satisfying $f(x,y,g(x,y)) = 0$

$|\vec{u} \cdot (\boxed{})|$ Volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Solution:
a) 0, area, volume, area, length, 0
b) $\rho^2 \sin(\phi)$, r, $-f_x(x,y,z)$, and $\vec{v} \times \vec{w}$

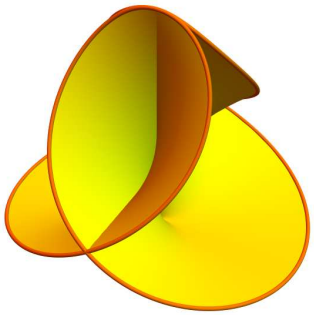
Problem 4) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), \frac{v^2}{2} \rangle \ ,$$

where (u,v) are in the domain $u^2+v^2 \le 9$.

P.S. You do not have to worry that this cool surface has self-intersections.



Solution:

$$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$$
$$\vec{r}_v = \langle -u\sin(v), u\cos(v), v \rangle \ .$$

The cross product

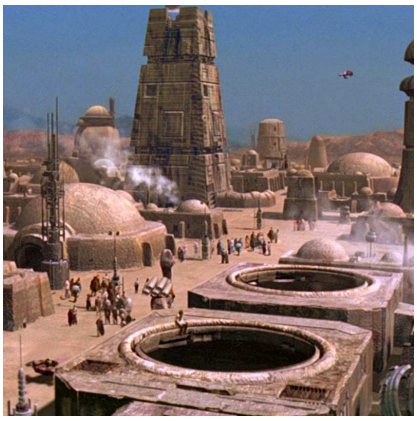
$$\vec{r}_u \times \vec{r}_v = \langle v\sin(v), -v\cos(v), u \rangle$$

has length $\sqrt{u^2+v^2}$. Now we can compute the integral

$$\int\int_R |\vec{r}_u \times \vec{r}_v| \ dudv = \int_0^{2\pi}\int_0^3 r \ r \ drd\theta = 18\pi \ .$$

Problem 5) (10 points)

On planet **Tatooine**, Luke Skywalker travels along a path C parametrized by



$$\vec{r}(t) = \langle t\cos(t), t\sin(t), 0 \rangle$$

from $t = 0$ to $t = 2\pi$. What is the work done

$$\int_C \vec{F} \cdot \vec{dr}$$

by the "force"

$$\vec{F} = \langle x^2 + y + z, y^3 + x, z^5 + x \rangle \ .$$

Solution:
This is a problem for the fundamental theorem of line integrals. It is applicable because the curl of $\vec{F}$ is zero. The potential f can be obtained by integration. We have

$$f(x,y,z) = x^3/3 + y^4/4 + z^6/6 + xy + xz \ .$$

Now, we just have to plug in the end point $\vec{r}(2\pi) = \langle 2\pi, 0, 0 \rangle$ and initial point $\langle 0,0,0 \rangle$ and get

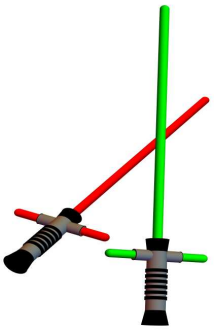
$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \ \vec{r}\,''(t) \ dt = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 8\pi^3/3 \ .$$

Problem 6) (10 points)

Tomorrow, on December 18, the "force awakens". There will be light sabre battles, without doubt.

a) (7 points) What is the distance between two light sabres given by cylinders of radius 1 around the line $\vec{r}(t) = \langle t, -t, t \rangle$ and the line connecting $(0, 14, 0)$ with $(3, 5, 6)$.

b) (3 points) A spark connects the two points of the sabre which are closest to each other. Find a vector in that direction.



Sabre by Connor (mathematica project).

Solution:
a) First compute the distance between the central lines. They contain the vectors $\langle 1, -1, 1 \rangle$ and $\langle 1, -3, 2 \rangle$. The cross product is $\vec{n} = \langle 1 - 1, -2 \rangle$. Pick a point $Q = (0, 14, 0)$ on one curve and $P = (0, 0, 0)$. Now

$$d = |\vec{PQ} \cdot \langle 1, -1, -2 \rangle| / |\langle 1, -1, -2 \rangle| = 7\sqrt{6}/3$$

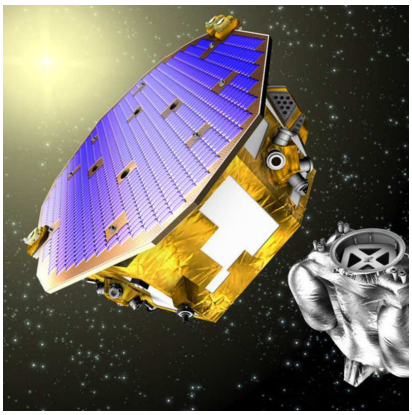
The distance between the cylinders is $7\sqrt{6}/3 - 2$.
b) The vector $\langle 1, -1, -2 \rangle$ found in a) is perpendicular to both lines. It already gives the connection between the two points!

Problem 7) (10 points)

100 years ago, Einstein proposed gravitational waves. To measure them, the LISA pathfinder was launched on December 3, 2015. It carries two cubes to the Lagrangian point between Earth and Moon aiming to measure the waves. Assume a gravitational wave from a black hole merger produces a force leading to an acceleration

$$\vec{r}\,''(t) = \langle \sin(t), \cos(t), \sin(t) \rangle \ .$$

What is $\vec{r}(t)$ at time $t = \pi$ if $\vec{r}\,'(0) = \langle 2, 0, 0 \rangle$ and $\vec{r}(0) = \langle 1, 0, 3 \rangle$.



LISA proof of concept will be followed by eLISA in 2034.

Solution:
Integrate

$$\vec{r}\,''(t) = \langle \sin(t), \cos(t), \sin(t) \rangle$$

to get

$$\vec{r}\,'(t) = \langle 3 - \cos(t), \sin(t), 1 - \cos(t) \rangle \ .$$

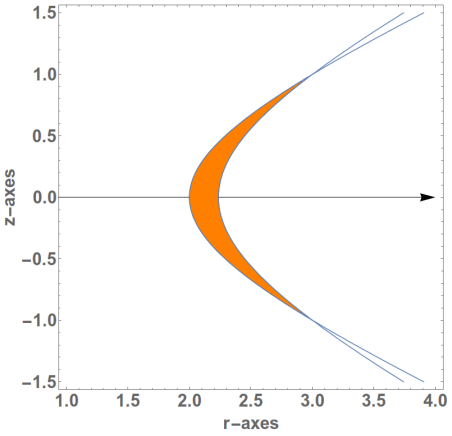
Now integrate again to get

$$\vec{r}\,(t) = \langle 1 + 3t - \sin(t), 1 - \cos(t), 3 + t - \sin(t) \rangle \ .$$

We have $\vec{r}(\pi) = \langle 1 + 3\pi, 2, 3 + \pi \rangle$.

Problem 8) (10 points)

a) (5 points) Find the integral $\int\int_G r \ drdz$, where G is the region enclosed by the curves $r^2 - 4z^2 = 5$ and $r^2 - 5z^2 = 4$ and contained in $r \geq 0$.



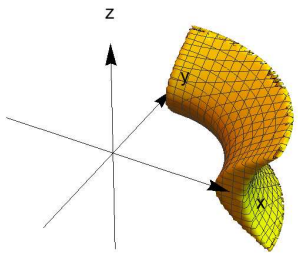
b) (5 points) The Galactic Empire builds a space craft E given as a solid in $x \geq 0, y \geq 0$, enclosed by

$$x^2 + y^2 - 4z^2 = 5$$

and

$$x^2 + y^2 - 5z^2 = 4 \ .$$

Find its volume. P.S. You can make use of problem a) to solve part b) as the problems are related.



Solution:
a) $\int_{-1}^{1}\int_{\sqrt{4+5z^2}}^{\sqrt{5+4x^2}} r dr dz = \int_{-1}^{1}(5+4z^2-4-5z^2)/2\, dz = 2/3.$
b) E is described in terms of cylindrical coordinates by

$$-1 \le z \le 1, \qquad \sqrt{4z^2+5} \le r \le \sqrt{5z^2+4}, \qquad 0 \le \theta \le 2\pi.$$

Hence, the volume of E is

$$\begin{aligned} &= \iiint_E 1\, dV \\ &= \int_{-1}^{1}\int_{0}^{\pi/2}\int_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} r\, dr\, d\theta\, dz \\ &= \int_{-1}^{1}\int_{0}^{\pi/2} \frac{r^2}{2}\Big|_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} dr\, d\theta\, dz \\ &= \pi/3\ . \end{aligned}$$

Since a) and b) are related, one an also use the result in a) directly. The volume is $\int_0^{\pi/2}\int_{-1}^{1} \frac{r^2}{2}\Big|_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} dr\, dr\, d\theta$. The inner integral is $2/3$ from a). So, the result is $\pi/2$ times $2/3$ which is $\pi/3$.

Problem 9) (10 points)



In September 2015, the west side of the Harvard Science center honored the concept of curl by displaying paddle wheels. One of the wheel tips moves on an oriented curve $\vec{r}(t) = \langle \cos(t), 0, \sin(t)\rangle$ bounding the disc parametrized by $\vec{r}(u,v) = \langle u\cos(v), 0, u\sin(v)\rangle$ with $0 \le u \le 1, 0 \le v \le 2\pi$. Let $\vec{F}$ be the wind vector field

$$\vec{F}(x,y,z) = \langle x^9 + y^7 + 3z, x^9 + y^9 + \sin(z), z^5 e^z\rangle\ .$$

Find the line integral $\int_C \vec{F}\cdot d\vec{r}$ measuring the energy gain during one rotation along the curve C parametrized by $\vec{r}(t)$.

Solution:
We use Stokes theorem with a disc surface bounded by the curve C. This surface can be parametrized by

$$\vec{r}(u,v) = \langle u, 0, v\rangle, \vec{r}_u \times \vec{r}_v = \langle 0, -1, 0\rangle \ ,$$

and where the parameter domain is $R : u^2 + v^2 \leq 1$. The curl is

$$\text{curl}(\vec{F})(x,y,z) = \langle -\cos(z), 3, 9x^8 - 5y^4\rangle \ .$$

The flux of the curl through the disc R

$$\int\int_R \langle -\cos(v), 3, 9u^8\rangle \cdot \langle 0, -1, 0\rangle \ dudv \ ,$$

which we can compute by using polar coordinates

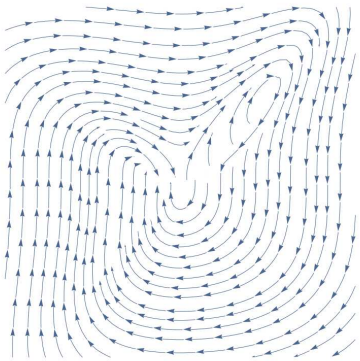
$$\int\int_R (-3) dudv = \int_0^{2\pi}\int_0^1 (-3r) \ drd\theta = -3\pi \ .$$

This is already the right sign as the parametrization of the surface and the line was already given in a compatible form.

Problem 10) (10 points)

The value of the line integral of the vector field $\vec{F}(u,v) = (2/\pi)\langle -uv^2 + v^3, uv - u^3\rangle$ along a curve $\vec{r}(t) = \langle x + \cos(t), y + \sin(t)\rangle$ depends only on the center point (x,y) and is given by

$$f(x,y) = -3 - 6x^2 + 2y + 4xy - 6y^2 \ .$$



a) (7 points) Find all critical points (x,y) for the function $f(x,y)$ and analyze them using the second derivative test.
b) (3 points) Given that

$$f(x,y) = -3 - (x-2y)^2 - 5x^2 - 2y^2 + 2y \ ,$$

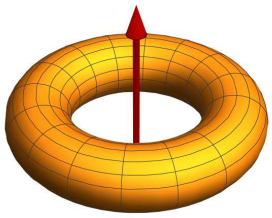
decide whether there is a global maximum for f.

Solution:
a) The gradient is $\langle -12x + 4y, 2 + 4x - 12y\rangle$ which when put to zero $\langle 0, 0\rangle$ gives $y = 3/16$ and $x = 1/16$. The discriminant is 128 and $f_{xx} = -12$. The function f has therefore a local maximum at $(1/16, 3/16)$. The value is $-(45/16)$.
b) Completing the square gives $f(x,y) = -2 - (x-2y)^2 - 5x^2 - y^2 - (y-1)^2$. Since this is a sum of negative squares, this goes to $-\infty$ in any direction. The maximum found in a) is therefore a global maximum. (P.S. there are functions of two variables with one critical point which is a local maximum without that this is a global maximum. So, the fact alone that we have only one critical point is not enough.)

Problem 11) (10 points)

The moment of inertia $f(x, y)$ of a torus of mass 4 with smaller tube radius x and bigger center curve radius y is

$$f(x, y) = 3x^2 + 4y^2 \ .$$



a) (7 points) Find the parameters (x_0, y_0) for the torus which have minimal moment of inertia under the constraint that

$$g(x, y) = x + 4y = 13 \ .$$

b) (3 points) Write down the equation of the tangent line to the level curve of f which passes through (x_0, y_0).

Solution:
a) This is a Lagrange problem. We have

$$\begin{aligned} 6x &= \lambda\, 1 \\ 8y &= \lambda\, 4 \\ x + 4y &= 13 \end{aligned}$$

Elminating λ gives $24x = 8y$ meaning $y = 3x$. Plugging into the constraint gives $x = 1, y = 3$.
b) Since the gradient of f is parallel to the gradient of g which we know and the tangent line through $(1, 3)$ is the constraint $x + 4y = 13$, the later equation is already the equation of the constraint.

Problem 12) (10 points)

A new **elliptical machine** has been designed to simulate running better. The leg of a runner moves on the curve parametrized by

$$\vec{r}(t) = \langle 8\cos(t), 2\sin(t) + \sin(2t) + \cos(2t) \rangle$$

with $0 \leq t \leq 2\pi$. Find the area of the region enclosed by the curve.



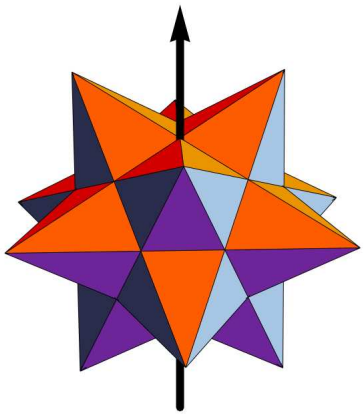
Solution:
We use Green's theorem with the vector field $\vec{F} = \langle 0, x \rangle$ which has curl 1. The line integral is

$$\int_0^{2\pi} \langle 0, 8\cos(t) \rangle \cdot \langle -8\sin(t), 2\cos(t) + 2\cos(2t) - 2\sin(2t) \rangle \, dt$$

This is $\int_0^{2\pi} 8\cos(t)(2\cos(t) + 2\cos(2t) - 2\sin(2t)) \, dt = 16\pi$.

Problem 13) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle y^5 + x, z^5 + y, x^5 + z \rangle$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = \langle x^3/3, y^3/3, 0 \rangle$$

through S?

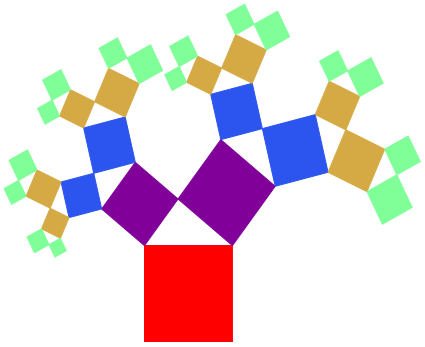
Solution:
a) We use the divergence theorem. The divergence of the vector field is 3. The flux therefore is 3 times the volume of the solid which is 30.
b) Again, we use the divergence theorem. The divergence is x^2+y^2. The integral $\int\int\int_E x^2+y^2 \, dxdydz$ is 1.

Problem 14) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = \langle -y + x^8, x - y^9 \rangle$$

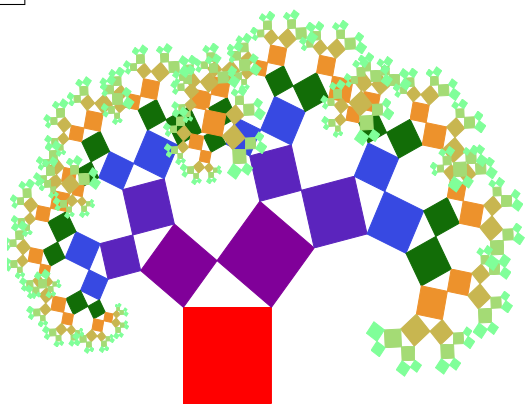
along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



Pythagoras tree theorem:

The generation n Pythagorean tree has area $n+1$.

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.



Solution:

We use the Green theorem. The curl of $\vec{F}$ is constant 2. The integral $\int\int_R \text{curl}(\vec{F})\, dxdy$ is therefore 2 times the area of R which is $2 \times 5 = \boxed{10}$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The parametrization $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$ describes a cone.

Solution:
It is a cylinder.

2) [T] [F] The vectors $\vec{v} = \langle 2, 1, 5 \rangle$ and $\vec{w} = \langle 2, 1, -1 \rangle$ are perpendicular.

Solution:
Their dot product is zero.

3) [T] [F] Let E be a solid region with boundary surface S. If $\iint_S \vec{F} \cdot d\vec{S} = 0$, then $\mathrm{div}(\vec{F})(x,y,z) = 0$ everywhere inside E.

Solution:
There could be a cancellation of divergence like for $\vec{F} = \langle x^2, 0, 0 \rangle$ with the unit sphere S.

4) [T] [F] If $\mathrm{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.

Solution:
This follows from the divergence theorem.

5) [T] [F] If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.

Solution:
The vector field $\vec{F}(x,y,z) = \langle x, 0, 0 \rangle$ is conservative but has divergence 1 everywhere.

6) [T] [F] If $\vec{F}, \vec{G}$ are two vector fields for which $\vec{F} - \vec{G} = \mathrm{curl}(\vec{H})$, then $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{G} \cdot d\vec{S}$ for any closed surface S.

Solution:
When adding two vector fields, the corresponding flux is added. Since the flux of the curl of a vector field through any closed surface is zero (follows from Stokes or from the divergence theorem), the flux of $\vec{F}$ and $\vec{G}$ is the same.

7) [T] [F] The linearization of $f(x,y) = x^2 + y^3 - x$ at $(2,1)$ is $L(x,y) = 4 + 4(x-2) + 3(y-1)$.

Solution:
Almost. It is just the constant which is wrong. The correct linearization is $L(x,y) = 3 + 4(x-2) + 3(y-1)$.

8) [T] [F] The volume of the solid E is $\int\int_S \langle x, 2x+z, x-y\rangle \cdot d\vec{S}$, where S is the surface of the solid E.

Solution:
Since the divergence of the vector field is 1, this follows from the divergence theorem.

9) [T] [F] The vector field $\vec{F}(x,y,z) = \langle x+y, x-y, 3\rangle$ has zero curl and zero divergence everywhere.

Solution:
yes, both are zero.

10) [T] [F] If $\vec{F}$ is a vector field, then the flux of the vector field curl(curl(F)) through a sphere $x^2 + y^2 + z^2 = 1$ is zero.

Solution:
By the divergence theorem and using the fact that the vector field is curl(G) for some other vector field $G = \text{curl}(F)$.

11) [T] [F] If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.

Solution:
It would follows from the divergence theorem if $\vec{F}$ were incompressible. It would also follow that the line integral along any closed curve is zero.

12) [T] [F] The equation div(grad(f)) = 0 is an example of a partial differential equation for an unknown function $f(x, y, z)$.

Solution:
If we write it out, it is.

13) [T] [F] The vector $(\vec{i}+\vec{j}) \times (\vec{i}-\vec{j})$ is the zero vector if $\vec{i} = \langle 1, 0, 0 \rangle$ and $\vec{j} = \langle 0, 1, 0 \rangle$.

Solution:
Either compute directly $\langle 1, 1, 0 \rangle \times \langle 1, -1, 0 \rangle = \langle 0, 0, -2 \rangle$ or foil out in your head to get the result is $-2\vec{i} \times \vec{j}$ which is equal to $-2\vec{k}$.

14) [T] [F] If f is maximal under the constraint $g = c$, then the angle between $\nabla f(x, y)$ and $\nabla g(x, y)$ is zero.

Solution:
It is zero or π. The second case is also possible.

15) [T] [F] Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P, L) \geq d(P, \Sigma)$.

Solution:
Yes, restricting to a line can make the distance not smaller.

16) [T] [F] The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.

Solution:
Yes. It is just the usual chain rule applied to the parametrization $\vec{r}\,'(t)$.

17) [T] [F] If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P, Q)$ is minimized.

Solution:
This point Q is the projection of the point onto the plane. Every other point has larger distance as we can draw a triangle which has a right angle at Q.

18) [T] [F] If $\vec{B}(t)$ is the bi-normal vector to an ellipse $\vec{r}(t)$ contained in the plane $x+y+z=1$, then $\vec{B}$ is parallel to $\langle 1,1,1\rangle$.

Solution:
It is always the normal vector

19) [T] [F] The parametrized surface $\vec{r}(u,v)=\langle u,v,u^2+v^2\rangle$ is everywhere perpendicular to the vector field $\vec{F}(x,y,z)=\langle x,y,x^2+y^2\rangle$.

Solution:
Since we do not see any theoretical reason why this should be true, lets experiment. At the point $\langle 1,1,2\rangle$, the vector field is $\langle 1,1,2\rangle$ and the normal vector to the surface is $\langle 1,0,2\rangle \times \langle 0,1,2\rangle = \langle -2,-2,1\rangle$. The dot product is not zero.

20) [T] [F] Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F}=\nabla f$. Then $\vec{r}\,'(t)=\vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

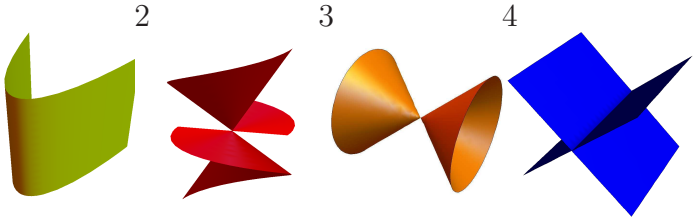
Solution:
By definition $\vec{r}'(t)=\vec{F}(\vec{r}(t))=\nabla f(\vec{r}(t))$. At a critical point, this is zero.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), uv\rangle$	
$\vec{r}(u,v) = \langle u, u^2, v\rangle$	
$x^2 - y^2 = z^2$	

1 2 3 4



b) (2 points) Match the expressions. There is an exact match.

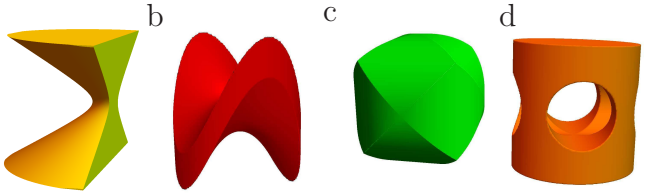
Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert\, dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$	
$\int_a^b \int_c^d \lvert\vec{r}_u \times \vec{r}_v\rvert\, dudv$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$0 \le x^2 - y^2 - z \le 1$	
$x^2 + y^2 - z^2 \le 1, x \ge 0$	
$x^2 + y^2 \le 3, y^2 + z^2 \ge 1, x^2 + z^2 \ge 1$	
$x^2 + y^2 \le 1, y^2 + z^2 \le 1, x^2 + z^2 \le 1$	

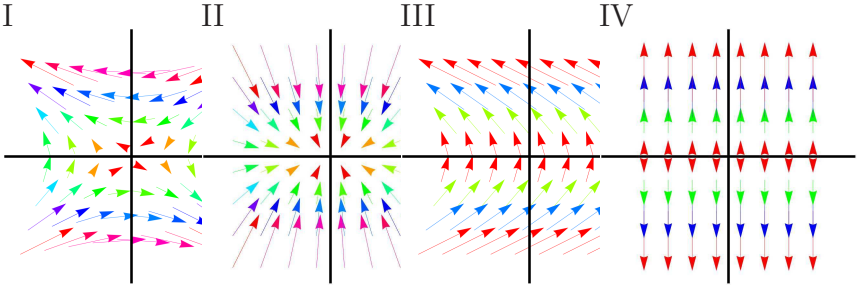
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle 0, 2y\rangle$	
$\vec{F}(x,y) = \langle -x, -2y\rangle$	
$\vec{F}(x,y) = \langle -2y, -x\rangle$	
$\vec{F}(x,y) = \langle -2y, 1\rangle$	

I II III IV



e) (2 points) Match the partial differential equations with formulas and functions $u(t,x)$. There is an exact match.

Equation	1-3	A-C
wave		
heat		
Laplace		

	Formulas
1	$u_{tt} = -u_{xx}$
2	$u_t = u_{xx}$
3	$u_{tt} = u_{xx}$

	Functions
A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

Solution:
a) 4,2,1,3
b) B,C,A,D
c) b,a,d,c
d) IV,II,I,III
e) 3,2,1 and B,C,A

Problem 3) (10 points)

a) (2 points) Mark the statement or statements which have a zero answer.

Statement	Must be zero
The curl of the gradient $\nabla f(x,y,z)$ at $(1,1,1)$	
The divergence of the curl $\nabla \times \vec{F}(x,y,z)$ at $(1,1,1)$	
The flux of a gradient field $\nabla f(x,y,z)$ through a sphere	
The dot product of $\vec{F}(1,1,1)$ with the curl$(F)(1,1,1)$	
The divergence of a gradient field $\nabla f(x,y,z)$ at $(1,1,1)$	

b) (2 points) Two of the following statements do not make sense. Recall that "incompressible" means zero divergence everywhere and that "irrotational" means zero curl everywhere.

Statement	Makes no sense
The discriminant of the gradient field	
A conservative and incompressible vector field	
The flux of the gradient of a function through a surface	
The gradient of the curl of a vector field	

c) (2 points) Match the following formulas with the geometric object they describe. Fill in the blanks as needed.

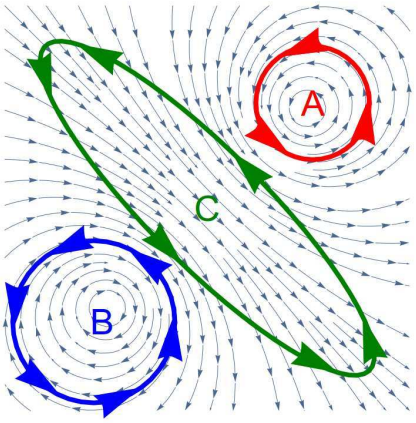
	Geometric object
A	unit tangent vector to a curve
B	unit normal vector to a surface
C	unit normal vector to a level curve
D	curvature of the curve
E	unit tangent vector to the surface

A-E	Formula
	$(\vec{r}_u \times \vec{r}_v)/\lvert\vec{r}_u \times \vec{r}_v\rvert$
	$\nabla f(x,y)/\lvert\nabla f(x,y)\rvert$
	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
	$(\vec{r}_u + \vec{r}_v)/\lvert\vec{r}_u + \vec{r}_v\rvert$
	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

d) (2 points) All three curves in the figure are oriented counter clockwise. Check whether in each of the three cases, the line integral is positive, negative or zero.

Check with a mark which applies. The line integral $\int_\gamma \vec{F} \cdot \vec{dr}$ is

γ	Positive	Negative	Zero
A			
B			
C			



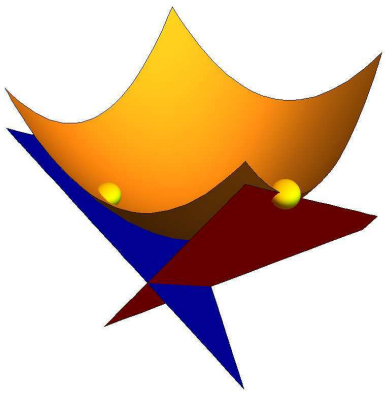
e) (2 points) Line or a plane? That is here the question.

U or V ?	Object
	$\vec{r}(0) + t\vec{B}(\vec{r}(0)) + s\vec{N}(\vec{r}(0))$
	$\vec{r}(0,0) + t\ \vec{r}_u \times \vec{r}_v$

	Dichotomy
U	is a normal plane to a curve
V	is a normal line to a surface

Solution:
a) The first two.
b) The first and last.
c) B,C,A,E,D
d) A - Positive, B - Negative, C - zero e) U,V

Problem 4) (10 points)



a) (6 points) Find the tangent planes at the points $A = (1,1,2)$ and $B = (-2,1,5)$ to the surface

$$x^2 + y^2 - z = 0\ .$$

b) (4 points) Find the parametric equation of the line of intersection of these two tangent planes.

Solution:
a) $\nabla f = \langle 2x, 2y, -1\rangle$.
We have $\nabla f(A) = \langle 2, 2, -1\rangle$ and $\nabla f(B) = \langle -4, 2, -1\rangle$. The equations of the tangent planes are therefore

$$2x + 2y - z = 2$$

and

$$-4x + 2y - z = 5 \ .$$

b) In order to find the intersection line, we have to find points P, Q on the intersection and form $\vec{r}(t) = \vec{OP} + t\vec{PQ}$. One can get points by plugging in one variable 0. For example when taking $y = 0$, one gets $P = (-1/2, 0, -3)$ or if $z = 0$, then $Q = (-1/2, 3/2, 0)$. Note that subtracting the two equations gives $x = -1/2$ so that we never can put x=0. Anyway, with the two points just found, we get

$$\vec{r}(t) = \langle -1/2, 0, -3\rangle + t\langle 0, 1, 2\rangle \ .$$

Of course there are many different ways to parametrize. A popular answer was also $\vec{r}(t) = \langle -1/2, 3/2 + 6t, 12t\rangle$, as many have computed a vector in the line using the cross product $\langle 2, 2, -1\rangle \times \langle -4, 2, -1\rangle = \langle 0, 6, 12\rangle$, which is an excellent way to get the line too. One still has the task to find a point on the intersection. An other popular intersection point was $(-1/2, 1, -1)$.

Problem 5) (10 points)

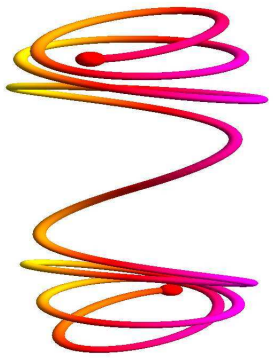
Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle 2x, 3y^3, 3z^2\rangle$$

along the curve

$$\vec{r}(t) = \langle \sin(t)\cos(t^3), \cos(t)\sin(t^3), t\rangle$$

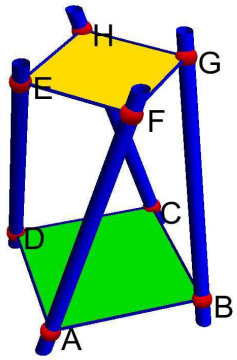
from $t = -\pi/2$ to $t = \pi/2$.



Solution:
This is a typical problem for the fundamental theorem of line integrals. A potential for the vector field is the scalar function $f(x, y, z) = x^2 + 3y^4/4 + z^3$. Now $A = \vec{r}(-\pi/2) = \langle -\cos(\pi^3/8), 0, -\pi/2\rangle$ and $B = \vec{r}(\pi/2) = \langle \cos(\pi^3/8), 0, \pi/2\rangle$ are the initial and end point. The result is $f(B) - f(A) = \pi^3/4$.

Problem 6) (10 points)

A monument has the form of a **twisted frustrum**. It is built from a base square platform $\{A = (-10, -10, 0), B = (10, -10, 0), C = (10, 10, 0), D = (-10, 10, 0)\}$ connected with an upper square $\{E = (-10, 0, 20), F = (0, -10, 20), G = (10, 0, 20), H = (0, 10, 20)\}$ using cylinders of radius 1. Find the distance between the pillar going through AF and the pillar going through BG.



Solution:
We have to compute the distance between the two lines and subtract the sum 2 of the two radii. The vector $\vec{v} = \vec{AF} = \langle 10, 0, 20\rangle$ is parallel to the first pillar. The vector $\vec{w} = \vec{BG} = \angle 0, 10, 20\rangle$ is parallel to the second pillar. The cross product $\vec{v} \times \vec{w}$ is $\langle -200, -200, 100\rangle$. The vector $\vec{AB} = \langle 20, 0, 0\rangle$ connects one line to the other. The distance between the center lines is given by the volume-area formula $|\vec{AB} \cdot (\vec{v} \times \vec{w})|/|\vec{v} \times \vec{w}|$ which is $40/3$. Now subtract the sum of the radii $1 + 1$ to get $\boxed{34/3} = 11.3333$.

Problem 7) (10 points)

The monkey saddle gets a cameo: find the volume of the solid contained in the cylinder

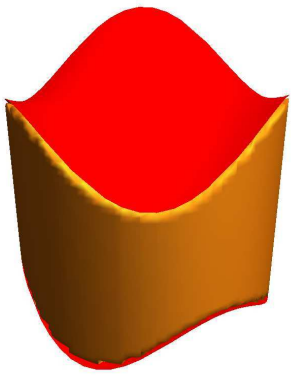
$$x^2 + y^2 < 1$$

above the surface

$$z = x^3 - 3xy^2 - 3$$

and below the surface

$$z = y^3 - 3yx^2 + 3\ .$$



Solution:
Set up the triple integral in cylindrical coordinates:

$$\int_0^{2\pi} \int_0^1 \int_{r^3\cos^3(\theta)-3r^3\cos(\theta)\sin^2(\theta)-3}^{r^3\sin^3(\theta)-3r^3\cos(\theta)\sin^2(\theta)+3} r \; dzdrd\theta$$

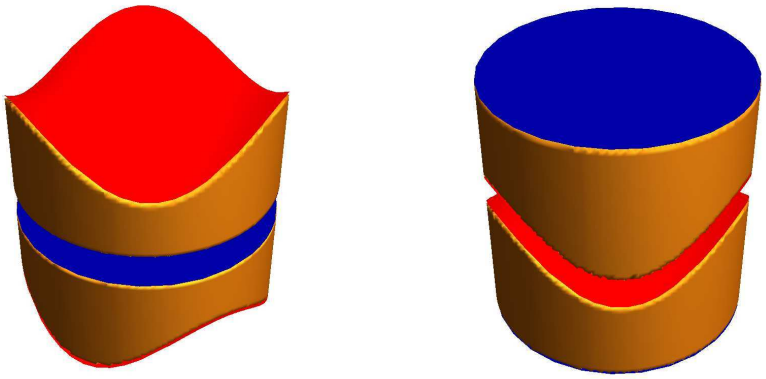
which is

$$\int_0^{2\pi} \int_0^1 6 + r^4(\cos^3(\theta) + 3\cos^2(\theta)\sin(\theta) + 3\cos(\theta)\sin^2(\theta) + \sin^3(\theta)) \; drd\theta \; .$$

One can integrate each term out, or use that each of the occuring trig functions have a symmetry rendering the integrals zero. In any case, we end up with

$$\int_0^{2\pi} \int_0^1 6 \; drd\theta = 6\pi \; .$$

You have seen all integrals which occur in homework: for example $\int \cos^3(t) \; dt = \int \cos(t)(1-\sin^2(t)) \; dt$ or $\int \sin^k(t)\cos(t) \; dt = \sin^{k+1}(t)/(k+1)$ for any integer $k \geq 0$, which if you don't see it directly can be done with the substitution $u = \sin(t)$.
P.S. A few students found an other cool way to compute the volume: cut the solid along the xy plane. Now take the part below and attach it above so that we end up with a cylinder of height 6. *This is how* **Archimedes** *would have done the computation if he would have known about the Monkey saddle.*

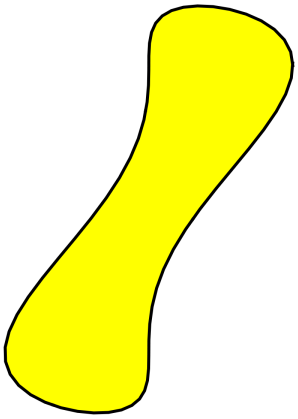


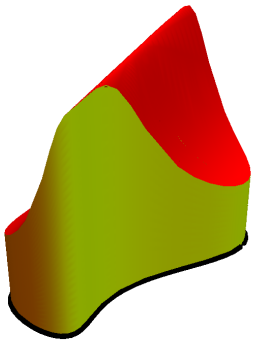
Problem 8) (10 points)

The volume of the solid above the region bound by the planar curve C : $\langle \cos^3(t), \sin(t) + \cos(t) \rangle$ and the graph of $f(x,y) = x^{-2/3}$ is given by the double integral

$$\int\int_G \frac{1}{\sqrt[3]{x^2}}\, dxdy \ .$$

Use the vector field $\vec{F} = \langle 0, 3x^{1/3} \rangle$ to compute this integral.





Solution:
This does at first not look like a Green's theorem problem. A hint was of course that a planar curve and a vector field in the plane was given. Since the curl is the function under consideration, by Greens theorem, we have to compute the line integral of the field $\vec{F}$ along the curve C. It is

$$\int_0^{2\pi} \langle 0, 3\cos(t) \rangle \cdot \langle 3\cos^2(t)\sin(t), \cos(t) - \sin(t) \rangle \ dt$$

which is

$$\int_0^{2\pi} 3\cos^2(t) + 3\cos(t)\sin(t) \ dt = 3\pi \ .$$

The answer is $\boxed{3\pi}$

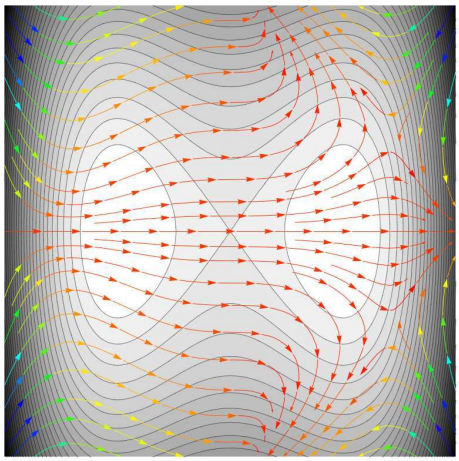
P.S. While we were proud of composing this problem, one discussion point in the team was whether the improper integral would be of any concern. Of course, mathematically there is no problem as $f(x) = x^{-2/3}$ is a function for which the improper integral $\int_{-a}^{b} f(x)\, dx$ exists for every $a < 0 < b$. (See Math 1a or Math 1b) As expected, no student worried about it and we did not expect this also, as the line integral is perfectly smooth. Mathematicians would justify the computation by smoothing out the function f at first (as we did to produce the picture) and then use Green, then take the limit. Integral theorems are a great way to regularize difficult integrals. We have seen in a homework that we can use Green theorem to compute the line integral along a fractal curve like the Koch curve, evenso this curve has infinite length and the line integral does not make sense on the Koch curve, the limit does. An other example is $\vec{F}(x,y) = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$, where the curl is not defined at the origin. It can be seen however as a "generalized function" which is localized at one point only satisfying that any integral containing this point gives one. Green-Stokes-Gauss can be pushed to this theory where it is a theory of "de Rham currents". Using powerful theorems to push computations into domains where it at first does not make sense is a common theme in mathematics and ads to its magic and power.

Problem 9) (10 points)

a) (8 points) The divergence of the vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle 1 - xy^2, 2 + 2yx^2 - yx^4\rangle$$

is the function $f(x,y) = P_x + Q_y$. Find and classify the critical points of $f(x,y)$.



b) (2 points) We have seen in general that the gradient field $\nabla f(x,y)$ is perpendicular to level curves $\{f(x,y) = c\}$ and that $\nabla f(x,y)$ is the zero vector at maxima or minima. Is the vector field $\vec{F}(x,y)$ zero at a maximum of $f(x,y) = \text{div}(F)(x,y)$?

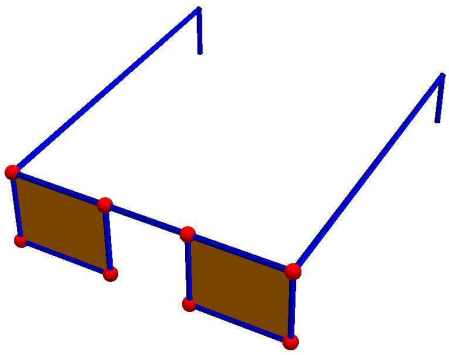
Solution:
a) The divergence is $f(x,y) = -y^2+2x^2-x^4$. The gradient is $\nabla f(x,y) = \langle 4x(1-x^2), -2y\rangle$. The critical points must satisfy $x = 0, \pm 1$ and $y = 0$. The critical points are $(0,0), (1,0)$ and $(-1,0)$. The second derivative test shows that $(0,0)$ is the saddle point and $(1,0), (-1,0)$ are maxima. We have $f_{xx} = 1 - 12x^2$ and $f_{yy} = -2$ and $f_{xy} = 0$ so that $D = -2(1 - 12x^2)$ is negative if $x = 0$ and positive $x = \pm 1$.

(x,y)	D	f_{xx}	Type	$f(x,y)$
(-1, 0)	16	-8	maximum	1
(0,0)	-8	4	saddle	0
(1,0)	16	-8	maximum	1

b) Lets just look at the vector field at a maximum like $(1,0)$. It is $\vec{F}(1,0) = \langle 1, 2\rangle$ which is not zero. The answer is No.

Problem 10) (10 points)

Economists know a **constraint duality principle**: "maximizing a first quantity while fixing the second is equivalent to minimizing the second when fixing the first". Let's experiment with that:



a) (5 points) Use the Lagrange method to find the **reading glasses** with maximal glass area $f(x,y) = 2xy$ and fixed frame material $4x + 15y = 120$.

b) (5 points) Use again the Lagrange method to find the reading glasses with minimal frame material $f(x,y) = 4x + 15y$ and fixed glass area $g(x,y) = 2xy = 120$.

Solution:
a) $\nabla f = \langle 2y, 2x\rangle$ and $\nabla g = \langle 4, 15\rangle$. The Lagrange equations are

$$\begin{aligned} 2y &= 4\lambda \\ 2x &= 15\lambda \\ 4x + 15y & \quad 120\ . \end{aligned}$$

It leads to the relation $y/x = 4/15$. Plugging in $y = 4x/15$ into the constraints solves the equations and gives $x = 15$ and $y = 4$.
b) We can reuse the same gradients. But now

$$\begin{aligned} 4 &= 2y\lambda \\ 15 &= 2x\lambda \\ 2xy & \quad 120\ . \end{aligned}$$

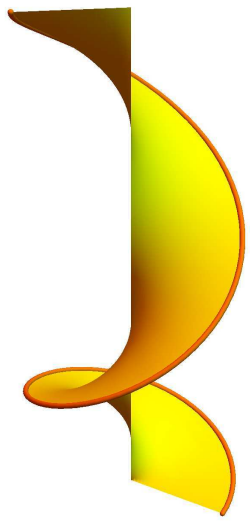
Now, we have again $x = 15$ and $y = 4$.
P.S. Interestingly enough there would be an other solution like $x = -15, y = -4$ which has not appeared in a). But these do not correspond to valid geometries of the glasses. But this example has shown that we can not prove a mathematical duality theorem as **maximizing a first quantity while fixing the second is not totally equivalent to minimizing the second when fixing the first**. A mathematician working in the field of mathematical economics would not get teterred from this experiment and ask "under which conditions does the duality principle hold?" or "is there at all an example, where the duality principle holds?"

Problem 11) (10 points)

a) (4 points) Find the arc length of the **helical curve**

$$\vec{r}(t) = \langle \cos(t), \sin(t), \frac{2t^{3/2}}{3}\rangle\ ,$$

where t goes from 0 to 9.



b) (3 points) Determine the angle between the velocity $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Write down the surface area integral for the surface $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), 2t^{3/2}/3\rangle$ contained inside the cylinder $x^2 + y^2 \leq 1$ and between $0 \leq z \leq 18$ containing the previous curve in its boundary. You do not have to compute the integral but write down an expression of the form $\int_{\square}^{\square}\int_{\square}^{\square} f(s,t)\, dsdt$ with a function $f(s,t)$ you determine.

Solution:
a) We have $|\vec{r}\,'(t)| = |\langle -\sin(t), \cos(t), \sqrt{t}\rangle| = \sqrt{1+t}$. The arc length is $\int_0^9 \sqrt{1+t}\, dt = (2/3)(10^{3/2}-1)$
b) We have $\vec{r}\,''(t) = \langle -\cos(t), -\sin 9t), t^{-1/2}/2\rangle$. At $t = 0$ we get $\langle\langle 0,1,0\rangle \cdot \langle -1, 0, undefined\rangle$. Actually, one can take the limit $t \to 0$ and get 0. so that the angle is $\pi/2$. We were generous and gave for an "undefined" answer full credit.
c) We have

$$|\vec{r}_u \times \vec{r}_v| = |\langle \sqrt{t}\sin(t), -\sqrt{t}\cos(t), s\rangle| = \sqrt{t+s^2}\ .$$

The integral is

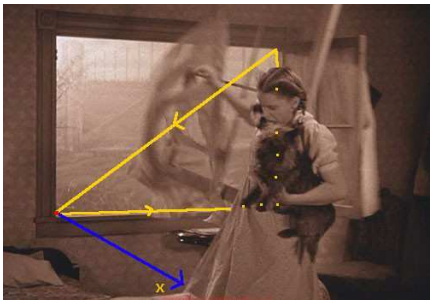
$$\int_0^9 \int_0^1 \sqrt{t+s^2}\, dsdt\ .$$

P.S. By the way, the integral can be done. It would be a good but challenging homework problem. But for an exam problem, with limited time , it would have been bad to ask for evaluating the integral. It would have been a lot of work for 3 points out of 140. The integral $\int \sqrt{1+s^2}ds$ appears quite often, like for example for the arc length of the parabola. You have seen it also in a homework and many of you have worked hard on it to get it. Integrating $\sqrt{t+s^2}$ is similar. Its anti derivative is obtained by integrating by parts and then use a partial fraction trick to get "Marry go round!" the same integral again leading to an equation which can be solved. So, $2\int \sqrt{t+s^2}\ ds = s\sqrt{t+s^2} + t\log(s+\sqrt{t+s^2})$. Now we have to integrate this with respect to t. The first part can be done directly, the second using integration by parts.

Problem 12) (10 points)

The tornado in the **wizard of Oz** induces the **force field**

$$\vec{F} = \langle \cos(x), -2x, y^3 + \sin(z^5)\rangle\ .$$



Dorothy's cardboard is picked up by the storm and pushed along the boundary of the triangle parametrized by

$$\vec{r}(u,v) = \langle 0, u, v\rangle$$

with $0 \le u \le 2$ and $0 \le v \le u/2$. Let C be the boundary of the triangle, oriented counter clockwise when looking from $(1,0,0)$ onto the window. Find the work $\int_C \vec{F}\cdot d\vec{r}$ which the tornado does onto the card board.

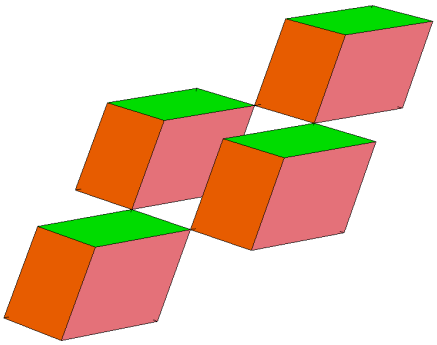
Solution:
This is a problem for Stokes theorem. The curl $\vec{G} = \text{curl}(\vec{F})$ is $\langle 3y^2, 0-2\rangle$. The normal vector to the parametrized surface $\vec{r}(u,v) = \langle 0, u, v\rangle$ is $\langle 1,0,0\rangle$ and points in the right direction. We compute the flux as

$$\int_0^2 \int_0^{y/2} 3y^2 \, dzdy = 6 \; .$$

Problem 13) (10 points)

A solid E is the union of 4 congruent, non-intersecting parallelepipeds. One of them is spanned by the three vectors

$$\vec{u} = \langle 1,0,0\rangle, \vec{v} = \langle 1,1,0\rangle, \vec{w} = \langle 0,1,1\rangle \; .$$



Find the flux of the vector field

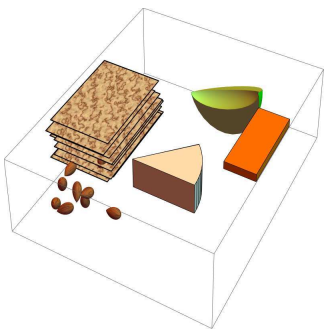
$$\vec{F} = \langle 4x + y^{2014}, z^{2014}, x^{2014}\rangle + \text{curl}(\langle -y^{2014}, x^{2014}, 3^{2014}\rangle)$$

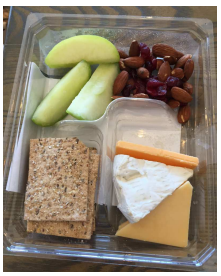
through the outwards oriented boundary surface of E.

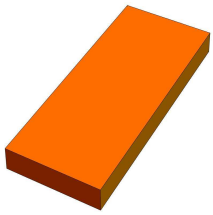
Solution:
The divergence is constant 4. The volume of the parallel epiped is the triple scalar product $\langle 1,0,0\rangle \cdot (\langle 1,1,0\rangle \times \langle 0,1,1\rangle) = 1$. By the divergence theorem, the flux is the triple integral of the divergence over the solid which is 4 times the volume of the solid. Because we have 4 parallel epipeds, the flux is $4 \cdot 4 = 16$.

Problem 14) (10 points) No justifications necessary

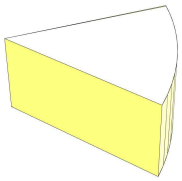
a) (5 points) **Cheese-fruit bistro boxes** can be rarely found in coffee shop these days because a "bistro monster" is eating them all. One of them contains apples, nuts, cheese and crackers. Lets match the objects and volume integrals:



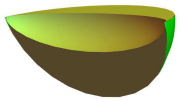




I



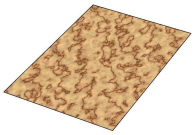
II



III



IV

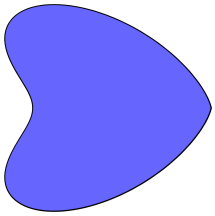


V

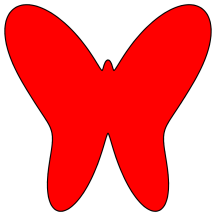
Enter I-V	volume formula
	$\int_1^2 \int_0^{\pi} \int_0^{\pi/5} \rho^2 \sin(\phi)\ d\theta d\phi d\rho$
	$\int_{-6}^{6} \int_{-10}^{10} \int_{-0.1}^{0.1} 1\ dz dx dy$
	$\int_0^{\pi/4} \int_0^{10} \int_{-5}^{5} r\ dz dr d\theta$
	$\int_{-1}^{1} \int_{-4}^{4} \int_{-8}^{8} 1\ dx dy dz$
	$\int_0^{\pi} \int_0^{2\pi} \int_0^{\cos(\phi/2)/4} \rho^2 \sin(\phi)\ d\rho d\theta d\phi$

b) (5 points) Biologist **Piet Gielis** once patented polar regions because they can be used to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when matching the following polar regions:

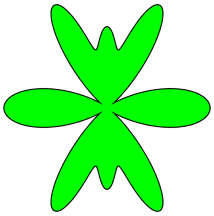
A



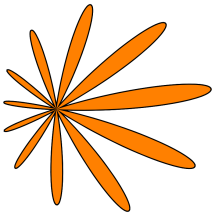
B



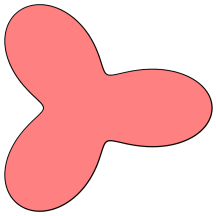
C



D



E



Enter A-E	polar region
	$r(t) \leq \lvert 2 + \cos(3t) \rvert$
	$r(t) \leq \lvert \cos(5t) - 5\cos(t) \rvert$
	$r(t) \leq \lvert 1 + \cos(t)\cos(7t) \rvert$
	$r(t) \leq \lvert 8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t) \rvert$
	$r(t) \leq \lvert \sin(11t) + \cos(t)/2 \rvert$

Solution:
a) III,V,II,I,IV
b) E,A,C,B,D
Symmetry can show the way. If the radial function is even, then there is a symmmetry along the x axes. This happens for A,C,E. Some can be figured out by counting. D has 11 petals. E has 3 petals. The symmetry has decided about B, but C and A were hard to distinguish. One way is to look at $t = \pi$ and $t = 0$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] If $f(x,y,z)$ is a function then the line integral of curl(∇f) around any closed circle is zero.

Solution:
The line integral of the gradient would be zero, but not of the curl. So, one could think, it is false, But it is true because $curl(grad(f)) = 0$.

2) [T] [F] If E is the solid half-sphere $x^2 + y^2 + z^2 \leq 1, z < 0$ then $\iiint_E x^4 \, dxdydz$ is positive.

Solution:
It is an integral of a positive function.

3) [T] [F] If the vector field $\vec{F}$ is incompressible and S and R are surfaces with the same boundary C and orientation then $\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot d\vec{S}$.

Solution:
This follows from the divergence theorem.

4) [T] [F] $\vec{r}(u,v) = \langle \cos(u), \sin(u), 0 \rangle + v\langle -\sin(u), \cos(u), 1 \rangle$ parametrizes a surface in the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
Plug in x, y, z into the equation.

5) [T] [F] The equation div(grad$f$) = $|\text{grad} f|^2$ is an example of a partial differential equation for the unknown function $f(x,y,z)$.

Solution:
If we write it out, it is.

6) [T] [F] The vector $(\vec{i} \times \vec{j}) \times \vec{i}$ is the zero vector.

Solution:
$\vec{i} \times \vec{j}$ is equal to $\vec{k}$.

7) [T] [F] The length of the gradient $|\nabla f|$ is minimal at a local minimum of $f(x,y)$ if a local minimum of f exists.

Solution:
It is zero at a minimum.

8) [T] [F] The length of the gradient $|\nabla f|$ is maximal at a local maximum of $f(x,y)$ if a local maximum of f exists.

Solution:
It is zero at a maximum.

9) [T] [F] If $\vec{r}(t)$ parametrizes the curve obtained by intersecting $y=0$ with $x^2+y^2+z^2=1$, then the bi-normal vector $\vec{B}$ is tangent to the surface.

Solution:
The normal vector points into the sphere towards the center. The binormal vector is both orthogonal to the tangent and normal vector. It is tangent to the sphere.

10) [T] [F] If the line integral of $\vec{F}$ along the closed loop $x^2+y^2=1, z=0$ is zero then the vector field is conservative.

Solution:
It needs to be with respect to all paths.

11) [T] [F] The vectors $\vec{v}=\langle 1,0,0\rangle$ and $\vec{w}=\langle 1,1,0\rangle$ have the property that $\vec{v}\cdot\vec{w}=|\vec{v}\times\vec{w}|$.

Solution:
Compute both, and it is almost true. In two dimensions, the cross product is also a scalar. But the sign is off.

12) [T] [F] The surface area of a sphere depends on the orientation of the sphere. It is positive if the normal vector points outwards and changes sign if the orientation is changed.

Solution:
It is the flux which depends on the orientation.

13) [T] [F] If $f(x,y,z) = (\text{div}(\vec{F})(x,y,z))$ has a maximum at $(0,0,0)$, then $\text{grad}(\text{div}(\vec{F}))(0,0,0) = \langle 0,0,0 \rangle$.

Solution:
Yes, f is a function.

14) [T] [F] The curl of a conservative vector field is zero.

Solution:
This is essentially Clairot.

15) [T] [F] The flux of the curl of $\vec{F}$ through a disc $x^2+y^2 \leq 1, z=0$ is always zero.

Solution:
By Stokes theorem, it would be zero through a closed surface. But the disc is not closed.

16) [T] [F] If the integral $\int\int\int_G \text{div}(\vec{F}(x,y,z))\ dxdydz$ is zero for the ball $G = \{x^2+y^2+z^2 \leq 1\}$, then the divergence is zero at $(0,0,0)$.

Solution:
Take $F = \langle x^3, 0, 0 \rangle$

17) [T] [F] The value $\sqrt{101 \cdot 10002}$ can by linear approximation be estimated as $1000 + 5 \cdot 1 + (1/20) \cdot 2$.

Solution:
The function $f(x,y) = \sqrt{101 \cdot 10002}$ is linearized by $L(x,y) = 1000 + 5(x-100) + (1/2)(y-10000)$.

18) [T] [F] If $\vec{F} = \text{curl}(\vec{G})$ and $\text{div}(\vec{F}) = 0$ everywhere in space, then $\text{div}(\vec{G}) = 0$ everywhere in space.

Solution:
$\text{div}(\vec{F}) = 0$ is always true in that case.

19) [T] [F] It is possible that $\vec{v} \cdot \vec{w} > 0$ and $\vec{v} \times \vec{w} = \vec{0}$.

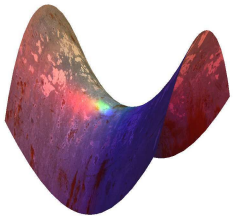
Solution:
Take $\vec{v} = \vec{w}$.

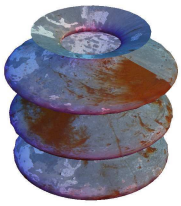
20) [T] [F] The directional derivative $D_{\vec{v}}(f)$ is defined as $\nabla f \times \vec{v}$.

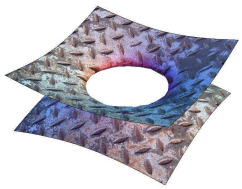
Solution:
It is the dot product not the cross product.

Problem 2) (10 points) No justifications are necessary.

a) (3 points) The following surfaces are given either as a parametrization or implicitly in some coordinate system (Cartesian, cylindrical or spherical). Each surface matches exactly one definition.


A


B

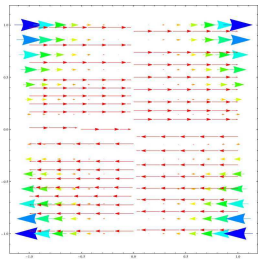

C

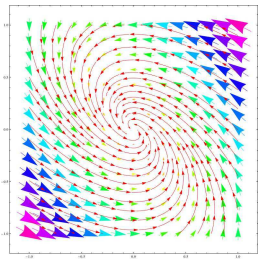

D

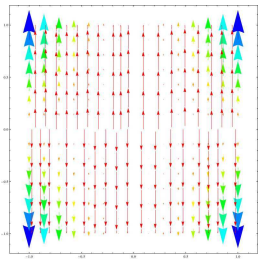
Enter A-D here	Function or parametrization
	$r = 3 + 2\sin(3z)$
	$\vec{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$
	$x^4 - zy^4 + z^4 = 1$
	$r^2 - 8z^2 = 1$

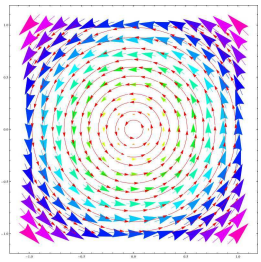
b) (3 points) The pictures display flow lines of vector fields in two dimensions. Match them.

Field	Enter 1-4
$\vec{F}(x, y) = \langle 0, x^2 y \rangle$	
$\vec{F}(x, y) = \langle x^2 y, 0 \rangle$	
$\vec{F}(x, y) = \langle -y - x, x \rangle$	
$\vec{F}(x, y) = \langle -y, x \rangle$	


1


2


3


4

c) (2 points) Match the following partial differential equations with functions $u(t, x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-C	1-3
Laplace		
wave		
heat		

A	$u(t, x) = t + t^2 - x^2$
B	$u(t, x) = t + t^2 + x^2$
C	$u(t, x) = x^2 + 2t$

1	$u_t(t, x) = u_{xx}(t, x)$
2	$u_{tt}(t, x) = -u_{xx}(t, x)$
3	$u_{tt}(t, x) = u_{xx}(t, x)$

d) (2 points) Two of the six expressions are **not** independent of the parametrization. Check them.

Velocity $\vec{r}\,'(t)$		Surface area $\int\int \lvert\vec{r}_u \times \vec{r}_v\rvert du dv$		Line integral $\int_a^b \vec{F} \cdot d\vec{r}$	
Arc length $\int \lvert\vec{r}\,'(t)\rvert\, dt$		Flux integral $\int\int_R \vec{F} \cdot d\vec{S}$		Normal vector $\vec{r}_u \times \vec{r}_v$	

Solution:
a) BADC
b) 3124
c) ABC, 231
d) Velocity and Normal vector. The Surface area and line integral does not depend on the parametrization for fixed orientation.

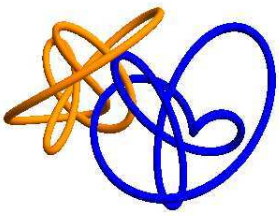
Problem 3) (10 points)

a) (4 points) The following objects are defined in three dimensional space.
Fill in either "surface", "curve", or "vector field" in each case.

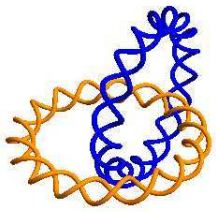
formula	surface, curve or field?
$x+y=1$	
$x+y=1, x-y=5$	
$\vec{F}(x,y,z)=\langle x, x+y, x-y\rangle$	
$\vec{r}(x,y)=\langle x, y, x-y\rangle$	
$\vec{r}(x)=\langle x, x, x^2-x\rangle$	

b) (2 points) Two closed curves $\vec{r}_1(t), \vec{r}_2(t)$ form a **link**. In our case, the curve $\vec{r}_2(t)$ is a copy of the other moved and turned around. Match three of them. Links and knots are relevant in biology: DNA strands can forms links or knots which need disentanglement.

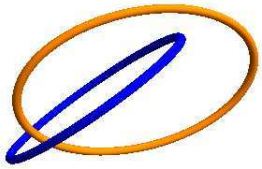
$\vec{r}_1(t)$	Enter A,B,C,D
$\langle 7\cos(t), 7\sin(t), 7\cos(t)\rangle$	
$\langle (7+\cos(17t))\cos(t), (7+\cos(17t))\sin(t), \sin(17t)\rangle$	
$\langle \cos(2t)+\sin(4t), \cos(4t)+\cos(3t), \cos(2t)+\sin(3t)\rangle$	



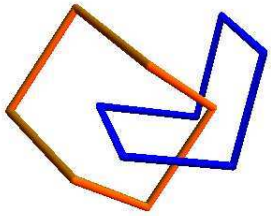
A



B



C



D

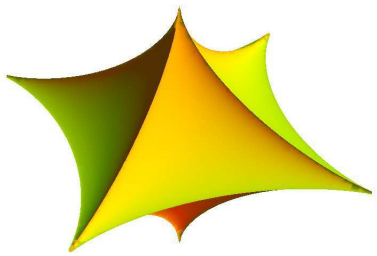
c) (4 points) Which of the following expressions are defined if $\vec{F}(x, y, z)$ is a vector field and $f(x, y, z)$ a scalar field in space. Is the result a scalar or vector field?

Formula	Defined	Not defined	Scalar	Vector
curl(grad(div($\vec{F}$)))				
curl(div(grad(f)))				
grad(div(curl($\vec{F}$)))				
grad(curl(div($\vec{F}$)))				
div(curl(grad(f)))				
div(grad(curl(f)))				

Solution:
a) surface, curve, field, surface, curve
b) CBA

Formula	Defined	Not defined	Scalar	Vector
curl(grad(div($\vec{F}$)))	x			x
curl(div(grad(f)))		x		
grad(div(curl($\vec{F}$)))	x			x
grad(curl(div($\vec{F}$)))		x		
div(curl(grad(f)))	x		x	
div(grad(curl(f)))		x		

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the surface S given by

$$3x^{2/3} + 3y^{2/3} + 6z^{2/3} = 12$$

at the point $(1, 1, 1)$.

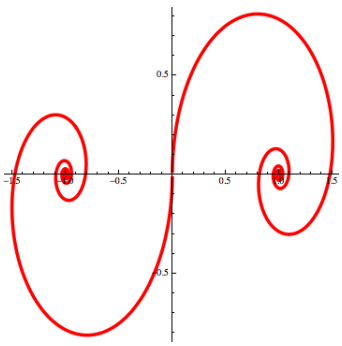
b) (5 points) When S is intersected with the plane $y = 1$, we get the curve

$$3x^{2/3} + 6z^{2/3} = 9 \, .$$

Find the tangent line of the form $ax + bz = d$ for the tangent line at $(x, z) = (1, 1)$.

Solution:
a) $\nabla f(1,1,1) = \langle 2,2,4\rangle$ so that $2x+2y+4z = 8$ is the solution.
b) $\nabla f(1,1) = \langle 2,4\rangle$ so that $x+2z = 3$ is the solution.

Problem 5) (10 points)



Find the line integral for the vector field

$$\vec{F}(x,y) = \langle x^6 + y + 3x^2y^3, y^7 + x + 3x^3y^2\rangle$$

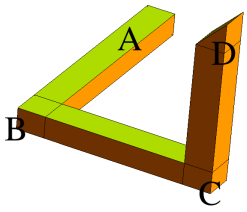
along the **ornamental curve**

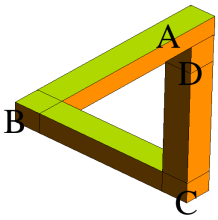
$$\vec{r}(t) = \langle \frac{t}{|t|}(1 - \frac{1}{1+t^2}\cos(4t)), \frac{1}{1+t^2}\sin(4t)\rangle$$

from $t = -\infty$ to $t = \infty$. This curve connects the point $(-1,0)$ with $(1,0)$ along an **infinite epic journey**.

Solution:
The vector field is a gradient field with $f(x,y) = x^7/7 + xy + y^8/8 + x^3y^3$. By the fundamental theorem of line integrals, we get $f(r(b)) - f(r(a)) = (1/7) - (-1/7) = 2/7$.

Problem 6) (10 points)





The **Penrose tribar** is a path in space connecting the points $A = (1,0,0), B = (0,0,0), C = (0,0,1), D = (0,1,1)$. While the distance between A and D is positive, we see an impossible triangle when the line of sight goes through A and D.

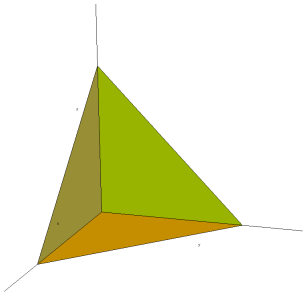
a) (2 points) Find the distance between the points A and D.

b) (4 points) Parametrize the line connecting A and D.

c) (4 points) Find the distance between the lines AB and CD.

Solution:
a) $\sqrt{3}$ by Pythagoras or distance formula.
b) $\vec{r}(t) = \langle 1-t, t, t\rangle$
c) Use the distance formula using $\vec{v} = \vec{AB} = \langle -1, 0, 0\rangle$ and $\vec{w} = \vec{CD} = \langle 0, 1, 0\rangle$ and two points $P = (0, 0, 0)$ and $Q = (0, 0, 1)$ to get $d = 1$.

Problem 7) (10 points)



You invent a **3D printing process** in which materials of variable density can be printed. To try this out, we take a tetrahedral region E:

$$x + y + z \leq 1; x \geq 0, y \geq 0, z \geq 0$$

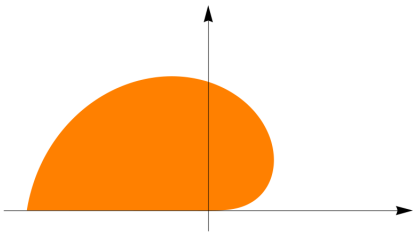
which has the density $f(x, y, z) = 24x$. Find the total mass

$$\int\int\int_E f(x, y, z)\, dxdydz\ .$$

Solution:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 24x\ dzdydx = 1$$

Problem 8) (10 points)



When we integrate the function $f(x, y) = 2y/(\sqrt{x^2+y^2}\arctan(y/x))$ over the **snail region** $r^2 \leq \theta \leq \pi$, we are led in polar coordinates to the integral

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} \frac{2r\sin(\theta)}{\theta}\ d\theta\ dr$$

Evaluate this integral.

Solution:

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} 2r \sin(\theta)/\theta \; d\theta \; dr$$

Changing the order of integration gives

$$\int_0^{\pi} \int_0^{\sqrt{\theta}} 2r \sin(\theta)/\theta \; dr \; d\theta = 2$$

Problem 9) (10 points)



Old Mc Donald wants to build a farm on a location, where the ground is as even as possible. Let $g(x, y) = y^2 + xy + x$ be the height of the ground. Find the point (x, y), where the **steepness**

$$f(x, y) = |\nabla g|^2$$

is minimal. Classify all critical points of f.

Solution:
The function under consideration is $f(x, y) = (1 + y)^2 + (x + 2y)^2$. There is one critical point $(2, -1)$ which has discriminant 4 and $f_{xx} = 2$. It is a minimum.

Problem 10) (10 points)



We build a **bike** which has as a frame a triangle of base length x and height y and a wheel which has radius y. Using Lagrange, find the bike which has maximal

$$f(x, y) = xy + 4\pi y^2$$

(which is twice the area) under the constraint

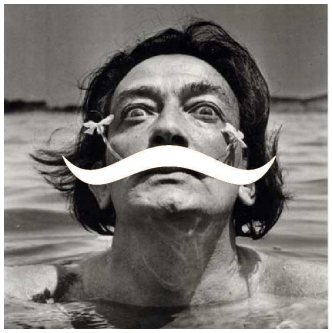
$$g(x, y) = x + 10\pi y = 3 \; .$$

Solution:
The Lagrange equations are

$$\begin{aligned} y &= \lambda \\ x + 8\pi y &= \lambda(10\pi) \\ x + 10\pi y &= 3 \end{aligned}$$

It has only one solution $(x, y) = (1/2, 1/(4\pi))$, which is a maximum.

Problem 11) (10 points)



Compute the area of the **moustache region** which is enclosed by the curve

$$\vec{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t) \rangle$$

with $0 \leq t \leq 2\pi$.

Hint. You can use without justification that integrating an odd 2π periodic function from 0 to 2π is zero.

Solution:
Use the vector field $\vec{F}(x, y) = \langle 0, x \rangle$ and use Greens theorem. We have

$$\int_0^{2\pi} \langle 0, 5\cos(t) \rangle \cdot \langle -5\sin(t), \cos(t) - 4\sin(4t) \rangle \, dt = 5\pi$$

We have used that $\int_0^{2\pi} \cos^2(t) \, dt = \pi$ (double angle formula) and the hint. The hint had not be justified and it is obvious. If you look at an odd function like $\sin(4x)\cos(x)$ [odd f means $f(-x) = -f(x)$], then the contribution on $[0, \pi]$ is canceled by the contribution on $[-\pi, 0]$. During the exam a proctor was asked what it means for a function to be "strange". An odd question! The proctor integrated it to zero.

Problem 12) (10 points)



We enjoy the pre-holiday season in a local Harvard square coffee shop, where **coffee aroma** diffuses in the air. Find the flux of the air velocity field

$$\vec{F}(x,y,z) = \langle y^2, x^2, z^2 \rangle$$

leaving a coffee box

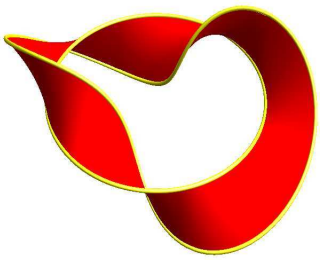
$$E : x^2 + y^2 \leq 1, x^2 + y^2 + z^2 \leq 4 \; .$$

Solution:
Use the divergence theorem and integrate

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2zr \; dzdrd\theta \; = 0 \; .$$

In this 21a coffee box, the total coffee aroma stays. Aroma leaves on the upper part and enteres in the lower part.

Problem 13) (10 points)



Find the line integral of

$$\vec{F}(x,y,z) = \langle -y, x, e^{\sin z} \rangle$$

along the positively oriented boundary of the **ribbon** $\vec{r}(u,v)$ parametrized on $0 \leq u \leq 4\pi$ and $0 \leq v \leq 1/2$ with

$$\vec{r}(u,v) = \langle (1+v\cos(2u))\cos(u), (1+v\cos(2u))\sin(u), v\sin(2u) \rangle$$

for which a **good fairy** gives you the normal vector

$$\begin{aligned} \vec{r}_u \times \vec{r}_v \; = \; & \langle -\sin(u)(v\cos(4u) + 2(v+1)\cos(2u) - 3v + 2)/2 \; , \\ & \cos(u)(v\cos(4u) - 2(v-1)\cos(2u) - 3v - 2)/2 \; , \\ & -\cos(2u)(v\cos(2u) + 1) \rangle \; . \end{aligned}$$

Solution:
We use Stokes, but this time, we compute the flux. We get

$$\int_0^{4\pi} \int_0^{1/2} \langle 0,0,2\rangle \cdot \langle \ldots, \ldots, -\cos(2u)(v\cos(2u)+1)\rangle \; dvdu$$

which simplifies to $-\pi/2$.
Remarks
1) One student was intimidated by the complexity of the normal vector and questioned the name "good fairy". While writing the exam, we actually debated for a while whether to give the normal vector. We were not persuaded by a "good fairy" but the prospect of having to grade 1000 pages of futile and probably wrong computations of $\vec{r}_u \times \vec{r}_v$. The above expressions are already simplified.
2) The ribbon is a variant of a Möbius strip. But unlike the famous Moebius strip, it is orientable. The surface has one side. We were actually surprised to see that the flux of the vector field $\langle 0,0,2\rangle$ is not zero because one would expect by symmetry to get zero. Actually, the above parametrization goes over the ribbon twice. We could go from 0 to 2π too.

Problem 14) (10 points)

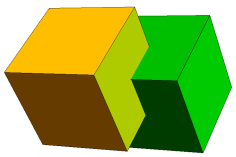


A computer can determine the volume of a solid enclosed by a triangulated surface by computing the fluxe of the vector field $\vec{F} = \langle 0,0,z\rangle$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = \langle 0,0,z\rangle$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\} \; \cup$$

$$\{1 \le x \le 3, 1 \le y \le 3, 1 \le z \le 3\} \; ,$$

where $\cup$ is the union and the surface is oriented outwards.



Ceci n'est pas une pipe

Ceci c'est une vache

Solution:
The divergence is 1. The volume of the cubistic cow is $2^3 + 2^3 - 1 = 15$. The divergence theorem gives the answer 15.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] For any two vectors $\vec{u}$ and $\vec{v}$ we have $|\vec{u}| \leq |\vec{v}| + |\vec{v} - \vec{u}|$.

Solution:
This is the triangle inequality.

2) [T] [F] The grid curves of a parametric surface are always perpendicular to each other at any point on the surface.

Solution:
Parametrize a plane with two nonperpendicular vectors.

3) [T] [F] The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ of $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 0, 1, 0 \rangle$ and $\vec{w} = \langle 2, 1, 1 \rangle$ is equal to 1.

Solution:
Compute it

4) [T] [F] For $f(x, y) = x^4 + y^4$ and $\vec{r}(t) = \langle t, t^2 \rangle$, we have $\frac{d}{dt} f(\vec{r}(t)) = \langle 4t^3, 4t^6 \rangle \cdot \langle 1, 2t \rangle$.

Solution:
This is the chain rule.

5) [T] [F] The flux of $\vec{F} = \langle x, 0, 0 \rangle$ through the outwardly-oriented boundary S of a parallelepiped spanned by edges $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle, \langle 1, 1, 3 \rangle$ is equal to 6.

Solution:
It is by the divergence theorem equal to the volume.

6) [T] [F] The differential equation $u_x = u_t$ for a function $u(x, t)$ is called the heat equation.

Solution:
It is called the transport equation

7) [T] [F] If a function $f(x,y)$ has a critical point at $(0,0)$ then $\text{div}(\text{grad}(f))(0,0)$ is zero.

Solution:
Take $f(x,y) = x^2 + y^2$. This is a counter example.

8) [T] [F] A function of two variables always has an odd number of critical points.

Solution:
There can be any number of critical points. Also zero.

9) [T] [F] If $f(x,y)$ is a function of two variables and $(0,0)$ is a maximum of $g(x) = f(x,0)$ and as well as a maximum of $h(y) = f(0,y)$ then $(0,0)$ is a maximum of f.

Solution:
$f_{xx} >$ and $f_{yy} > 0$ does not imply $D = f_{xx}f_{yy} - f_{xy}^2$ is positive.

10) [T] [F] The divergence of a vector field $\vec{F}$ is always equal to the divergence of the curl of $\vec{F}$.

Solution:
Take $\vec{f}(x,y,z) = \langle x,y,z \rangle$. The curl is zero but the divergence is 3.

11) [T] [F] The flux of a vector field $\vec{F}$ of length $|\vec{F}| = 1$ through a triangular surface S can not be larger than the surface area of the triangle.

Solution:
Use $|\vec{F}(r(u,v) \cdot \vec{r}_u \times \vec{r}_v| \leq |\vec{F}(\vec{r}(u,v))||\vec{r}_u \times \vec{r}_v| = |\vec{r}_u \times \vec{r}_v|$.

12) [T] [F] The arc length of the boundary of a surface is independent of the parametrization of the surface.

Solution:
The arc length is independent of parametrization in general.

13) [T] [F] The vector field $\text{curl}(\vec{F})$ is at every point (x, y, z) perpendicular to $\vec{F}(x, y, z)$.

Solution:
A counter example is $\langle 0, x, 1\rangle$ which has curl $\langle 0, 0, 1\rangle$.

14) [T] [F] The scalar function $f = \text{div}(\vec{F})$ has the property that $\text{grad}(f(x, y, z))$ is perpendicular to $\vec{F}(x, y, z)$.

Solution:
Take $\vec{F}(x, y, z) = \langle x^2, 0, 0\rangle$ which has the divergence $2x$, the gradient of which is $\langle 1, 0, 0\rangle$.

15) [T] [F] The Lagrange equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = x^2 + y^2 = 1$ have infinitely many solutions if $f = g$.

Solution:
All the points are critical points.

16) [T] [F] If a vector is perpendicular to itself, then it is the zero vector.

Solution:
Yes, $\vec{v} \cdot \vec{v} = 0$ implies $|\vec{v}| = 0$.

17) [T] [F] The gradient of the divergence of the curl of a vector field $\vec{F}$ is the vector field which assigns the zero vector to each point.

Solution:
Take $\vec{F}(x, y, z) = \langle x^2, y^2, z^2\rangle$. It has divergence $2x + 2y + 2z$ and the gradient $\langle 2, 2, 2\rangle$.

18) [T] [F] The identity $\text{Proj}_{\vec{v}}(\vec{w}) = \text{Proj}_{\vec{w}}(\vec{v})$ holds for all vectors $\vec{v}, \vec{w}$.

Solution:
If the two vectors are not parallel, then the projection vectors are not even parallel.

19) [T] [F] The function $f(x, y) = \sin(xy)$ is a solution to the Laplace equation $f_{xx} + f_{yy} = 0$.

Solution:
Differentiate and see it is not the same.

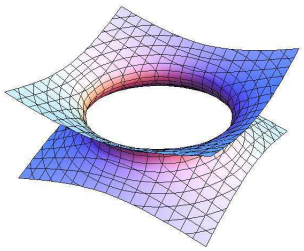
20) [T] [F] The formula $\vec{r}(u, v) = \langle 2u, (9 + u^2)\cos(v), (9 + u^2)\sin(v)\rangle$ gives a parametrization of a one-sheeted hyperboloid.

Solution:
$y^2 + z^2 = 9 + u^2 = 9 + (x/2)^2$ would be true if there was $\sqrt{9 + u^2}$ instead of $9 + u^2$.

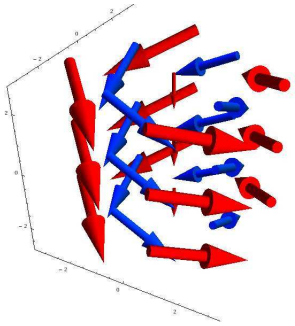
Problem 2) (10 points)

a) (6 points) Match the following objects.

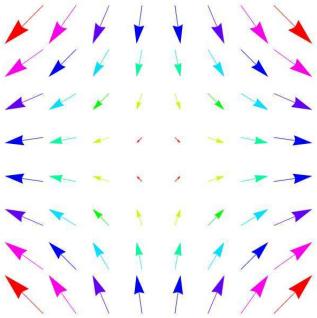
Formula	Enter 1-6
$\rho(\phi,\theta) \leq e^{\phi}$	
$\vec{F}(x,y,z) = \langle -y, x, -2 \rangle$	
$x^2 + y^2 - 5z^2 = 1$	
$z + x^2 - y^2 = 2$	
$\vec{F}(x,y) = \langle x, -y \rangle$	
$x^4 + 2y^4 \leq 3$	



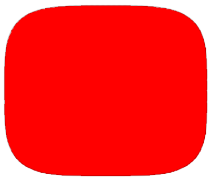
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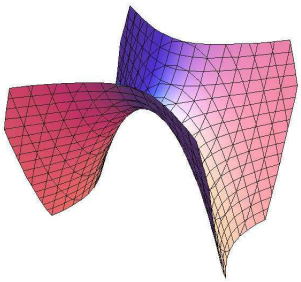
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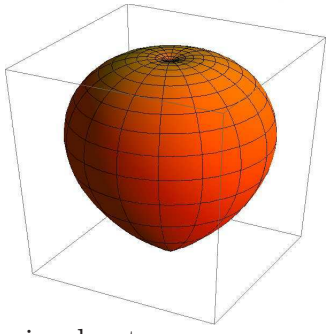
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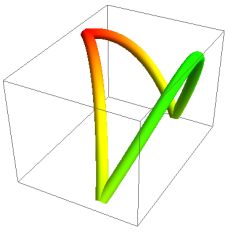
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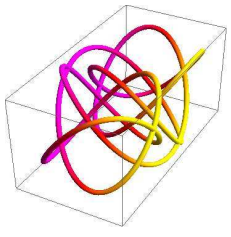
6

b) (4 points) A knot is a closed curve in space. Match the following knots

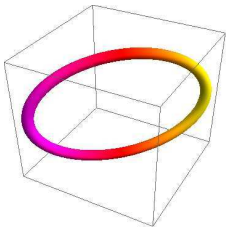
Formula	Enter A,B,C,D
$\vec{r}(t) = \langle \cos(5t), \cos(t) + \sin(5t), \cos(7t) \rangle$	
$\vec{r}(t) = \langle \lvert\cos(t)\rvert, \lvert\sin(t) + \cos(t)\rvert, \lvert\cos(2t)\rvert \rangle$	
$\vec{r}(t) = \langle (2 + \cos(\frac{3t}{2}))\cos(t), (2 + \cos(\frac{3t}{2}))\sin(t), \sin(\frac{3t}{2}) \rangle$	
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$	



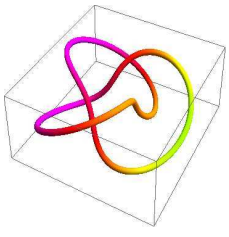
A



B



C



D

Solution:

a) 6,2,1,5,3,4

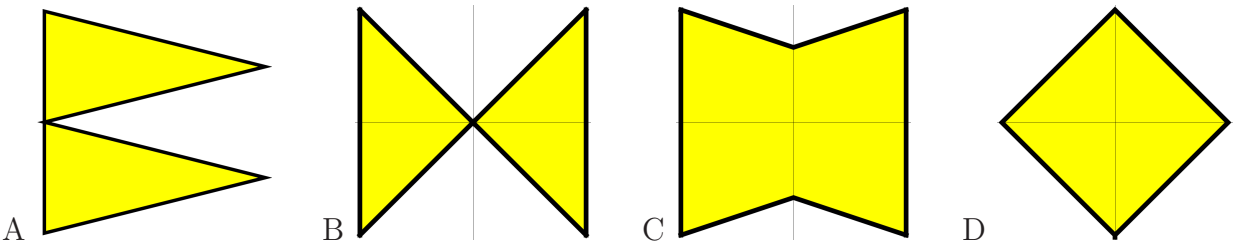
b) B,A,D,C

Problem 3) (10 points)

a) (4 points) It is Hobbit time. The following regions resemble ancient runes of the Anglo Saxons studied by JRR Tolkien. (A is "b", B is "d", C is "st", and D is "oe").

Integral	Enter A,B,C,D
$\int_{-1}^{1}\int_{-|x|}^{|x|} f(x,y)\,dydx$	
$\int_{-1}^{1}\int_{-1-|x|/2}^{1+|x|/2} f(x,y)\,dydx$	
$\int_{-1}^{1}\int_{0}^{||y+\frac{1}{2}|-|y-\frac{1}{2}||-|y|} f(x,y)\,dxdy$	
$\int_{-1}^{1}\int_{|y|-1}^{1-|y|} f(x,y)\,dxdy$	

[1]



b) (4 points) Matching polar regions

Formula	Enter E,F,G,H
$r \le \theta(2\pi - \theta)$	
$r \le \lvert\cos(5\theta)\rvert$	
$r \le 2 + \cos(5\theta)$	
$r \le 2\pi - \theta$	



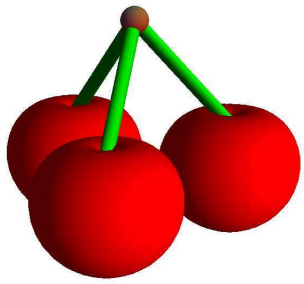
c) (2 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

Integral theorem	Grad	Curl	Div	Line integral	Flux integral
Divergence theorem					
Stokes' theorem					

[1] More information on http://www.theshorterword.com/anglo-saxon-runes

Solution:
a) B,C,A,D
b) F,E,G,H
d) Divergence theorem: divergence and flux integral.
Stokes theorem: curl, line integral and flux integral.

Problem 4) (10 points)



Three cherries have centers at $A = (-1,-1,-1)$, $B = (1,0,-2)$ and $C = (0,1,-2)$ and are tied together at the origin $O = (0,0,0)$.[2]
Find the distance between O and the plane through A, B, C.

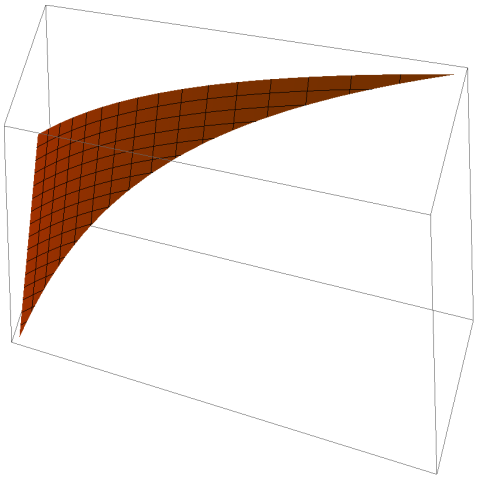
Solution:
The vector $\vec{n} = \vec{AB} \times \vec{AC} = \langle 1,1,3 \rangle$ is normal to the plane. The vector $\vec{AO} = \langle 1,0,-2 \rangle$ connects a O with a point on the plane. We have

$$d = \frac{|\vec{AO} \cdot \vec{n}|}{|\vec{n}|} = \langle 1,1,3 \rangle \cdot \langle 1,1,1 \rangle / \sqrt{11} = 5/\sqrt{11} \; .$$

The distance is $\boxed{5/\sqrt{11}}$.

Problem 5) (10 points)

[2]Picture from Mathematica project by Alissa Zhang



Find the surface area of the surface

$$\vec{r}(u,v) = \langle u^2 + v, u, v \rangle$$

for which $0 \le v \le 4$ and $\frac{v}{4} \le u \le 1$.

Solution:
We compute $\vec{r}_u = \langle 2u, 1, 0 \rangle, \vec{r}_v = \langle 1, 0, 1 \rangle$. Their cross product is $\langle 1, -2u, -1 \rangle$ which has length $\sqrt{2+4u^2}$. To integrate

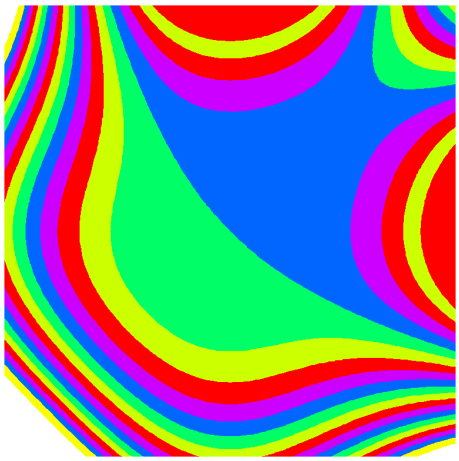
$$\int_0^4 \int_{v/4}^1 \sqrt{2+4u^2}\, dudv \ ,$$

we draw the triangle in the uv plane over which we integrate, then change the order of integration

$$\int_0^1 \int_0^{4u} \sqrt{2+4u^2}\, dvdu = \int_0^1 4u\sqrt{4u^2+2}\, du = (6^{3/2} - 2^{3/2})/3 \ .$$

It could also be written as $\boxed{2\sqrt{6} - \frac{2}{3}\sqrt{2}}$.

Problem 6) (10 points)



The function $f(x,y) = 2x^3 + 2y^3 - 3x^2y^2$ is called the "happy function" as you can see when you turn your head clockwise by $\pi/4$. Find and classify its extrema.

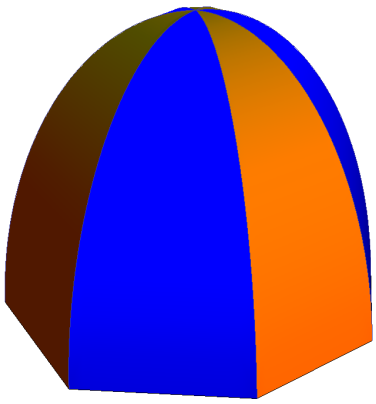
In one of the critical points, the discriminant D is zero. We want you nevertheless to decide whether this point is a "local maximum" a "local minimum" or "neither of them".

Solution:
The gradient is $\langle 5x^2 - 6xy^2, 6y^2 - 6x^2y\rangle$. If one of the x, y is zero, both are and $(0,0)$ is a critical point. If none are zero, then $x = y^2$ and $y = x^2$. Plugging in the second to the first gives $x = x^4$ and so $x = 1$. The discriminant D is

$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x - 6y^2)(12y - 6x^2) - (-12xy)^2 \ .$$

At the first critical point $(0,0)$ this is zero. At the second critical point $(1,1)$, then $D = 36 - 144 < 0$ and the point is a saddle point.
To analyze the behavior at $(0,0)$, set $y = 0$ to see that $f(x,0) = 2x^3$. This function takes both positive and negative values arbitrarily close to 0. It is neither a local maximum, nor a local minimum. To summarize, $\boxed{\text{there is one saddle point } (1,1)}$. It is at the nose of the face. Furthermore, the point $(0,0)$ with $D = 0$ is **neither maximum,nor minimum.** This is the critical point on the lips.

Problem 7) (10 points)



Archimedes computed volumes of solids which now bear his name. He showed that, as for the sphere, each "Archimedean globe" has volume equal to two thirds of the prism in which it is inscribed. Later it was discovered that also the surface area is two thirds of the surface area of a circumscribing prism. To find globes of minimal surface we are led to the problem:

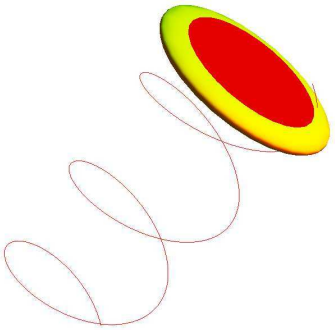
Find values of r, h satisfying $g(r,h) = r^2h = 3$ so that $f(r,h) = 3r^2 + 2rh$ is minimal.

Solution:
The Lagrange equations are

$$\begin{aligned} 6r + 2h &= \lambda 2rh \\ 2r &= \lambda r^2 \\ r^2h &= 3 \ . \end{aligned}$$

Eliminating λ gives $3r = h$. Plugging into the third equation gives $\boxed{r = 1 \text{ and } h = 3}$.

Problem 8) (10 points)



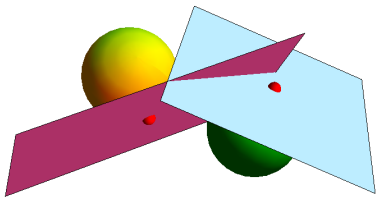
The frisbee was invented by Harvard students in 1845 when a student threw a cake plate to George Frisbie Hoar and shouted "Frisbie, catch!".
A point on the outer rim of a frisbee moves on a curve $\vec{r}(t)$ satisfying [3]

$$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t)\rangle \ .$$

We know that $\vec{r}(0) = \langle 0, 1, 0\rangle$ and $\vec{r}\,'(0) = \langle 1, 0, 1\rangle$. Find $\vec{r}(t)$ and the arc length of the curve $\vec{r}(t)$ from $t = 0$ to $t = 2\pi$.

Solution:
$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t)\rangle$
$\vec{r}\,'(t) = \langle 1, -\sin(t), \cos(t)\rangle$
$\vec{r}(t) = \langle t, \cos(t), \sin(t)\rangle$ The speed is $\sqrt{2}$. The arc length is $\sqrt{2}(2\pi)$ which is $\boxed{\sqrt{8}\pi}$.

Problem 9) (10 points)



Find a parametrization of the line of intersection of the tangent plane at the point $(1, -1, 0)$ of the sphere

$$x^2 + y^2 + z^2 = 2$$

and the tangent plane to the point $(5, 1, 1)$ of the sphere

$$(x-5)^2 + y^2 + z^2 = 2 \ .$$

[3] Some claim this incident happened at Yale, which is a fairy tale. **George Frisbie Hoar** graduated from Harvard in 1846 and became later a United States Senator. He fought against political corruption, and campaigned for the rights of African and Native Americans.

Solution:
The tangent plane to the first is obtained by computing the gradient at $(1,-1,0)$ and fixing the constant

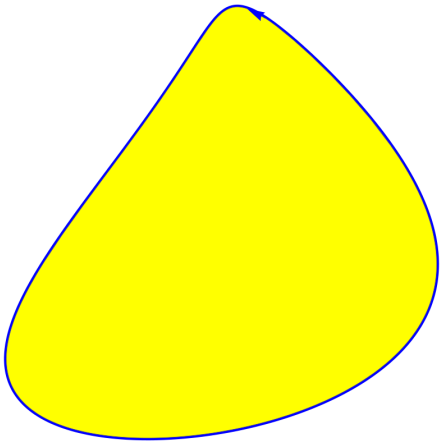
$$2x - 2y = 4$$

The tangent plane to the second is

$$2y + 2z = 4\ .$$

To find the intersecting line, we find two poins like $(2,0,2)$ or $(0,-2,4)$ on the line and parametrize $\boxed{\vec{r}(t) = \langle 2,0,2\rangle + t\langle -2,-2,2\rangle}$. Note that there are many solutions. An other convenient solution is to take the cross product of $\langle 2,-2,0\rangle$ and $\langle 0,2,2\rangle$ to get a vector $\langle -4,-4,4\rangle$ in the intersection and to find one point in the intersection.

Problem 10) (10 points)



Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle 3\cos(t) - \sin(2t), 4\sin(t) + \cos(t)\rangle\ ,$$

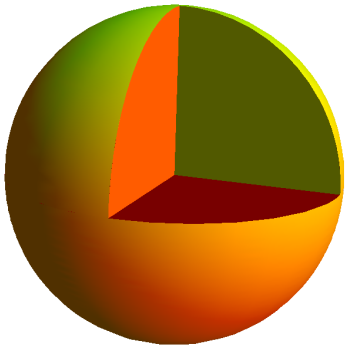
where $0 \le t \le 2\pi$.

Solution:
We use Greens theorem using $\vec{F} = \langle 0, x\rangle$. We get

$$\begin{aligned}\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt &= \int_0^{2\pi} \langle 0, 3\cos(t) - \sin(2t)\rangle \cdot \langle -3\sin(t) + 2\cos(2t), 4\cos(t) - \sin(t)\rangle \\ &= \int_0^{2\pi} 12\cos^2(t)\ dt = 12\pi\ .\end{aligned}$$

The answer is $\boxed{12\pi}$.

Problem 11) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3 + x^2, y^3 - xy, z^3 - xz \rangle$$

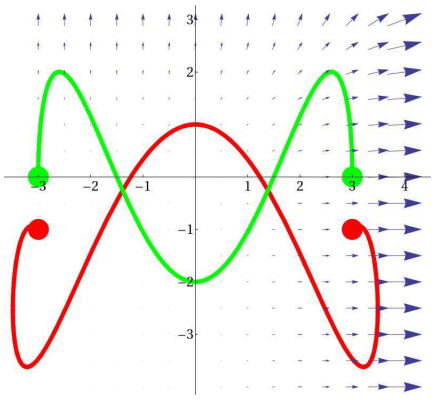
through the boundary surface of E (oriented outwards), where the solid E is a unit sphere from which the first octant has been removed.

Solution:
The divergence is $3x^2 + 3y^2 + 3z^2 = 3\rho^2$. By symmetry, we can compute $\int\int\int_E 3\rho^2 \, dV$ over the entire sphere and multiply with 7/8.

$$(7/8)\int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2 \, \rho^2 \sin(\phi) \, d\rho d\phi d\theta = (7/8)2\pi 23/5 = 21\pi/10 \, .$$

Thw answer is $\boxed{\frac{21\pi}{10}}$

Problem 12) (10 points)



Assume the wind velocity on the Charles is

$$\vec{F}(x,y) = \langle e^x, e^y \rangle \, .$$

A sail boat takes the path

$$C_1 : \vec{r}(t) = \langle -3\cos(t) - \sin(2t), 2\sin(t) + 2\cos(4t) - 3 \rangle$$

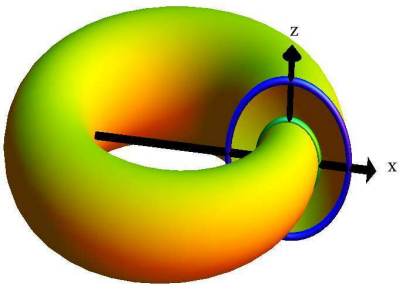
from $(-3,-1)$ to $(3,-1)$. An other boat follows the path

$$C_2 : \vec{r}(t) = \langle -3\cos(t), 2\sin(3t) \rangle$$

from $(-3,0)$ to $(3,0)$. To find out which path needs more energy, compute both line integrals $\int_{C_1} \vec{F} \cdot \vec{dr}$ and $\int_{C_2} \vec{F} \cdot \vec{dr}$.

Solution:
The vector field is a gradient field with potential $f(x,y) = e^x + e^y$. By the fundamental theorem of line integrals, we can just compute the potential at the beginning and end points. The first integral is $f(3,-1) - f(-3,-1) = e^3 - e^{-3}$. The second is $f(-3,0) - f(3,0) = e^3 - e^{-3}$. The two line integrals are the same. The answer in both cases is $\boxed{e^3 - e^{-3}}$.

Problem 13) (10 points)



The "foot in the mouth" surface S seen in the picture is parametrized by

$$\vec{r}(u,v) = \langle \quad (4 + g(u,v)\cos(v))\cos(u) - 4, \\ (4 + g(u,v)\cos(v))\sin(u), \\ g(u,v)\sin(v)\rangle$$

with $g(u,v) = (2 - \frac{u}{2\pi})$ and $0 \le u, v \le 2\pi$. It is oriented outwards. Its boundary consists of two circles in the xz-plane centered at the origin, with radius 1 and 2. Find the flux of the curl of

$$\vec{F}(x,y,z) = \langle z + yz, x, \sin(x^3 y) + y^2 + z^4 \rangle$$

through S.

Solution:
We use Stokes theorem. The boundary consists of two curves. The outer one is oriented clockwise the inner one counter clockwise.

$$\vec{r}_1(t) = \langle 2\cos(t), 0, -2\sin(t)\rangle$$

$$\vec{r}_2(t) = \langle \cos(t), 0, \sin(t)\rangle$$

In the xz-plane, we have $y = 0$ and $\vec{F} = \langle z, x, z^4\rangle$. Write down the two line integrals

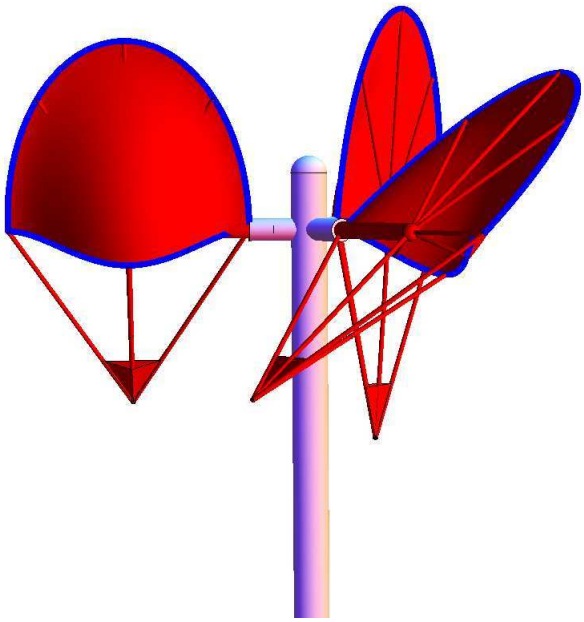
$$\int_{C_1} \vec{F} \cdot \, d\vec{r} = \int_0^{2\pi} \langle -2\sin(t), 2\cos(t), 16\sin(t)^2\rangle \cdot \langle -2\sin(t), 0, 2\cos(t)\rangle \, dt = 4\pi$$

$$\int_{C_2} \vec{F} \cdot \, d\vec{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t), \sin(t)^2\rangle \cdot \langle \sin(t), 0, -\cos(t)\rangle \, dt = -\pi$$

The answer is $\boxed{3\pi}$.

Problem 14) (10 points)





Porter square in Cambridge features a moving art project. It proves that paddle wheels and curl are omni present. To build that model for google earth, we idealized one of the blades as the surface

$$\vec{r}(t,s) = \langle 2\cos(t)\sin(s), 4\sin(t)\sin(s), \cos^4(s)\rangle$$

with $t, s \in [0, \pi]$. The surface S is oriented so that the boundary consists of two parts. The surface boundary can be parametrized as

$$\vec{r}_1(t) = \langle 2\sin(t), 0, \cos^4(t)\rangle, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

and

$$\vec{r}_2(t) = \langle 2\cos(t), 4\sin(t), 0\rangle, 0 \leq t \leq \pi \ .$$

The wind force is given by the curl of the vector field $\vec{F}(x,y,z) = \langle y, z, 0\rangle$. Find the flux of curl$(\vec{F})$ through the surface S.

Solution:
Use Stokes theorem and compute two line integrals.

$$\int_{-\pi/2}^{\pi/2} \langle 0, \cos^4(t), 0\rangle \cdot \langle 2\cos(t), 0, 4\cos^3(t)\rangle \ dt = 0$$

$$\int_0^{\pi} \langle 4\sin(t), 0, 0\rangle \cdot \langle -2\sin(t), 4\cos(t), 0\rangle \ dt = -8\int_0^{\pi} \sin^2(t) \ dt = -4\pi \ .$$

The result is the sum of these two line integrals which is $\boxed{-4\pi}$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The tangent plane of the surface $z = f(x, y)$ at a local maximum of f is parallel to the xy-plane

Solution:
The tangent plane to the function $g(x, y, z) = z - f(x, y)$ can be written $\langle -f_x, -f_y, 1\rangle$ which is $\langle 0, 0, 1\rangle$ at critical points.

2) [T] [F] For any smooth functions $f(x, y), x(t), y(t)$, we have $\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$.

Solution:
This is the chain rule.

3) [T] [F] At a local maximum of a function $f(x, y)$ we always have $f_{xx} \leq 0$ and $f_{yy} \leq 0$.

Solution:
Indeed if one of the conditions is not satisfied, then the function increases in some direction.

4) [T] [F] If $(0, 0)$ is a critical point for a function $f(x, y)$ as well as for a function $g(x, y)$ then $(0, 0)$ is a critical point of the function $f(x, y) + g(x, y)$.

Solution:
The gradient of the sum is also zero at $(0, 0)$.

5) [T] [F] The curves $\vec{r}(t) = \langle t, 2t\rangle$ and $\vec{s}(t) = \langle 2t, -t\rangle$ intersect at a right angle at $(0, 0)$.

Solution:
The velocity vectors are perpendicular at $(0, 0)$.

6) [T] [F] The quadric $x - y^2 + z^2 = 5$ is a hyperbolic paraboloid.

Solution:
It is, shifted and turned a bit

7) [T] [F] If $\vec{u}, \vec{v}, \vec{w}$ are unit vectors, then the length of the vector projection of $\vec{u} \times \vec{v}$ onto $\vec{w}$ is the same as the length of the vector projection of $\vec{v} \times \vec{w}$ onto $\vec{u}$.

Solution:
In both cases, we have the same triple scalar product

8) [T] [F] The partial differential equation $u_{tt} = u_{xx}$ is called the Clairaut equation.

Solution:
This is the wave equation

9) [T] [F] $\iint_R \sqrt{1 - x^2 - y^2}\, dx\, dy = \frac{2\pi}{3}$, where R is the region $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy-plane.

Solution:
The integral is the volume of half of the unit sphere.

10) [T] [F] There exists a vector field $\vec{F}(x, y, z)$ in space such that $\text{curl}(\vec{F}) = \langle 5x, -11y, 7z \rangle$.

Solution:
The divergence would have to be zero.

11) [T] [F] Let S is the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ with normal pointing away from the center. Then the flux integral is $\iint_S \langle 0, 0, 1 \rangle \cdot \, d\vec{S} = 2\pi$.

Solution:
The flux integral is the same than the flux integral through the disc by the divergence theorem which is π, not 2π.

12) [T] [F] The points that satisfy $\theta = \pi/4$ and $\phi = \pi/4$ form a surface which is part of a cone.

Solution:
This is a curve, not a surface

13) [T] [F] The curvature of the curve $\vec{r}(t) = \langle t, t, t^2 \rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle t^3, t^3, t^6 \rangle$ at $t = 0$.

Solution:
Different parametrization of the same curve

14) [T] [F] If $f(x, y, z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere (i.e. $\vec{F}$ is incompressible).

Solution:
No, the divergence of the gradient is the Laplacian and not encessarily zero.

15) [T] [F] For any function $f(x, y, z)$ we have $\text{curl}(\text{curl}(\text{grad}(f))) = \vec{0}$.

Solution:
Already the inner part $\text{curl}(\text{grad}(f)) = 0$ everywhere.

16) [T] [F] For any vector field $\vec{F}$ and any curve $\vec{r}$ parametrized on $[a, b]$ we have $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$.

Solution:
If $\nabla f = \vec{F}$ and we would replace $\vec{F}$ on the right hand side with f, we would get the fundamental theorem of line integrals. As it is, it already does not make sense because the left hand side is a scalar and the right hand side is a vector field.

17) [T] [F] There exist vector fields $\vec{F}$ and $\vec{G}$ in space such that $\text{curl}(\vec{F}) = \text{grad}(\vec{G})$.

Solution:
The right expression $\text{grad}(\vec{G})$ is not even defined.

18) [T] [F] If $\vec{F}$ is a smooth vector field in space and S is a closed oriented surface, then $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:
This follows from the divergence theorem or from Stokes theorem.

19) [T] [F] The solid enclosed by the surfaces $z = 2 - \sqrt{x^2 + y^2}$ and $z = \sqrt{x^2 + y^2}$ has the volume $\int_0^{2\pi} \int_0^1 \int_r^{2-r} r \, dz dr d\theta$.

Solution:
This is the volume in cylindrical coordinates.

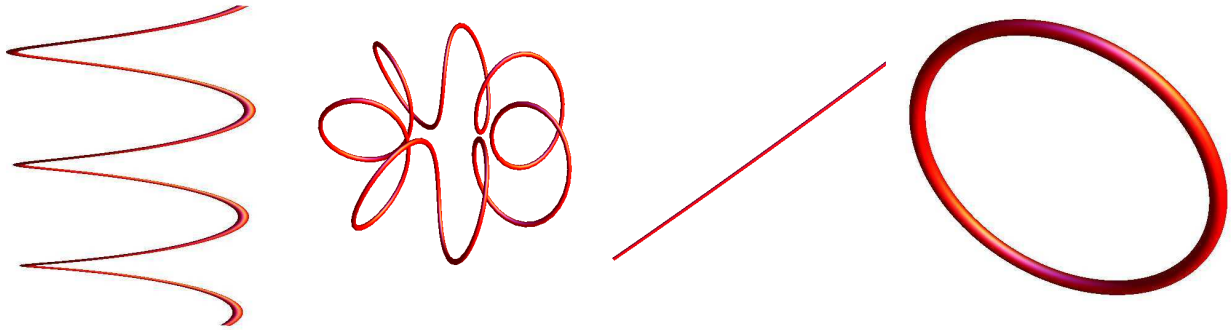
20) [T] [F] If $\vec{r}\,''(t) = \langle 0, 0, \sin(t) \rangle$, $\vec{r}(0) = \langle 0, 1, 0 \rangle$, $\vec{r}\,'(0) = \langle 1, 0, 0 \rangle$, then $\vec{r}(t) = \langle t, 1 + t, t - \sin(t) \rangle$.

Solution:
It is almost right. But already $\vec{r}\,'(0)$ does not fit.

Problem 2) (6 points)

a) (4 points) Match the curves. There is an exact match.

1 2 3 4

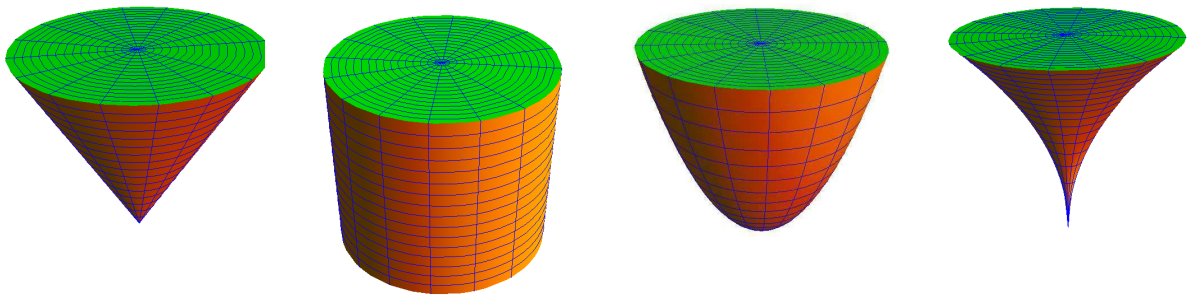


Enter 1-4	Object definition
	$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$
	$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$
	$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle t, t, t \rangle$

b) (4 points) Match the solids with the triple integrals. Also here, there is an exact match:

Enter A-D	3D integral computing volume
	$\int_0^{2\pi}\int_0^1\int_r^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_{r^2}^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_{\sqrt{r}}^1 r\ dzdrd\theta$
	$\int_0^{2\pi}\int_0^1\int_0^1 r\ dzdrd\theta$

A B C D



c) (2 points) What was the name again?

Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = -g_{yy}$

Solution:
a) 1,4,2,3
b) A,C,D,B
c) transport and Laplace

Problem 3) (10 points)

a) (5 points) For the following quantities, decide whether they are vector fields or scalar fields (functions) or nonsense. Here $\vec{F} = \langle P, Q, R \rangle$ is a vector field in space, $f(x, y, z)$ is a scalar function and $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$. Recall that $\nabla \times \vec{F} = \text{curl}(\vec{F}), \nabla \cdot \vec{F} = \text{div}(\vec{F})$ and $\nabla f = \text{grad}(f)$.

object	scalar	vector	not defined
$\nabla \vec{F}$			
$\nabla \cdot \vec{F}$			
$\nabla \times \vec{F}$			
$\nabla(\nabla \cdot \vec{F})$			
$\nabla \times (\nabla \times \vec{F})$			
$\nabla \times (\nabla \cdot \vec{F})$			
∇f			
$\nabla f \times \vec{F}$			
$\nabla f \cdot \vec{F}$			
$\nabla \times (\nabla f)$			

Solution:
a)

object	scalar	vector	not defined
$\nabla \vec{F}$			*
$\nabla \cdot \vec{F}$	*		
$\nabla \times \vec{F}$		*	
$\nabla(\nabla \cdot \vec{F})$		*	
$\nabla \times (\nabla \times \vec{F})$		*	
$\nabla \times (\nabla \cdot \vec{F})$			*
∇f		*	
$\nabla f \times \vec{F}$		*	
$\nabla f \cdot \vec{F}$	*		
$\nabla \times (\nabla f)$		*	

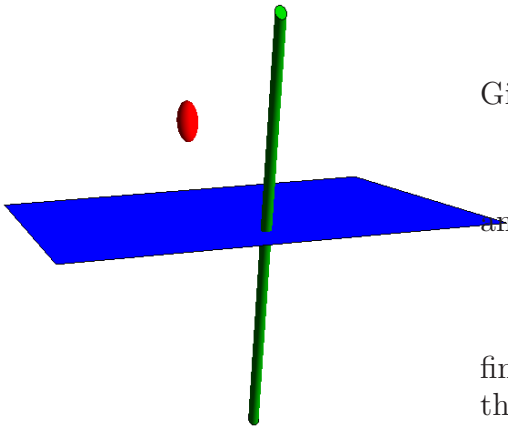
b) (5 points) Match the formulas for the position vector $\vec{r}(t)$ of a curve in space:

label	formula
A	$\vec{r}\,''(t)$
B	$\int_a^b \lvert\vec{r}\,'(t)\rvert\, dt$
C	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
D	$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$
E	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

expression	enter A-E
curvature	
unit normal vector	
unit tangent vector	
arc length	
acceleration	

Solution:
b) E,D,C,B,A

Problem 4) (10 points)



Given a point $P = (4, 3, 1)$, a plane

$$\Sigma : 3x + 4y - 12z = 0$$

and a line

$$L : \frac{x-1}{3} = \frac{y-2}{4} = \frac{z-1}{12}\,,$$

find the sum $d(P, L) + d(P, \Sigma)$ of the distances of P to the line and plane.

Solution:
A point on the plane is $(0,0,0)$. The distance of the point to the plane is

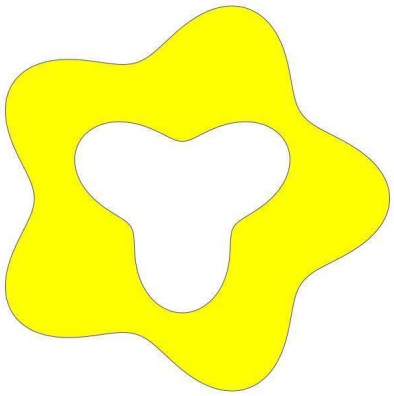
$$d(P,\Sigma) = \frac{|\langle 4,3,1\rangle \cdot \langle 3,4,12\rangle|}{|\langle 3,4,12\rangle|} = 12/13 \; .$$

A point on the line is $(1,2,1)$. The distance of the point to the line is

$$d(P,L) = \frac{|\langle 3,1,0\rangle \times \langle 3,4,12\rangle|}{|\langle 3,4,12\rangle|} = \sqrt{1521}/13 = 39/13 \; .$$

The sum is $\boxed{51/13}$.

Problem 5) (10 points)



a) (5 points) Find the double integral

$$\int_0^3 \int_y^3 \frac{\sin(2x)}{x} \, dx \, dy \; .$$

b) (5 points) What is the area of the polar region

$$3 + \sin(3\theta) \leq r \leq 6 + \cos(5\theta) \; ?$$

Solution:
a) This is a typical switching of the order of integration problem. Writing it as a type I integral gets rid of the x

$$\int_0^3 \int_0^x \frac{\sin(2x)}{x} \, dydx = \frac{1-\cos(6)}{2} \; .$$

b) The integral in polar coordinates is

$$\int_0^{2\pi} \int_{3+\sin(3\theta)}^{6+\cos(5\theta)} r \; drd\theta = \int_0^{2\pi} \frac{(6+\cos(5\theta))^2}{2} - \frac{(3+\sin(3\theta))^2}{2} \, d\theta = 27\pi \; .$$

The result is $\boxed{27\pi}$.

Problem 6) (10 points)

a) (8 points) Locate and classify all the local maxima, minima and saddle points of the function

$$f(x,y) = x^4 + y^4 - 8x^2 - 8y^2 \ .$$

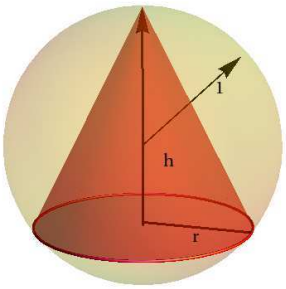
b) (2 points) Is there a global maximum or a global minimum of f? Explain.

Solution:
a)

x	y	D	f_{xx}	Type	f
-2	-2	1024	32	minimum	-32
-2	0	-512	32	saddle	-16
-2	2	1024	32	minimum	-32
0	-2	-512	-16	saddle	-16
0	0	256	-16	maximum	0
0	2	-512	-16	saddle	-16
2	-2	1024	32	minimum	-32
2	0	-512	32	saddle	-16
2	2	1024	32	minimum	-32

b) There is no global maximum since for $y = 0$ already the function blows up for $x \to \infty$. There is a global minimum since the function goes to $+\infty$ at infinity in all directions. These global minima are critical points and are the points, where $f = -32$.

Problem 7) (10 points)



For which base radius r and height h does a cone inscribed into the unit sphere have maximal volume $f(r,h) = \pi r^2 h/3$? The constraint is given by Pythagoras as $g(r,h) = r^2 + (h-1)^2 = 1$. Use the Lagrange method.

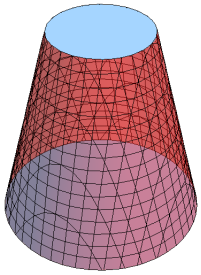
Solution:
The gradient of f is $\nabla f(r,h) = \langle \pi 2rh/3, \pi r^2/3 \rangle$, the gradient of g is $\nabla g(r,h) = \langle 2r, 2h-2 \rangle$. The Lagrange equations are

$$\begin{aligned} 2\pi hr/3 &= \lambda 2r \\ \pi r^2/3 &= \lambda(2h-2) \\ r^2+(h-1)^2 &= 1\ . \end{aligned}$$

Eliminating λ gives $2h/r = r/(h-1)$ or $r^2 = 2h(h-1)$. Plugging this into the third equation gives $\boxed{h=4/3}$. Therefore $\boxed{r = 2\sqrt{2}/3}$.

Problem 8) (10 points)





A bird's feeding cage E is part of a cone $x^2+y^2 = 4(3-z)^2$ with $1 < z < 2$. The cage is filled with different kind of seeds, the heavier have gone down and the density is $(3-z)$. We want to find the moment of inertia

$$\iiint_E (x^2+y^2)(3-z)\ dxdydz$$

so that we can know how much energy the feeding cage has if a squirrel spins it. You do not have to worry in this problem that squirrels are not birds.

Solution:
The integral in cylindrical coordinates is

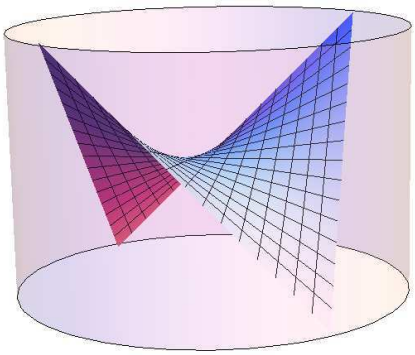
$$\int_0^{2\pi}\int_1^2\int_0^{2(3-z)} r^3(3-z)\ drdzd\theta\ .$$

Start at the inner integral $\int_0^{2(3-z)} r^3(3-z)dr$ which is $2^4(3-z)^4(3-z)/4 = 4(3-z)^5$. Now integrate this (do not multiply out the polynomial!)

$$\int_1^2 4(3-z)^5 dz = -4(3-z)^6/6\Big|_1^2 = 4(2^6-1^6)/6 = 42\ .$$

Integrating over θ gives $\boxed{84\pi}$.

Problem 9) (10 points)



a) (4 points) Find the surface area of the surface

$$\vec{r}(s,t) = \langle s, -t, 2st \rangle$$

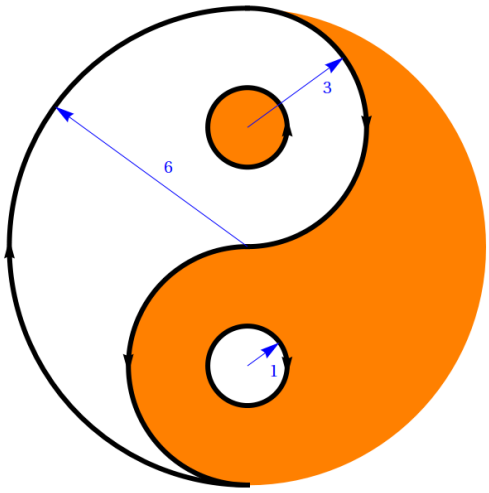
with $s^2 + t^2 \leq 9$.
b) (4 points) The coordinates of the surface satisfies $2xy + z = 0$. Find the tangent plane at $(1, 1, -2)$.
c) (2 points) What is the formula for the linearization of $f(x, y) = 2xy$ at the point $(1, 1)$.

Solution:
a) $r_s = \langle 1, 0, 2t \rangle$, $r_t = \langle 0, -1, 2s \rangle$ and $r_s \times r_t = \langle 2t, -2s, -1 \rangle$ with length $\sqrt{4t^2 + 4s^2 + 1}$. To integrate this over a disc of radius 3, we use polar coordinates:

$$\int_0^3 \int_0^{2\pi} \sqrt{1 + 4r^2} r \, d\theta dr = 2\pi(4r^2 + 1)^{3/2}/12\Big|_0^3 .$$

This is $\boxed{\pi(37^{3/2} - 1)/6}$.
b) The gradient of $g(x, y, z) = z + 2xy = 0$ is $\langle 2y, 2x, 1 \rangle$ which is at the point $(1, 1, -2)$ equal to $\langle 2, 2, 1 \rangle$ so that the equation of the tangent plane is $2x+2y+z = d$. The constant d is obtained by plugging in the point $(1, 1, -2)$ which gives $\boxed{2x + 2y + z = 2}$.
c) To get the linearization, compute the gradient of $f(x, y) = 2xy$ which is $2, 2$ if $f(1, 1) + 2(x - 1) + 2(y - 1) = 2x + 2y - 2$. We have $\boxed{L(x, y) = 2x + 2y - 2}$.

Problem 10) (10 points)



Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{F}(x, y) = \langle -y + \sin(e^x), x \rangle$$

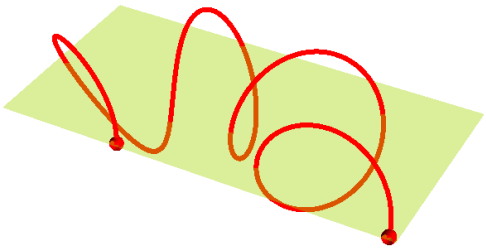
around C. Notice that the Ying and the Yang have the same area.

Solution:
We use Greens theorem noticing however that the orientation of the curve is negative:

$$\int_C F(r(t)) \cdot r'(t)\, dt = -\int\int \text{curl}(\vec{F})(x,y)\, dxdy$$

Because the curl of $\vec{F}$ is constant 2, the right hand side is $-2\text{Area(Yang)} = -2\pi 6^2/2$ which is $\boxed{-36\pi}$.

Problem 11) (10 points)



Let C be the curve

$$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$$

parametrized by $0 \leq t \leq \pi$ starting at $t = 0$ and ending at $t = \pi$. Calculate the line integral

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt\ ,$$

where $\vec{F}$ is the vector field

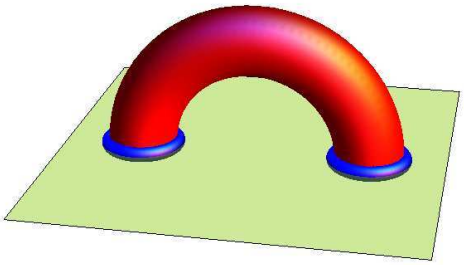
$$\vec{F}(x,y,z) = \langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2} \rangle\ .$$

Solution:
We notice that

$$\vec{F} = \nabla f, f(x,y,z) = e^{2x^2+3y^2+4z^2}\ .$$

The fundamental theorem of line integrals assures that the line integral is the potential difference $f(\vec{r}(\pi)) - f(\vec{r}(0))$. We see that $\vec{r}(\pi) = \langle -1, 0, 0 \rangle$ and $\vec{r}(0) = \langle 3, 0, 0 \rangle$. Now $f(\vec{r}(\pi)) - f(\vec{r}(0)) = \boxed{e^2 - e^{18}}$.

Problem 12) (10 points)



Find the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x^2, 0\rangle$ through a half torus surface S given by $(\sqrt{x^2+z^2}-3)^2+y^2=1, z\geq 0$ which intersects the xy-plane $z=0$ in two circles $C_1 : (x-3)^2+y^2=1$ and $C_2 : (x+3)^2+y^2=1$. The torus S is oriented outwards.

Solution:
By Stokes theorem, it is the sum of the line integrals along the two circles. Parametrize them in the right orientation as

$$\vec{r}_1(t) = \langle 3+\cos(t), \sin(t), 0\rangle, \vec{r}_2(t) = \langle -3+\cos(t), \sin(t), 0\rangle \ .$$

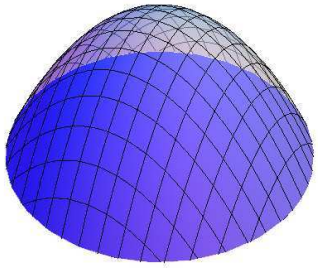
Now compute the line integrals

$$\int_0^{2\pi} \langle -\sin(t), (3+\cos(t))^2, 0\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle \ dt = 7\pi$$

$$\int_0^{2\pi} \langle -\sin(t), (-3+\cos(t))^2, 0\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle \ dt = -5\pi$$

The sum is $\boxed{2\pi}$. **Remark.**It was also possible to compute these line integrals by applying Stokes again and using the flux of the curl through the discs. Alternatively, one could have seen with the divergence theorem that the flux of the curl through the torus is the sum of the fluxes through the bottom closure discs. Computing the flux through the disc needs a parametrization of the discs and a computation of the curl. It could also be done by applying only then Stokes. A wide variety of solutions applied here and the class has explored essentially all possible cases.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3 z, y^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part S of the boundary of the solid

$$G : z + x^2 + y^2 \leq 1,\ z \geq 0\ .$$

The paraboloid surface S is oriented upwards.

Solution:
Use the divergence theorem. The divergence of $\vec{F}$ is $(3x^2 + 3y^2)z$. Denote by G the solid bound by the xy-plane and the paraboloid. We have

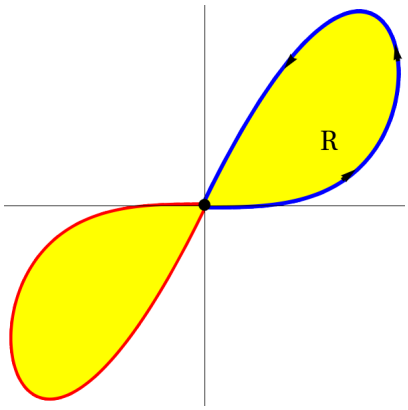
$$\int\int\int_G \mathrm{div}(\vec{F})(x,y,z)\ dxdydz = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r(3r^2)z\ dzdrd\theta = \pi/8\ .$$

Denote by D the floor surface. It is parametrized as $\vec{r}(s,t) = \langle s,t,0\rangle$ with $s^2+t^2 \leq 1$. The vector field on the floor is $\vec{F}(x,y,z) = \langle 0,0,1+e^{x^2+y^2}\rangle$. The flux of this through the floor D parametrized by $r(u,v) = \langle u,v,0\rangle$ is

$$\int\int_D 1 + \exp(u^2+v^2)\ dudv = \int_0^{2\pi}\int_0^1 (1+\exp(r^2)r\ drd\theta = e\pi\ .$$

The total result is $\boxed{\pi/8 + e\pi}$.

Problem 14) (10 points)



Find the area of the **propeller** shaped region enclosed by the figure 8 curve

$$\vec{r}(t) = \langle t - t^3, 2t^3 - 2t^5 \rangle\ ,$$

parametrized by $-1 \leq t \leq 1$. To find the total area compute the area of the region R enclosed by the right loop $0 \leq t \leq 1$ and multiply by 2.

Solution:
This is a typical Green problem. We compute the line integral of the vector field $\vec{F}(x,y) = \langle 0, x\rangle$ along the curve

$$2\int_0^1 \langle 0, t-t^3\rangle \cdot \langle 1-3t^2, 6t^2-10t^4\ dt = 1/6\ .$$

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The functions $e^{x^2+y^3-y}$ and x^2+y^3-y have the same critical points.

Solution:
The gradient of the first function is $e^{x^2+y^3-y}$ times the gradient of the second.

2) [T] [F] The line $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$ hits the plane $x+y+z = 100$ at a right angle.

Solution:
Yes, the normal vector to the plane is parallel to $\langle 1, 1, 1 \rangle$.

3) [T] [F] The quadric $x^2 - 2y^2 + z^2 = 5$ is a one sheeted hyperboloid.

Solution:
It is stretched a bit in the y direction by a factor $\sqrt{2}$ and scaled by a factor $\sqrt{5}$ but otherwise, it is a standard hyperboloid

4) [T] [F] The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.

Solution:
It is also possible if they are nonzero but form a 45 degree angle.

5) [T] [F] The partial differential equation $u_x = u_{tt}$ is called the **Heat equation**.

Solution:
The variables are reversed.

6) [T] [F] The curvature of the curve $\vec{r}(t) = \langle \sin(2t), \cos(2t)/\sqrt{2}, \cos(2t)/\sqrt{2} \rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$ at $t = 0$.

Solution:
Both are circles of radius 1 which have curvature 1.

7) [T] [F] The space curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.

Solution:
We can check $x^2 + z^2 = 1$.

8) [T] [F] If a smooth function $f(x, y)$ has a global maximum, then it has a global minimum.

Solution:
A counter example is $f(x, y) = -x^2 - y^2$.

9) [T] [F] If $L(x, y)$ is the linearization of $f(x, y)$ at (x_0, y_0) and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ on $f = c$ at the point $\vec{r}(t_0) = \vec{s}(t_0) = (x_0, y_0)$ so that $|\vec{r}\,'(t_0)| = |\vec{s}\,'(t_0)| = 1$, then $|d/dtL(\vec{s}(t))| = |d/dtf(\vec{r}(t))|$ at the time $t = t_0$.

Solution:
Locally, near $(x-0, y_0)$ the linearization $L(x, y)$ is close to the function $f(x, y)$. The level surface to f through (x_0, y_0) and the level surface of $L(x, y)$ have the same normal vector $\vec{n} = \nabla f = \nabla L$. The vectors $\vec{v} = \vec{s}\,'(t_0)$ and $\vec{w} = \vec{r}\,'(t_0)$ are both normal to the same vector and so parallel. Since they have the same length, either $\vec{v} == \vec{w}$ or $\vec{v} = -\vec{w}$. By the chain rule $d/dtL(\vec{s}(t)) = \vec{n} \cdot \vec{v} = \pm \vec{n} \cdot \vec{w} = \pm d/dtf(\vec{r}(t))$. The absolute values agree.

10) [T] [F] If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.

Solution:
The power $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$ is positive and so is the integral.

11) [T] [F] The flux of the vector field $\vec{F} = \nabla f$ through the surface $f(x, y, z) = x^4 + y^4 + z^4 = 1$ is positive if the surface is oriented so that $\vec{r}_u \times \vec{r}_v$ points in the direction of the gradient of f.

Solution:
If we parametrize the surface with $\vec{r}(u, v)$, then the flux is $\int\int_S \nabla f(\vec{r}(u, v)) \cdot (r_u \times r_v)\, dudv$. The integrand is positive by assumption.

12) [T] [F] If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, all points on the constraint curve are extrema for f.

Solution:
Every point on the curve $g(x, y) = 1$ is a solution to the Lagrange equations because $\nabla f = \nabla g$.

13) [T] [F] If a point (x_0, y_0) is a minimum of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a local minimum of the function $f(x, y)$ without constraints.

Solution:
The gradient of f does not have to be the zero vector.

14) [T] [F] If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any closed ellipse is zero.

Solution:
This follows from the fundamental theorem of line integrals.

15) [T] [F] The flux of an irrotational vector field is zero through any surface S in space.

Solution:
First of all, the surface does not need to be closed. But even if the surface is closed, this would still be false: the field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ is irrotational but the flux through any closed surface is zero.

16) [T] [F] The divergence of a gradient field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is everywhere zero.

Solution:
While the curl of gradient is zero and the divergence of a curl is zero, the divergence of a gradient is the Laplacian of f and not necessarily zero.

17) [T] [F] The line integral of the vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a circle in the $xy-$ plane is zero.

Solution:
Yes, by the fundamental theorem of line integrals.

18) [T] [F] For any solid E, the moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ is always larger than the volume $\iiint_E 1 \, dxdydz$ of E.

Solution:
For small solids, the moment of inertia is small, for large solids, the moment of inertia is large.

19) [T] [F] The curvature of a parametrized curve satisfying $|\vec{r}\,'(t)| = 1$ is bounded above by the length $|\vec{r}\,''|$ of the acceleration.

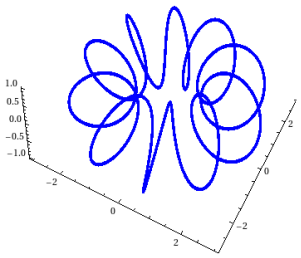
Solution:
One can deduce this directly from the formula for curvature $|\vec{r}\,' \times \vec{r}\,''| \leq |\vec{r}\,''|$ in that case. Actually, more is true, the curvature is in this case the length of the acceleration.

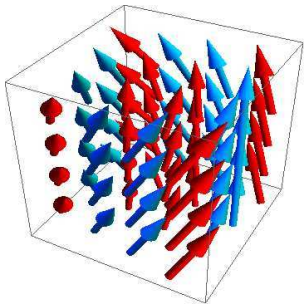
20) [T] [F] Given a vector field $\vec{F} = \langle P, Q, R \rangle$, the directional derivative of $\operatorname{div}(\vec{F}(x, y, z))$ in the direction $\vec{v} = \langle 1, 0, 0 \rangle$ is $P_{xx} + Q_{xy} + R_{xz}$.

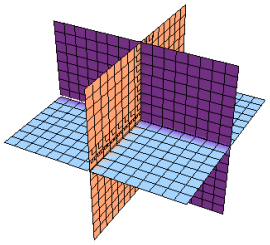
Solution:
Yes by definition $\operatorname{div}(\vec{F}(x, y, z)) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z)$. The directional derivative in the $\langle 1, 0, 0 \rangle$ direction is the partial derivative with respect to x. We use also Clairot.

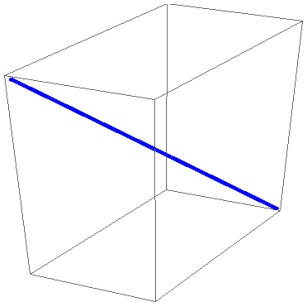
Problem 2) (6 points)

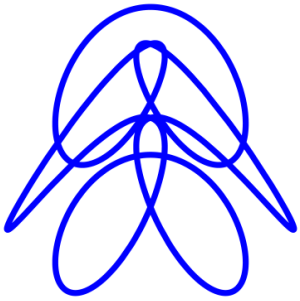
a) (6 points) Match the objects with their definitions

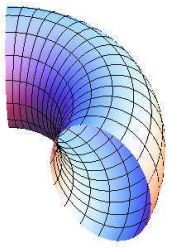
1 

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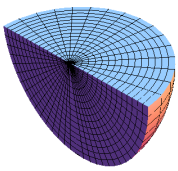
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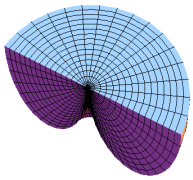
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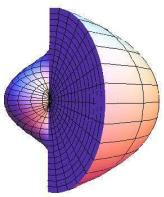
Enter 1-6	Object definition
	$\vec{r}(t) = \langle (2+\cos(10t))\cos(t), (2+\cos(10t))\sin(t), \sin(10t) \rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 2 \rangle$
	$\vec{r}(t,s) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
	$x^2y^2z^2 = 0$
	$(x-1)/5 = (y-2)/10 = (z-1)/3$
	$\vec{r}(t) = \langle \sin(t)+\cos(5t), \cos(t)+\cos(6t) \rangle$

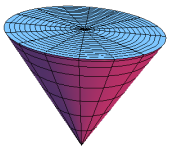
b) (4 points) Match the solids with the triple integrals:

Enter A-D	3D integral computing volume
	$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1/\cos(\phi)} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{\sin(\phi)} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{1} \rho^2\sin(\phi)\, d\rho d\phi d\theta$
	$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi-\theta} \rho^2\sin(\phi)\, d\rho d\phi d\theta$

A 

B 

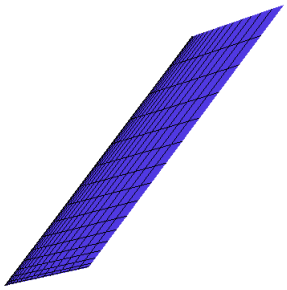
C 

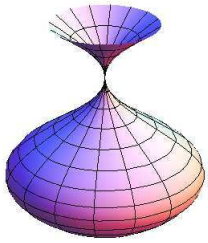
D 

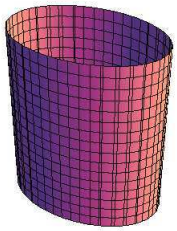
Solution:
a) 1,2,6,3,4,5
b) D,B,A,C

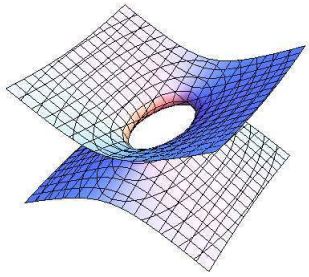
Problem 3) (10 points)

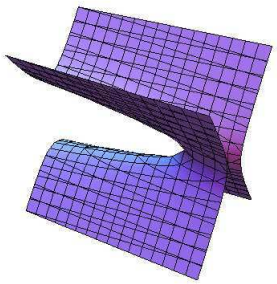
a) (6 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.

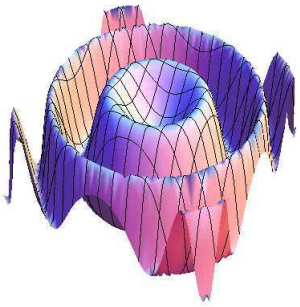

A


B


C


D


E


F

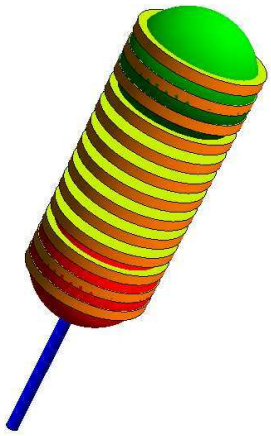
Enter A-F here	Function or parametrization
	$\vec{r}(u,v) = \langle u^2, v^2, u^2+v^2 \rangle$
	$\vec{r}(u,v) = \langle (1+\sin(u))\cos(v), (1+\sin(u))\sin(v), u \rangle$
	$4x^2+y^2-9z^2=1$
	$x-9y^2+4z^2=1$
	$\vec{r}(u,v) = \langle u, v, \sin(u^2+v^2) \rangle$
	$4x^2+9y^2=1$

b) (4 points) If the blank box is replaced by $\nabla f(5,6)$ the statement becomes true or false. Determine which case we have. The function $f(x,y)$ is an arbitrary nice function like for example $f(x,y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x,y) = f(5,6)$ and has the property that $\vec{r}(0) = \langle 5, 6 \rangle$.

True/False	Topic	Statement
	Linearization	$L(x,y) = f(5,6) + \square \cdot \langle x-5, y-6 \rangle$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t))\|_{t=0} = \square \cdot \vec{r}\,'(0)$
	Steepest descent	f decreases at $(5,6)$ most in the direction of $\square$
	Estimation	$f(5+0.1, 5.99) \sim f(5,6) + \square \cdot \langle 0.1, -0.01 \rangle$
	Directional derivative	$D_{\vec{v}} f(5,6) = \square \cdot \vec{v}$, $\|\vec{v}\| = 1$
	Level curve	of f through $(5,6)$ has the form $\square \cdot \langle x-5, y-6 \rangle = 0$
	Vector projection	of $\nabla f(5,6)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \square)/\|\vec{v}\|^2$
	Tangent line	of $\vec{r}(t)$ at $(5,6)$ is parametrized by $\vec{R}(s) = \langle 5,6 \rangle + s \square$

Solution:
a) A,B,D,E,F,C
b) T,T,F,T,T,F,T,F

Problem 4) (10 points)



Two ice cream scoops given by spheres

$$x^2 + y^2 + (z+1)^2 = 1$$

and

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 1$$

are enclosed by a cylinder which is tangent to both spheres. Find the equation of the cylinder.

Hint: consider the distance of a general point (x, y, z) to the line passing through the centers of the spheres.

Solution:
The line L passes through the points $A = (0, 0, -1)$ and $B = (1, 1, 2)$ which are connected by the vector $\vec{v} = \langle 1, 1, 3\rangle$. The distance from a point $P = (x, y, z)$ to the line is given by

$$d(P, L) = \frac{|\langle x - 0, y - 0, z + 1\rangle \times \langle 1, 1, -3\rangle|}{|\langle 1, 1, -3\rangle|} .$$

The cross product in that formula

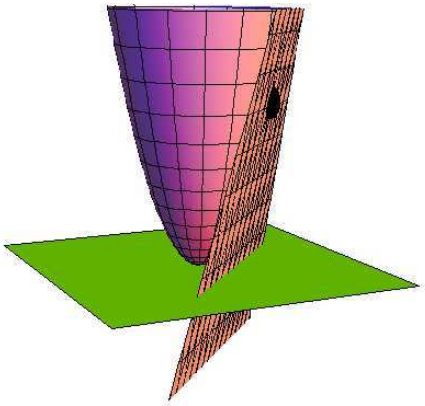
$$\langle x, y, z + 1\rangle \times \langle 1, 1, 3\rangle = \langle 3y - z - 1, z + 1 - 3x, x - y\rangle .$$

The equation $d(P, L) = 1$ is equivalent to $d(P, L)^2 = 1$ which is

$$(3y - z - 1)^2 + (z + 1 - 3x)^2 + (x - y)^2 = 11 .$$

Its fine to leave the result like this but one could write it as $\boxed{-6x + 10x^2 - 6y - 2xy + 10y^2 + 4z - 6xz - 6yz + 2z^2 = 9}$.

Problem 5) (10 points)



Find a parametrization

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

for the line obtained by intersecting the tangent plane Σ to the surface

$$x^2 + y^2 - z = 0$$

at $(-1, -1, 2)$ with the xy-plane.

Solution:
The gradient of f is $\nabla f(x,y,z) = \langle 2x, 2y, -1\rangle$ and at the given point $\nabla f(-1,-1,2) = \langle -2,-2,-1\rangle$. The tangent plane is

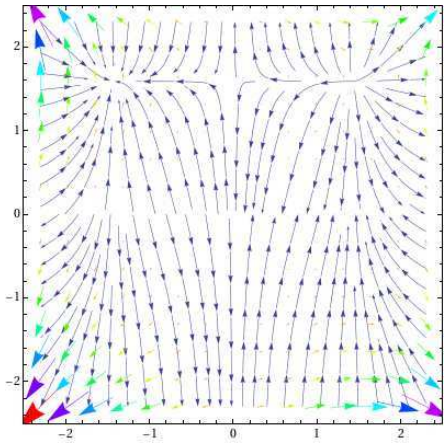
$$-2x - 2y - z = 2 \; .$$

For $z = 0$, we have $x + y = -1, z = 0$. While these two equations describe the line, we were asked to compute a parametrization. It is most conveniently found by picking two points on the line like $(-1,0,0)$ and $(0,-1,0)$. The vector connecting these points is $\langle 1,-1,0\rangle$. The parametrization of the line is

$$\vec{r}(t) = \langle -1,0,0\rangle + t\langle 1,-1,0\rangle \; .$$

The answer is $\boxed{\vec{r}(t) = \langle t-1, -t, 0\rangle}$.

Problem 6) (10 points)



The vector field

$$\vec{F}(x,y) = \langle P, Q\rangle = \langle y(x^4 - 2x^2), x(y^4 - 4y)\rangle$$

has the curl

$$f(x,y) = \text{curl}(\vec{F})(x,y) = Q_x(x,y) - P_y(x,y) \; .$$

Find and classify all critical points of f by deciding whether they are local maxima, local minima or saddle points. Is there a global maximum or global minimum of f?

Solution:
We have $Q_x = y^4 - 4y, P_y = x^4 - 2x^2$ so that we have to find the extrema of the function

$$f(x, y) = y^4 - 4y - x^4 + 2x^2 \ .$$

The gradient of f is

$$\nabla f(x, y) = \langle -4x^3 + 4x, 4y^3 - 4 \rangle \ .$$

The critical points satisfy the two equations

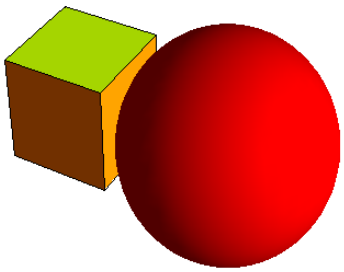
$$\begin{aligned} -4x^3 + 4x &= 0 \\ 4y^3 - 4 &= 0 \ . \end{aligned}$$

This leads to $y = 1$ and $x = -1, 0, 1$ so that we have three critical points. To apply the **second derivative test**, we have to compute $f_{xx} = -12x^2 + 4$ and the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = (-12x^2 + 4)(12y^2)$.

Point	D	f_{xx}	Nature of critical point	local or global
$(-1, 1)$	-96	-8	saddle	neither max nor min
$(0, 1)$	48	4	min	local minimum
$(1, 1)$	-96	-8	saddle	neither max nor min

There are no global maxima nor global minima because the function takes arbitrarily large negative values $x = 0, x \to -\infty$ and arbitrarily large values $x = 0, y \to \infty$.

Problem 7) (10 points)



We want to minimize the volume of the union of a **sphere** of radius x and a **cube** of side length y under the constraint that the sum of the two surface areas is equal to 4. Find the minimal value using the Lagrange method.

Remark: You do not have to show any derivations of the volume and surface area of the sphere.

Solution:
We want to minimize the function

$$f(x,y) = 4\pi x^3/3 + y^3$$

under the constraint

$$g(x,y) = 4\pi x^2 + 6y^2 = 4 \ .$$

The Lagrange equations are

$$\begin{aligned} 4\pi x^2 &= \lambda 8\pi x \\ 3y^2 &\quad \lambda 12y \\ 4\pi x^2 + 6y^2 &= 4 \ . \end{aligned}$$

These equations have a solution with $x = 0$ which is $(x,y) = (0, \sqrt{2/3})$ and a solution with $y = 0$ which is $(x,y) = (\sqrt{1/\pi}, 0)$ and a solution, where both x and y are nonzero. To get this solution, divide the first equation by the second, to get $(4/3)\pi x^2/y^2 = (2/3)\pi x/y$ or $2x = y$. This gives $x = (6+\pi)^{-1/2}$ and $y = 2(6+\pi)^{-1/2}$. This point is the minimum. The answer is $\boxed{(x,y) = ((6+\pi)^{-1/2}, 2(6+\pi)^{-1/2})}$.

Remark: this problem is inspired by the corresponding two dimensional problem presented in the planar exam review. The 2D problem in the plane has been described in the new book "The Mathematical Mechanic: Using Physical Reasoning to Solve Problems by Mark Levi, 2009". The book shows how the solution can be explained using physical reasoning. Both in the two and three dimensional situation, the cube length is the same than the diameter of the circle or sphere.

Remark. Some more work had been required here, if the formula for the surface area of the sphere were not known. In that case, one had to do the computation $\int_0^{2\pi}\int_0^{\pi} R^2 \sin(\phi)\ d\phi d\theta = 4\pi R^2$ and $\int_0^{2\Pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin(\phi) d\rho d\phi d\theta = 4\pi R^3/3$.

Problem 8) (10 points)

A solid E in space is determined by the inequalities

$$0 \le z \le 9 \ ,$$

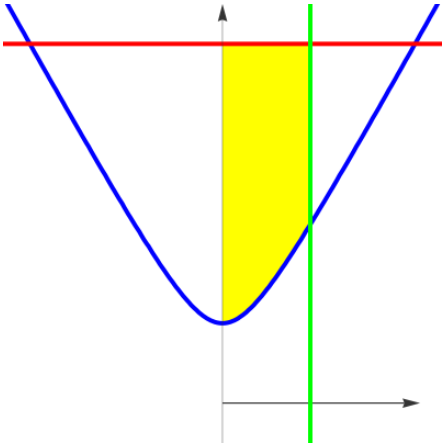
$$z^2 - x^2 - y^2 \ge 4$$

and

$$x^2 + y^2 \le 1 \ .$$

Find the volume of E.

Solution:

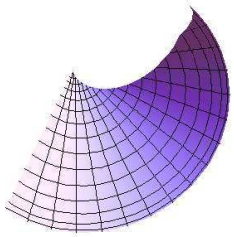


Use **cylindrical coordinates**. It is crucial to **draw the situation** in the rz-plane to read off the integration bounds: the region is bounded from below by the hyperboloid $z^2 - r^2 = 4$ (a hyperbola in the rz-plane) from above by the plane $z = 9$ (a horizontal line in the rz plane) and to the side by the cylinder $r = 1$ (a vertical line in the rz-plane). The computation is:

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{\sqrt{4+r^2}}^9 r dz dr d\theta &= 2\pi \int_0^1 r(9 - \sqrt{4+r^2})\, dr \\ &= \pi(9 + 16/3 - 2 \cdot 5^{3/2}/3)\ . \end{aligned}$$

The final result is $\boxed{\pi(9 + 16/3 - 2 \cdot 5^{3/2}/3)}$.

Problem 9) (10 points)



A surface S is parametrized by

$$\vec{r}(u, v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle$$

where

$$0 \leq u \leq \sqrt{\pi}, u^2 \leq v \leq \pi\ .$$

Find its surface area.

Solution:
We have $|r_u \times r_v| = \sqrt{8} u e^{-2u^2}$. The integral

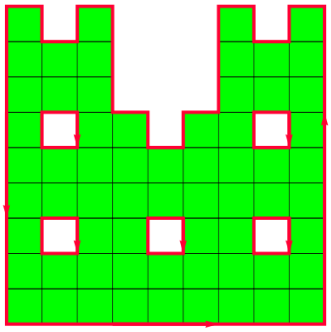
$$\int_0^{\sqrt{\pi}} \int_{u^2}^{\pi} \sqrt{8} u e^{-2u^2} dv du$$

becomes after a **switch of variables** (make a picture!)

$$\int_0^{\pi} \int_0^{\sqrt{v}} \sqrt{8} u e^{-2u^2} du dv\ .$$

The result is $\boxed{(2\pi - 1 + e^{-2\pi})/(2\sqrt{2})}$.

Problem 10) (10 points)



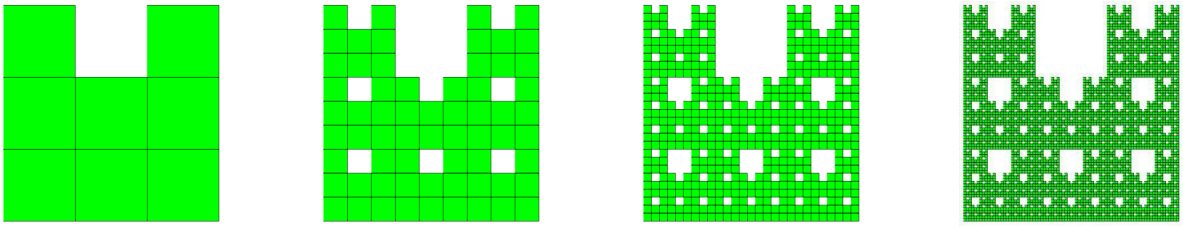
What is the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field

$$\vec{F}(x,y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

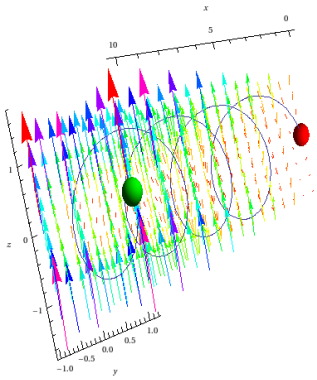
along the boundary C of the planar "castle region" shown in the picture? Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

Solution:
The curl $Q_x - P_y$ of the vector field is constant -1. The line integral is already given in the correct orientation so that it is by **Green's theorem** equal to the double integral of the curl over the region G. Since the castle has $81 - 9 - 8 = 64$ "bricks", the result is $-area(G) = \boxed{-64}$.

The castle by the way is a second step to a fractal construction. You can stack 8 such castles on each other to get a new one, then rescale. In the limit, we get a **fractal** a self similar structure, where one 1/8'th of the castle is similar to the castle itself. Because dividing things up into 3 parts leads to 8 times more bricks, the dimension is fractional $1 < \log(8)/\log(3) < 2$, hence the name fractal. [The dimensionality formula appears in the Star treck movie, when young spock learns math.] Without the hole, it would be a 2D object because $\log(9)/\log(3) = 2$. We could solve the exam problem at any stage. At stage n, there are 8^n bricks and the line integral would be -8^n.



Problem 11) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x,y,z) = \langle \cos(x), 2 + \cos(y), e^z + x(y^2 + z^2) \rangle$$

along the curve $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ with $0 \leq t \leq 3\pi$.

Hint: you might want to find a split $\vec{F} = \vec{G} + \vec{H}$ and compute line integrals of $\vec{G}$ and $\vec{H}$ separately.

Solution:
The vector field is the sum of a gradient field $\vec{G} = \langle \cos(x), 2 + \cos(y), e^z \rangle = \nabla f$ with $f(x, y, z) = \sin(x) + 2y + \sin(y) + e^z$ and the rest $\vec{H} = \langle 0, 0, x(y^2 + z^2) \rangle$. We have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r} + \int_C \vec{H} \cdot d\vec{r} \ .$$

The first line integral is by the **fundamental theorem of line integrals** $f(3\pi, -1, 0) - f(0, 1, 0) = -4 - 2\sin(1)$. The second line integral can be computed directly (note that $y^2 + z^2 = 1$ on the curve)

$$\begin{aligned} \int_0^{3\pi} \vec{F}(\vec{r}(t) \cdot \vec{r}\,'(t)\ dt &= \int_0^{3\pi} \langle 0, 0, t \rangle \cdot \langle 1, -\sin(t), \cos(t) \rangle\ dt \\ &= \int_0^{3\pi} t\cos(t)\ dt \\ &= t\sin(t)|_0^{2\pi} - \int_0^{3\pi} \sin(t)\ dt \\ &= 0 + \cos(t)|_0^{3\pi} = -2 \ . \end{aligned}$$

The sum of the two integrals is $\boxed{-6 - 2\sin(1)}$.

Problem 12) (10 points)



A biker in the Harvard Hemenway gym pedals. Assume that the force of a foot is

$$\vec{F} = \langle 0, 0, x^3 - x^2 + \sqrt{2 + \sin(z)} \rangle$$

and that one of the feet moves on a path $C : \vec{r}(t) = \langle 2\cos(t), 0, 2\sin(t) \rangle$. How much work

$$\int_C \vec{F} \cdot d\vec{r}$$

is done by this foot, when pedaling 10 times which means $0 \leq t \leq 20\pi$?

Solution:
By **Stokes theorem**, the line integral from 0 to 2π is equal to the flux of the curl of $\vec{F}$ through any surface which has C as a boundary. We take the simplest surface S which the disc parametrized by $\vec{r}(u,v) = \langle u, 0, v\rangle$ which has the normal vector $\vec{r}_u \times \vec{r}_v = \langle 1,0,0\rangle \cdot \langle 0,0,1\rangle = \langle 0,-1,0\rangle$. The curl of the vector field is

$$\text{curl}(\vec{F})(x,y,z) = \langle 0, -3x^2 + 2x, 0\rangle \ .$$

The flux of the curl is is

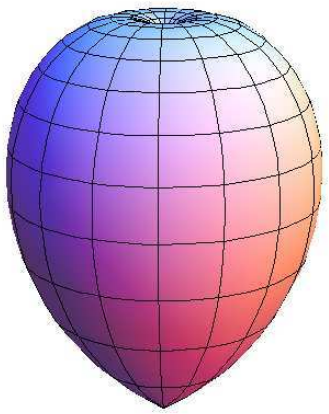
$$\int\int_{u^2+v^2\leq 4} \langle 0,-3u^2+2u,0\rangle \cdot \langle 0,-1,0\rangle \ dudv = \int\int_{u^2+v^2\leq 4} 3u^2 - 2u \ dudv \ .$$

This double integral is best computed in polar coordinates $u = r\cos(\theta), v = r\sin(\theta)$ leading to

$$\int_0^{2\pi}\int_0^2 [3r^2\cos^2(\theta) - 2r\cos(\theta)]r \ drd\theta \ = \pi(3\cdot 2^4/4) = 12\pi \ .$$

Peddaling 10 rounds leads to the work $\boxed{120\pi}$.

Problem 13) (10 points)



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x,y,z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y)\rangle \ .$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field $\vec{F}$ through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.

Solution:
The divergence of the vector field is constant 2. The flux through the tonsil surface is therefore by the divergence theorem 2 times the volume of the tonsil. This volume integral is best done in **spherical coordinates**:

$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{\phi}\rho^2\sin(\phi)\,d\rho d\phi d\theta = (4\pi/3)\int_0^{\pi}\phi^3\sin(\phi)\,d\phi\,.$$

This integral needs integration by parts:

$$\begin{aligned}
2\text{ Volume(tonsil)} &= \frac{4\pi}{3}[-\phi^3\cos(\phi)|_0^{\pi} + \int_0^{\pi} 3\phi^2\cos(\phi)\,d\phi] \\
&= \frac{4\pi}{3}[\pi^3 - 3\int_0^{\pi}\sin(\phi)2\phi\,d\phi] \\
&= \frac{4\pi}{3}[\pi^3 + 6\phi\cos(\phi)_0^{\pi} - 6\int_0^{\pi}\cos(\phi)\,d\phi] \\
&= \frac{4\pi}{3}(\pi^3 - 6\pi)\,.
\end{aligned}$$

The computation could be simplified a bit by switching the order of integration (for the triangle in the ϕ, ρ plane) requiring less integration by parts. The result is still $\boxed{\frac{4\pi}{3}(\pi^3 - 6\pi)}$.

Name:

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TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v}+\vec{w}$ has length $1/3$.

Solution:
Look at the diagonal of the parallelogram spanned by $\vec{v}$ and $\vec{w}$. It can have any length between 0 and 2.

2) [T] [F] For any three vectors, we have $|(\vec{u}\times\vec{v})\times\vec{w}| = |(\vec{v}\times\vec{w})\times\vec{u}|$.

Solution:
For $\vec{u}=\vec{i}, \vec{v}=\vec{j}$ and $\vec{w}=\vec{j}$, the first expression is 1 the last is 0.

3) [T] [F] Denote by $d(P,L)$ the distance from a point P to a line L in space. For any point P and any two lines L, K in space, we have $d(P,L)+d(P,K) \geq d(L,K)$.

Solution:
For any point A on L and any point B on K, we have $d(P,A)+d(P,B) \geq d(A,B)$ by the triangle inequality. This is especially true when A is the point on the line L such that $d(P,A)=d(P,L)$ and when B is the point on the line K such that $d(P,K)=d(P,B)$.

4) [T] [F] For any three vectors $\vec{u},\vec{v},\vec{w}$, the relation $|\vec{u}\times(\vec{v}\times\vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.

Solution:
Use the formula $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\alpha)$ twice.

5) [T] [F] If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature $1/2$ everywhere.

Solution:
While the statement is true for the speed, the curvature does not change under reparametrization.

6) [T] [F] If the curvature of a space curve is constant 1 and the speed $|\vec{r}\,'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}\,''(t)| = 1$ everywhere.

Solution:
The assumption implies $|\vec{T}| = |\vec{r}'|$ and $\kappa = |\vec{T}'|/|\vec{r}'| = |\vec{T}'| = |\vec{r}'| = |\vec{r}''|$. An other way to think about it: $|r'| = 1$ implies $r'' \cdot r' = 0$ so that $1 = |r'' \times r'|/|r'|^3 = |r''|$.

7) [T] [F] If a vector field $\vec{F} = \langle P, Q \rangle$ has $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\text{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.

Solution:
Given vector field $\vec{F} = \nabla f$, there is a solution f to $\text{div}(\text{grad}(f) = 0$ like $f(x, y) = x^2 - y^2$. The vector field $\langle 2x, -2y \rangle$ has divergence zero and curl zero.

8) [T] [F] If the level curve $f(x, y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0, 0)$ must be a critical point for which $D < 0$.

Solution:
It can be a surface with $D = 0$ like $x^4 - y^4$.

9) [T] [F] The surface $\vec{r}(u, v) = \langle u^3 \cos(v), u^3 \sin(v), u^3 \rangle$ with $v \in [0, 2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.

Solution:
$x^2 + y^2 = z^2$.

10) [T] [F] There is a non-constant function $f(x, y, z)$ of three variables such that $\text{div}(\text{grad}(f)) = f$.

Solution:
For example $f(x, y, z) = e^x$.

11) [T] [F] If $\text{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\text{div}(\vec{F}) = 0$ everywhere.

Solution:
Indeed, then $\text{div}(\vec{F}) = \text{div}(\text{curl}(\vec{F}) = 0$.

12) [T] [F] The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.

Solution:
It is a single cone.

13) [T] [F] The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0, 3, 0)$ is $y = 3$.

Solution:
The gradient is $\langle 0, 6, 0\rangle$ so that the equation is $6y = d$. Plugging in the point into the equation gives $y = 18$.

14) [T] [F] Assume (x_0, y_0) is not a critical point of $f(x, y)$. It is possible that f increases at (x_0, y_0) most rapidly in the direction $\langle 1, 0\rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5\rangle$.

Solution:
The direction of maximal increase is the opposite direction of the direction of maximal decrease.

15) [T] [F] Assume $\vec{F}(x, y, z)$ is defined everywhere except on the z-axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z-axis, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all curves C.

Solution:
You can have $\text{curl}(\vec{F})$ to be singular on the z axes. An example is $\langle -y/(x^2 + y^2), x/(x^2 + y^2), 0\rangle$.

16) [T] [F] A point (x_0, y_0) is an extremum of $f(x, y)$ under the constraint $g(x, y) = 0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0, y_0) can not be a local maximum on the constraint curve.

Solution:
The discriminant D has no significance for extremization problems under constraints. Take $x^2 + y^2$ or $-x^2 - y^2$ to get minima or maxima evenso $D > 0$.

17) [T] [F] The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2\rangle$ can be the curl of another vector field $\vec{G}$.

Solution:
We would need that the divergence of $\vec{F}$ is zero.

18) [T] [F] If $f(x,y)$ and $g(x,y)$ are two functions and $(2,3,3)$ is a critical point of the function $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$, then $(2,3)$ is a solution of the Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y) = 0$.

Solution:
Being a critical point of F means $g(x,y) = 0$ (partial derivative of F with respect to λ is zero) and $f_x = \lambda g_x, f_y = \lambda g_y$ (partial derivatives of F with respect to x, y are zero).

19) [T] [F] Assume $(0,0)$ is a global maximum of $f(x,y)$ on the disc $D = \{x^2+y^2 \leq 1\}$, then $\int\int_D f(x,y)\, dxdy \leq \pi f(0,0)$.

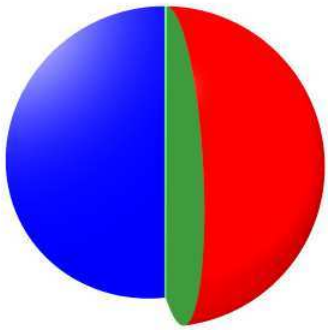
Solution:
$\int\int_D f(x,y)\, dxdy \leq \int\int_D f(0,0)\, dxdy = f(0,0)\pi$.

20) [T] [F] Let C be a curve parametrized by $\vec{r}(t), 0 \leq t \leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f \cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}\,''(t)}(f(\vec{r}(t))\, dt$.

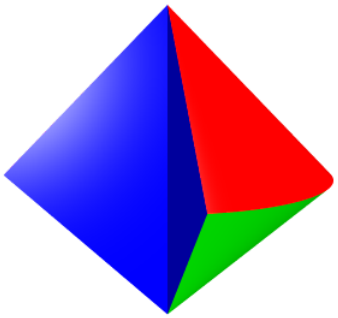
Solution:
It would be true for the velocity, not for the acceleration.

Problem 2) (10 points)

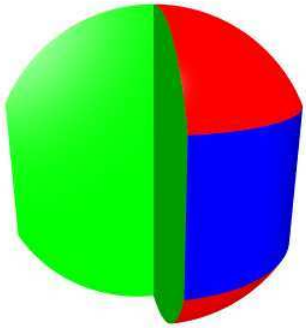
a) (4 points) Match the following triple integrals with the regions.



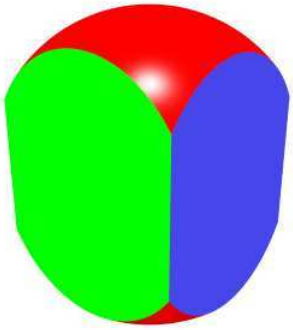
I



II



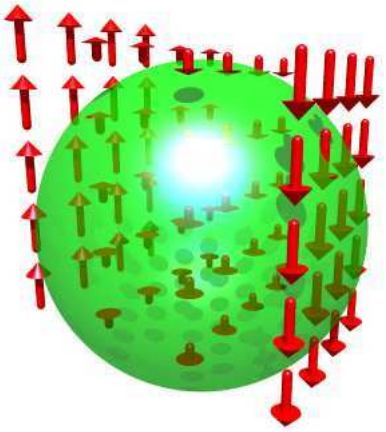
III



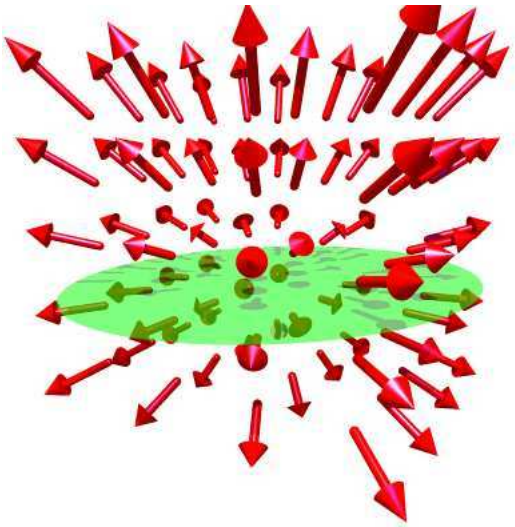
IV

Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \; dzdydx$

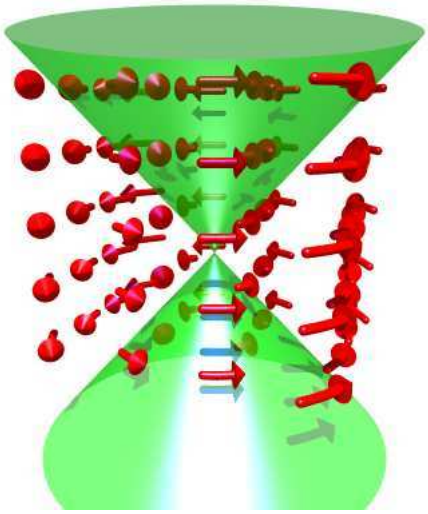
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



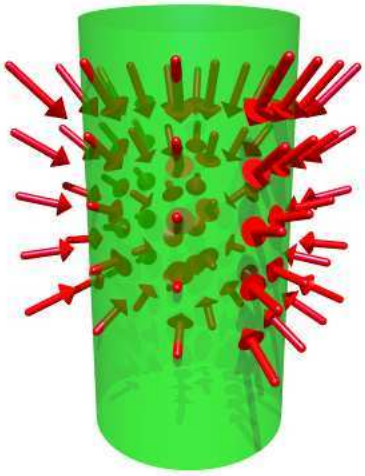
A



B



C



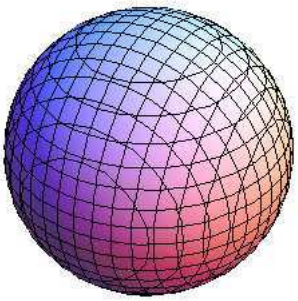
D

Enter A,B,C,D	Field	Surface	Flux zer
	$\vec{F}(x,y,z) = \langle x,y,z\rangle$	$\vec{r}(u,v) = \langle u,v,0\rangle$	
	$\vec{F}(x,y,z) = \langle 0,0,y\rangle$	$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$	
	$\vec{F}(x,y,z) = \langle -x,-y,-z\rangle$	$\vec{r}(\theta,z) = \langle \cos(\theta), \sin(\theta), z\rangle$	
	$\vec{F}(x,y,z) = \langle -y,x,0\rangle$	$\vec{r}(\theta,z) = \langle z\cos(\theta), z\sin(\theta), z\rangle$	

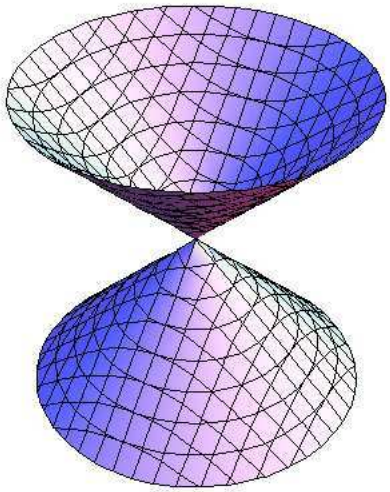
Solution:
a) III,II,I,IV.
b) B,A,D,C. The flux is zero for B,A,C. In the plane and cone situation the vector field is tangent to the surface. In the sphere case, there is part of the surface where the flux is positive and part of the surface where it is just the opposite. Only for the cylinder, the flux is nonzero. All vectors point inwards.

Problem 3) (10 points)

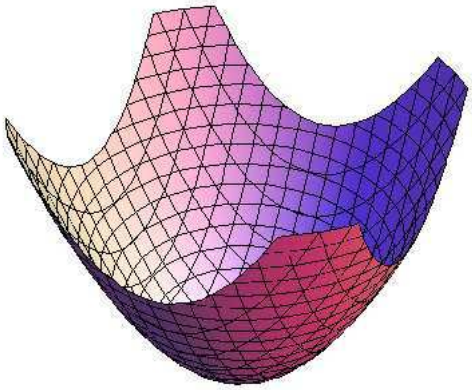
a) (6 points) Match the following level surfaces with functions $f(x, y, z)$ and also match the parametrization of part of the surface $f(x, y, z) = 0$.



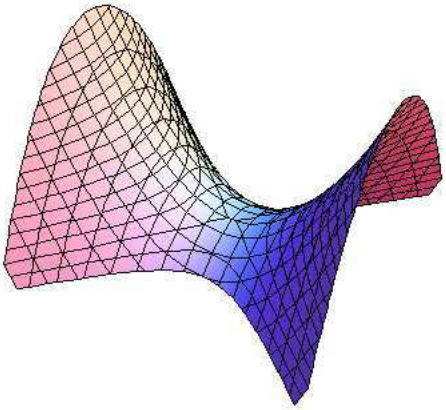
I



II



III



IV

Enter I,II,III,IV	$f(x,y,z)=0$	Enter I,II,III,IV	parametrization
	$f(x,y,z)=-x^2+y^2+z$		$\langle u,v,u^2-v^2\rangle$
	$f(x,y,z)=x^2+y^2+z^2-1$		$\langle u,v,u^2+v^2\rangle$
	$f(x,y,z)=-x^2-y^2+z$		$\langle u,v,\sqrt{1-u^2-v^2}\rangle$
	$f(x,y,z)=-x^2-y^2+z^2$		$\langle s\cos(t),s\sin(t),s\rangle$

b) (2 points) We know that $\vec{r}\,''(t)=\langle-\cos(t),-\sin(t),0\rangle$, $\vec{r}(0)=\langle 2,3,4\rangle$ and $\vec{r}\,'(0)=\langle 0,1,1\rangle$. The expression $\langle\cos(t)+1,\sin(t)+3,t+4\rangle$ is equal to:

Check which applies	result
	the velocity $\vec{r}\,'(t)$
	the position $\vec{r}(t)$
	the curvature $\kappa(\vec{r}(t))$
	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of the partial differential equation $\mathrm{div}(\mathrm{grad}(f))=0$ for $f(x,y)$?

Check which applies	PDE
	Transport equation
	Wave equation
	Heat equation
	Laplace equation

Solution:
a) IV,I,III,II and IV,III,I,II
b) It is the position. (The problem had a typo with $\langle 0,1,0\rangle$. We did not take any points off in b).)
c) This is the Laplace equation.

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2+y^2+(z-6)^2=1$ and the cylinder of radius 2 around the line $x=y=z$.

Solution:
The points $A=(0,0,0)$ and $B=(1,1,1)$ are on the line so that the perametrization of the line is $\vec{r}(t)=A+t\vec{v}$ with $\vec{v}=\vec{AB}=\langle 1,1,1\rangle$. We first compute the distance between the point $P=(4,0,6)$, the center of the sphere, and the line. This distance is

$$d=\frac{|\vec{PA}\times\vec{v}|}{|\vec{v}|}=\frac{|\langle 4,0,6\rangle\times\langle 1,1,1\rangle|}{|\langle 1,1,1\rangle|}$$

$=\frac{|\langle -6,2,4\rangle|}{\sqrt{3}}$. We have to subtract from this the radius of the sphere and the radius of the cylinder, so that the result is now $\boxed{\sqrt{56/3}-3}$.

Problem 5) (10 points)

a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

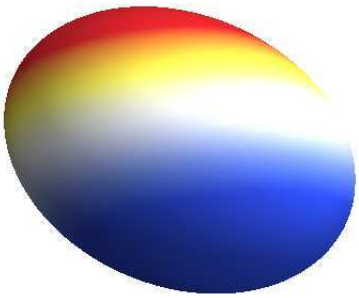
b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

Solution:
a) The gradient of $f(x, y, z) = 4xy - z^2$ is $\nabla f(x, y, z) = \langle 4y, 4x, -2z \rangle$. The gradient at $\langle 1, 1, 2 \rangle$ is $\langle 4, 4, -4 \rangle$. The equation of the tangent plane is $4x + 4y - 4z = d$, where the constant $d = 0$ can be obtained by plugging in the point $(x, y, z) = (1, 1, 2)$. The plane is $\boxed{x + y - z = 0}$.
b) Since $f(1, 1, 2) = 0$, we get with the linearization $L(x, y, z) = 0 + 4(x - 1) + 4(y - 1) - 4(z - 2) = 0.004 - 0.04 - 0.004 = \boxed{-0.04}$
c) $\vec{r}(t) = \langle 1, 1, 2 \rangle + t\langle 4, 4, -4 \rangle$ which is $\boxed{\vec{r}(t) = \langle 1 + 4t, 1 + 4t, 2 - 4t \rangle}$.

Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



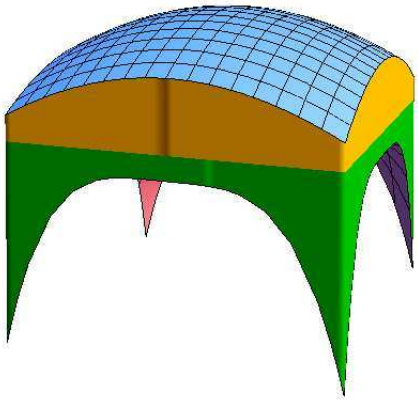
Solution:
The Lagrange equations are

$$\begin{aligned} 5 &= \lambda 10x \\ -2 &= \lambda 2y \\ 9 &= \lambda 6z \\ 5x^2 + y^2 + 3z^2 &= 9 \end{aligned}$$

Dividing the first equation by the second we get $y = -2x$. If we divide the first by the third, we get relation $z == 3x$. Plugging this into the third 4th equation gives $36x^2 = 9$ or $x = \pm 1/2$. The critical points are $(1/2, -1, 3/2)$ and $(-1/2, 1, -3/2)$. We evaluate f at these two points to get $f(1/2, -1, 3/2) = 768, f(-1/2, 1, -3/2) = 732$. The point $\boxed{(1/2, -1, 3/2)}$ is the maximum.

Problem 7) (10 points)

The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

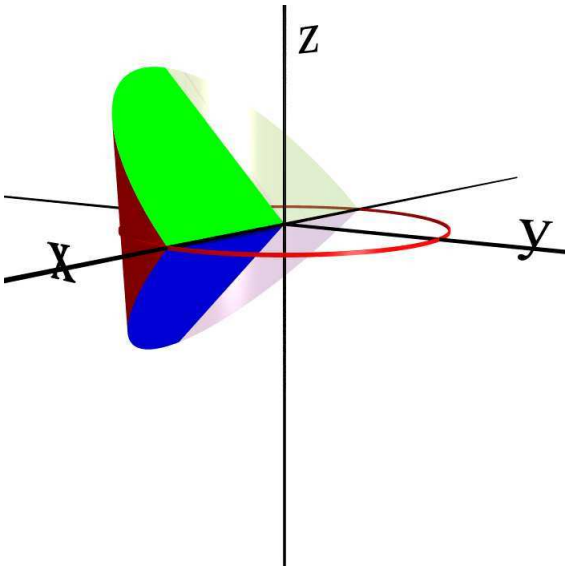
Solution:
The function to extremize is $f(x,y) = 12 + x^4 + y^4 - 2x^2 - 2y^2$ Its gradient is $\nabla f(x,y) = \langle 4x^3 - 4x, 4y^3 - y\rangle$. This gradient is equal to $\langle 0,0\rangle$ if $x \in \{0,1,-1\}$ and $y \in \{0,1,-1\}$. There are 9 critical points. Now we proceeed and use the second derivative test. We compute the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2-4)(12y^2-4)$ and $f_{xx} = 12x^2 - 4$. D is negative if exactly one of the x, y is zero. Otherwise, it is positive. f_{xx} is negative if $x = 0$.

x	y	D	$f_x x$	Type	f(x,y)=
-1	-1	64	8	minimum	10
-1	0	-32	8	saddle	11
-1	1	64	8	minimum	10
0	-1	-32	-4	saddle	11
0	0	16	-4	maximum	12
0	1	-32	-4	saddle	11
1	-1	64	8	minimum	10
1	0	-32	8	saddle	11
1	1	64	8	minimum	10

The minimal value 10 occurs at 4 places. These are $\boxed{(-1,-1),(-1,1),(1,-1),(1,1)}$. These are local minima. But they are also global minima because $f(x,y) = (x^2-1)^2 + (y^2+1)^2 + 10$ is always positive and goes to infinity for $(x,y) \to \infty$.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Solution:
We use cylindrical coordinates. The base region in the xy plane is the forth quadrant. Its roof is $z = -y$, its floor is $z = y$. We have to integrate $f(x, y) = -2y = -2r\sin(\theta)$ over the fourth quadrant and get:

$$\int_0^1 \int_{3\pi/2}^{2\pi} -2r\sin(\theta) r \; d\theta dr = 2/3 \; .$$

The volume of the cheese is $\boxed{2/3}$.

Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

$$\vec{r}(u, v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle \; ,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Solution:
We have $\vec{r}_u(u, v) = \langle 3, 4, 0 \rangle$ and $\vec{r}_v(u, v) = \langle 2, 1, v^{5/2} \rangle$, so that

$$\vec{r}_u \times \vec{r}_v = \langle 4v^{5/2}, -3v^{5/2}, -5 \rangle \; .$$

Its magnitude is $|\vec{r}_u \times \vec{r}_v| = \sqrt{25v^5 + 25} = 5\sqrt{v^5 + 1}$. The surface area is

$$\int\int |\vec{r}_u \times \vec{r}_v| \; dvdu = \int_0^1 \int_{u^{1/4}}^1 5\sqrt{v^5 + 1} \; dvdu \; .$$

Since this can not be solved directly, we have to change the order of integration:

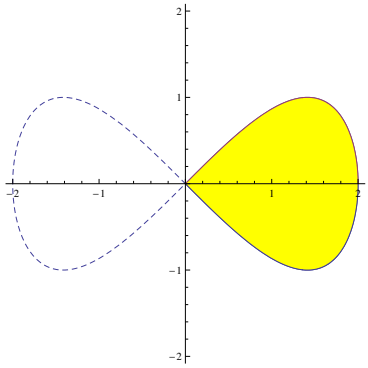
$$\int_0^1 \int_0^{v^4} 5\sqrt{v^5 + 1} \; dudv = \int_0^1 5v^4\sqrt{v^5 + 1} \; dv = \frac{2}{3}(2\sqrt{2} - 1) \; .$$

The result is $\boxed{\frac{2}{3}(2\sqrt{2} - 1)}$.

Problem 10) (10 points)

Find the area $\int\int_R 1\, dxdy$ of the region R inside the right leaf of the **Gerono lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2\sin(t), 2\sin(t)\cos(t)\rangle \ .$$



Solution:
We use the vector field $\vec{F}(x,y) = \langle 0, x\rangle$ and use **Greens theorem**. We get

$$\int_0^\pi \langle 0, 2\sin(t)\rangle \cdot \langle 2\cos(t), 4\cos^2(t) - 2\rangle = \int_0^\pi 8\sin(t)\cos^2(t) - 4\sin(t)\ dt = \cos^3(t)(8/3) - 4\cos(t)_0^\pi = 8/3$$

The area is $\boxed{8/3}$.

Problem 11) (10 points)

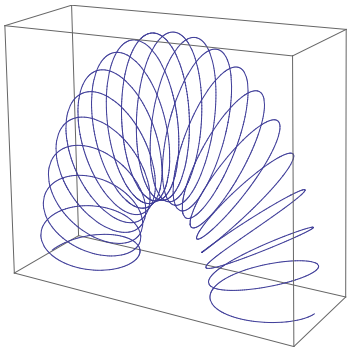
Find the line integral of the vector field

$\vec{F}(x,y,z) = \langle \cos(x+z), 2yze^{y^2z} + 7, \cos(x+z) + y^2e^{y^2z}\rangle$

along the **slinky** curve

$\vec{r}(t) = \langle \sin(40t), (2+\cos(40t))\cos(t), (2+\cos(40t))\sin(t)\rangle$

with $0 \leq t \leq \pi$.



Solution:
The vector field $\vec{F}$ is a **gradient field** with $\vec{F}(x,y,z) = \nabla f(x,y,z)$ with $f(x,y,z) = \sin(x+z) + 7y + e^{y^2z}$. The curve starts at $\vec{r}(0) = \langle 0, 3, 0\rangle$ and ends with $\vec{r}(\pi) = \langle 0, -3, 0\rangle$. The line integral is therefore by the **fundamental theorem of line integrals** $f(r(\pi)) - f(r(0)) = -20 - 22 =$ -$\boxed{42}$. This is the answer to the "ultimate question".

Problem 12) (10 points)

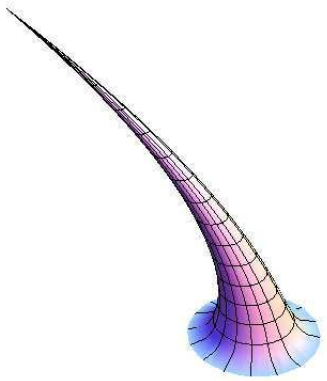
Find the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz\rangle$$

and S is the **thorn** surface parametrized by

$$\vec{r}(s,t) = \langle (1-s^{1/3})\cos(t) - 4s^2, (1-s^{1/3})\sin(t), 5s\rangle$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.



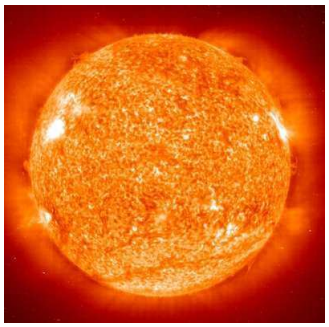
Solution:
This problem can be solved in three different ways. 1. solution. The vector field $\vec{F}$ is the sum of the gradient of $f(x,y,z) = \cos(\pi y)e^{2x} + z^2x$ and $\vec{G}(x,y,z) = \langle 0, x^2\cos(z\pi/2), 0\rangle$. By Stokes theorem, the flux of $\text{curl}(F) = \pi x^2\sin(\pi z)/2)/2, 0, 2x\cos(\pi z/2)$ is the line integral of $\langle 0, x^2\cos(z\pi/2), 0\rangle$ along the boundary curve $\vec{r}(t) = \langle \cos(t), \sin(0), 0\rangle$ which is $\int_0^{2\pi}\langle 0, \cos^2(t), 0\rangle \cdot \langle \sin(t), \cos(t), 0\rangle = 0$.
2. Solution. The flux is by Stokes theorem the line integral along the boundary $\vec{r}(t)$ which is by Stokes theorem the flux integral of $\text{curl}(\vec{F}) = \pi x^2\sin(\pi z)/2)/2, 0, 2x\cos(\pi z/2)$ through the disc with that boundary. This flux integral is zero because on the disc $\text{curl}(\vec{F}(x,y,z) = \langle 0, 0, 2x\rangle$ so that the flux is the double integral of $2x$ over the disc which is zero.
3. Solution. The flux through the "thorn" together with the flux through the bottom disc (oriented downwards) closing the surface is zero because $\text{div}(\text{curl}(\vec{F}) = 0$. Therefore, the flux through the thorn is the same as the flux through the disc (oriented upwards) which is zero as in the 2. Solution. The result is again $\boxed{0}$.

Problem 13) (10 points)

Assume the vector field

$$\vec{F}(x,y,z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int\int_S \vec{F} \cdot d\vec{S}$.

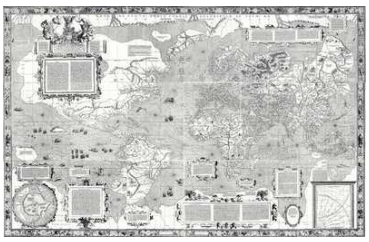


Solution:
The divergence is $15x^2 + 15y^2 + 15z^2$. We integrate this over the sphere to get by the divergence theorem the flux through the surface. To compute the triple integral we use spherical coordinates

$$\int_0^{2\pi} \int_0^{\pi} \int_0^3 15\rho^2 \rho^2 \sin(\phi)\, d\rho d\phi d\theta \; .$$

The result is $15(3^5/5)4\pi = 4 \cdot 3^6 \cdot \pi = \boxed{2916\pi}$.

Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u,v) = \langle \cos(u)\cos(\arctan(\sinh(v))), \sin(u)\cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle \; .$$

a) (3 points) Show that $|\vec{r}(u,v)| = 1$ verifying so that $\vec{r}(u,v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
b) (3 points) Show that $|\vec{r}_u(u,v)| = |\vec{r}_v(u,v)| = 1/\cosh(v)$ and that $\vec{r}_u(u,v) \cdot \vec{r}_v(u,v) = 0$.
c) (2 points) Use b) to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
d) (2 points) Use $\int 1/\cosh^2(x)\, dx = 2\arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π.

Hint for b): you can use the identity $\cos(\arctan(\sinh(v)) = 1/\cosh(v)$.

Solution:
a) With $w = \pi/2 - \arctan(\sinh(v))$ we can rewrite this as

$$\vec{r(u, w)} = \langle \cos(u)\sin(w), \sin(u)\sin(w), \cos(w) \rangle$$

which is the standard parametrization of the sphere.
b) This is a direct computation using the one-dimensional chain rule.
c) Use the formula $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$ if the vectors a, b are perpendicular.
d) $\int_{-\infty}^{\infty} 1/\cosh^2(x)\, dx = 2$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] The function $f(x,y,z) = x^2 - y^2 - z^2$ increases in the direction $\langle -3,-1,2\rangle/\sqrt{14}$ at the point $(1,1,1)$.

Solution:
Because the directional derivative is $\nabla f(1,1,1) \cdot \langle -3,-1,2\rangle/\sqrt{14} = \langle 2,-2,-2\rangle \cdot \langle -3,-1,2\rangle/\sqrt{14} = -6$, it decreases.

2) [T] [F] The unit tangent vector of the curve $\vec{r}(t) = \langle 3t, 4t, t^2\rangle$ at time $t = 0$ is $\langle 3/5, 4/5, 0\rangle$.

Solution:
The velocity is $\langle 3, 4, 2t\rangle$ and at time $t = 0$ this has length 5. Normalize the velocity vector and get the unit tangent vector.

3) [T] [F] There exist two nonzero vectors $\vec{a}$ and $\vec{b}$ such that the length of the vector projection of $\vec{a}$ to $\vec{a} \times \vec{b}$ is $\frac{1}{2}|\vec{b}|$.

Solution:
The projection from $\vec{a}$ onto the vector $\vec{a} \times \vec{b}$ perpendicular to $\vec{a}$ is always 0.

4) [T] [F] The arc length of the curve $\vec{r}_1(t) = \langle e^{3t^3} - 1, t^6 + 2, \sin(2t^3)\rangle$, $0 \le t \le 1$ is larger than that of $\vec{r}_2(t) = \langle e^{3t} - 1, t^2 + 2, \sin(2t)\rangle$, $0 \le t \le 1$.

Solution:
The arc length is the same because the second curve is just a reparametrization of the first.

5) [T] [F] The tangent plane of the graph of $f(x,y) = \sin(x) + y^3$ at $(0,1,1)$ is $x + 3y = 3$.

Solution:
In order to find the tangent plane, first write the graph as a level curve of a function of 3 variables: $g(x,y,z) = \sin(x) + y^3 - z = 0$. Its gradient is $\langle \cos(x), 3y^2, -1\rangle = \langle 1, 3, -1\rangle$.

6) [T] [F] There exists a curve C on the level surface of $f(x,y,z) = x^3 + e^{yz} + \cos(y) = 2$ such that the line integral $\int_C \nabla f \cdot d\vec{r} > 0$.

Solution:
The velocity vector to the curve is always perpendicular to the gradient vector so that the line integral is zero.

7) [T] [F] If Q is the point away from the plane $3x + 5y + z = 7$ and P is the point on the plane closest to Q, then $\vec{PQ}$ is parallel to $\langle 3, 5, 1\rangle$.

Solution:
The vector PQ is normal to the plane and so parallel to the gradient of the linear function defining the plane.

8) [T] [F] The vector field $\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz, x^4 + y^2 - z^2\rangle$ is the curl of a vector field $\vec{G}$.

Solution:
If $F = \text{curl}(G)$, then $\text{div}(\text{curl}(G) = 0$. But the difvergence of $\vec{F}$ is not zero.

9) [T] [F] Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ and C be the unit circle oriented counterclockwise. Since $Q_x = P_y$ everywhere, Green implies $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:
Green's theorem can not be applied because at the origin, the curl is not defined.

10) [T] [F] By linear approximation of the function $f(x, y, z) = e^{x+y+z}$ we can estimate $f(0.1, 0.01, 0.001)$ as 1.111.

Solution:
The gradient of f at $(0, 0, 0)$ is $\langle 1, 1, 1\rangle$. The approximation is $f(0, 0, 0) + 1 \cdot 0.1 + 1 \cdot 0.01 + 1 \cdot 0.001$ which is what we have given.

11) [T] [F] If $\vec{F}(x, y, z)$ is a vector field defined on $0 < x^2 + y^2 + z^2 < 4$ and $\text{curl}(\vec{F}) = 0$ everywhere on this solid, then $\vec{F} = \nabla f$ for some function f.

Solution:
It is simply connected because we can pull together any closed loop to a point.

12) [T] [F] The tangent plane of the surface $x^2+y^4+z^6=6$ at $(2,1,1)$ is perpendicular to the line $\vec{r}(t)=\langle 1+2t, 3+2t, -4+3t\rangle$.

Solution:
The tangent plane is perpendicular to the gradient which is $\langle 4,4,6\rangle$.

13) [T] [F] Given two curves $C_1 : \vec{r}_1(t) = \langle t, t^3\rangle, 0 \le t \le 1$ and $C_2 : \vec{r}_2(s) = \langle s, s^5\rangle, 0 \le s \le 1$ $f(x,y) = \sin(x^2y)$. Then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals.

14) [T] [F] If $f(x,y)$ has a global maximum, then the discriminant function $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ has a global maximum.

Solution:
There is no relation between critical points of f and D. Take $-x^2-y^2+x^2y$ for example. It has a global maximum but $D(x,y) = 4-4y-4x^2$ does not.

15) [T] [F] Let $\vec{F}(x,y,z) = \langle x,y,z\rangle$ and S the surface boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ oriented by outward normal vectors. Then $\int\int_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
Use the divergence theorem. It says that the result is 3, not zero.

16) [T] [F] Let $\vec{F}(x,y,z) = \langle x/3, y/3, z/3\rangle$ and S the unit sphere oriented by the outward normal vectors. Then $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ is the volume of the unit ball.

Solution:
By Stokes theorem, the result is zero. One can also see it with the divergence theorem.

17) [T] [F] In three dimensional space there exist two nonzero vector fields $\vec{F}$ and $\vec{G}$ such that $\text{curl}(\vec{F}) = \text{div}(\vec{G})$.

Solution:
One is a vector, the other is a scalar

18) [T] [F] The vector field $\vec{F}(x,y,z) = \langle \cos(y), \cos(z), \cos(x) \rangle$ has the property that $\vec{F} = \text{curl}(\text{curl}(\vec{F}))$.

Solution:
This is a direct computation: $\vec{G} = \text{curl}\vec{F} = \langle \sin(z), \sin(x), \sin(y) \rangle$ and $\text{curl}(\vec{G}) = \vec{F}$.

19) [T] [F] There exists a vector field $\vec{F}(x,y,z)$ defined on $\mathbf{R}^3$ such that every line integral $\int_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ over a closed curve C is equal to 0, but not every surface integral $\int \int_S \vec{F} \cdot dS$ over a closed surface S is equal to 0.

Solution:
Take a gradient field which is not divergence free like $\langle x, y, z \rangle$.

20) [T] [F] Whenever $\vec{F} = \nabla f$, for some function $f(x,y)$ defined on the annulus $\frac{1}{2} \leq x^2 + y^2 \leq 2$, then $\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the circle $x^2 + y^2 = 1$.

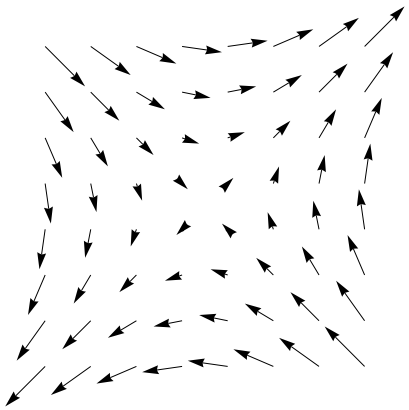
Solution:
Even so the function is not defined at the origin, the fundamental theorem of line integrals applies.

Problem 2) (10 points)

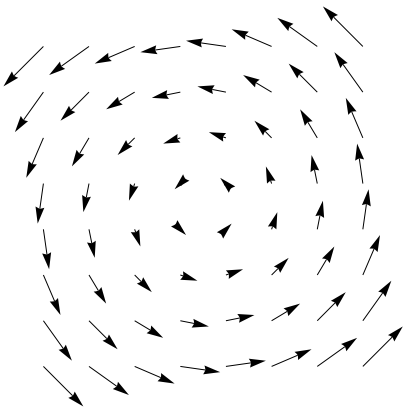
a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field $\vec{F}$ is **divergence free** if $\text{div}(\vec{F}) = 0$ everywhere in the plane. A field $\vec{F}$ is **irrotational**, if $\text{curl}(\vec{F}) = \vec{0}$ everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

field	enter I-IV	divergence free	irrotational
$\vec{F}(x,y) = \langle -y, x \rangle$			
$\vec{F}(x,y) = \langle y, x \rangle$			
$\vec{F}(x,y) = \langle -x-y, x-y \rangle$			
$\vec{F}(x,y) = \langle x+y, x+y \rangle$			

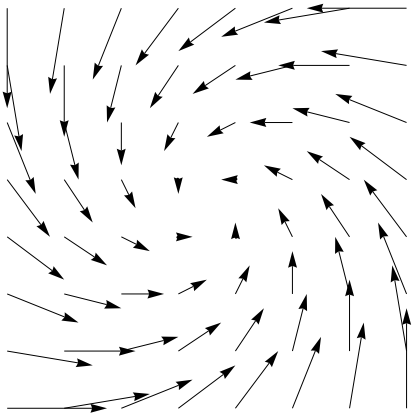
I



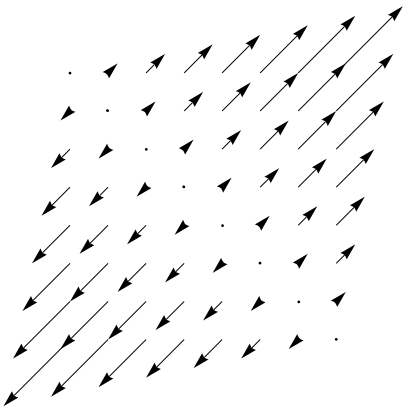
II



III



IV



b) (5 points) Match the following names of partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t,x) = t^2 + x^2$
B	$u(t,x) = t^2 - x^2$
C	$u(t,x) = \sin(x+t)$
D	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_x(t,x)$
2	$u_{tt}(t,x) = u_{xx}(t,x)$
3	$u_{tt}(t,x) = -u_{xx}(t,x)$
4	$u_t(t,x) = u_{xx}(t,x)$

Solution:
2a) II divergence free, I divergence free and irrotational, III , IV irrotational, 2b) ADCB, 2413

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals $A - H$ in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\int\int_R x^2 - y^2 \; dxdy$	A
$\int\int_R 1 \; dxdy$	B
$\int\int\int_R 1 \; dxdydz$	C
$\int\int\int_R x^2 + z^2 \; dxdydz$	D

$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	F
$\int_a^b \lvert\vec{r}\,'(t)\rvert \; dt$	G
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \; dudv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x,y,z) = \langle -y, x, x \rangle$	$\langle 0, -1, 2 \rangle$	A
$\vec{F}(x,y,z) = \langle x^2, y, x \rangle$	$2x + 1$	B
$f(x,y,z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1 \rangle$	C
$f(x,y,z) = x^3 + 5y^2$	$10y$	D

Solution:
a) FEHGCB, b) BACD

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0, 1, -1), B = (4, 0, -1), C = (2, 1, 3), \text{ and } D = (2, 2, 0) .$$

a) (3 points) What is the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AD}$?

b) (3 points) Find the volume of the parallelepiped spanned by $\vec{AC}, \vec{AB}$ and $\vec{AD}$.

c) (4 points) Determine the distance between the two skew lines AB and CD.

Solution:
$\vec{AC} = \langle 2, 0, 4\rangle$.
$\vec{AB} = \langle 4, -1, 0\rangle$.
$\vec{AD} = \langle 2, 1, 1\rangle$.
a) The area of the parallelogram is $|\langle -1, -4, 6\rangle| = \sqrt{53}$.
b) The volume of the parallelepiped is 22.
c) The distance is $|\vec{AD} \cdot (\vec{AB} \times \vec{CD})|/|\vec{AB} \times \vec{CD}| = 22/13$.

Problem 5) (10 points)

a) (5 points) The curl of $\vec{F}(x, y) = \langle -e^{xy}, y\rangle$ is equal to a scalar function $f(x, y)$. Estimate $f(1.1, 0.001)$ by linear approximation.

b) (5 points) Using the same function as in a), the equation $f(x, y) = \text{curl}(\vec{F})(x, y) = 1$ defines y as a function $g(x)$ of x near $x = 1$. Find $g'(1)$.

Solution:
The function is $f(x, y) = xe^{xy}$. We have $\nabla f(x, y) = \langle e^{xy} + xye^{xy}, x^2e^{xy}\rangle$. Compute $f(1, 0) = 1$ and $\nabla f(1, 0) = \langle 1, 1\rangle$.
a) We have $L(1.1, 0.001) = 1 + 0.1 + 0.001 = 1.101$.
b) We use the implicit differentiation formula which comes from $f(x, g(x)) = 1$. We have $g'(x) = -f_x(1, 0)/f_y(1, 0) = -1$.

Problem 6) (10 points)

Find all the critical points of the function $f(x,y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

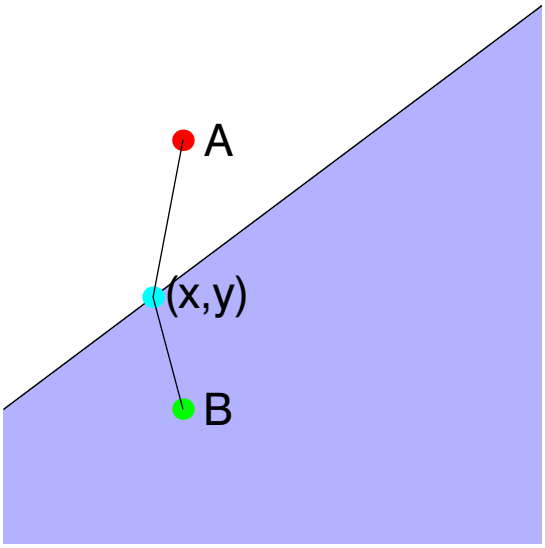
Solution:
The critical points are $P = (-2, 0)$ and $Q = (-2, 2)$. The Discriminant at P is $D = -12$ so that P is a saddle point. The Hessian at Q is 12 and $f_{xx} = 2$ which is a local minimum.

$P = (-2, 0)$	D=-12	Saddle point
$Q = (-2, 2)$	D=12, $f_{xx} = 2$	local minimum

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from $A = (0, 2)$ to $B = (0, 0)$ along a straight line segment from A to (x, y) and swims through the lake $x - y \geq -1$ from (x, y) to a gold chest located at $B = (0, 0)$ again on a straight line segment. The effort from A to (x, y) is the square of the distance from A to (x, y). Her effort from (x, y) to B is 2 times the squared distance from (x, y) to B. Using the Lagrange method, find the choice of a drop point (x, y) on the lake shore that minimizes her effort.





Solution:
This is a Lagrange problem. We have to extremize the total effort $f(x,y) = x^2 + (y-2)^2 + 2(x^2+y^2) = 3x^2 + 3y^2 - 4y + 4$ under the constraint $g(x,y) = x - y + 1 = 0$ that the drop point is on the shore. The Lagrange equations are

$$\begin{aligned} 6x &= \lambda \\ 6y - 4 &= -\lambda \\ x - y &= -1 \, . \end{aligned}$$

To solve it, the first two equations give $6x = 4 - 6y$. Together with the third, we end up with the solution $\boxed{x = -1/6, y = 5/6}$.

Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \, ,$$

where C is the curve given by

$$\vec{r}(t) = \langle \frac{t\pi}{2}, 1-t, t^3 \rangle, 0 \leq t \leq 1$$

and

$$\vec{F}(x,y,z) = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz) \rangle \, .$$

Solution:
The vector field is a gradient field with gradient $f(x,y) = xe^{y^2} + \sin(xz)$. Apply the fundamental theorem of line integrals. We have $\vec{r}(1) = \langle \pi/2, 0, 1 \rangle$ and $\vec{r}(0) = \langle 0, 1, 0 \rangle$. The result is $\boxed{\pi/2 + 1}$.

Problem (9) (10 points)

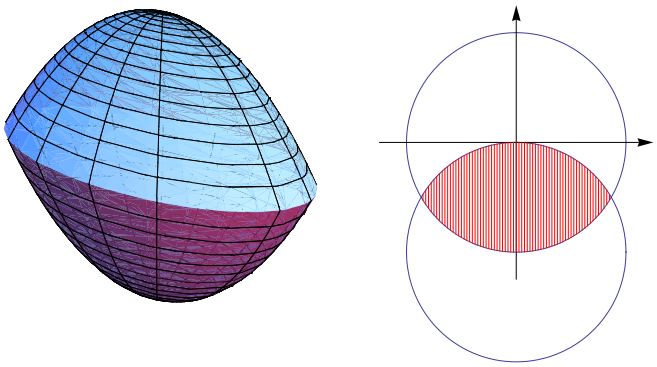
The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z+2)^2 = 4 \, .$$

Find the surface area of the UFO.



Solution:
The two surfaces forming the UFO hull are parts of spheres of radius 2. We use spherical coordinates. The angle θ goes from 0 to 2π. The angle ϕ ranges from 0 to $\pi/3$. We know $|r_\theta \times r_\phi| = \rho^2 \sin(\phi) = 4\sin(\phi)$. Lets compute half of the surface:

$$2\pi \int_0^{\pi/3} 4\sin(\phi)\, d\phi = 2\pi 4\cos(\phi)|_0^{\pi/3} = (2\pi)4(1-1/2) = 4\pi\ .$$

Both surfaces together have surface area $\boxed{8\pi}$.

Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\, dydxdz\ .$$

Solution:
We are stuck with the integral. Change the order of integration using Fubini (for the most inner two integrals, where z is a constant and does not depend on x, y):

$$\int_0^2 \int_{z^2}^4 \int_1^3 xz\cos(y^2)\, dxdydz\ .$$

Now we can solve the most inner integral using $\int_1^3 x\, dx = 4$

$$\int_0^2 \int_{z^2}^4 4z\cos(y^2)\, dydz\ .$$

Again, we are stuck and we change the order of integration again. It helps to make a 2D picture to find:

$$\int_0^4 \int_0^{\sqrt{y}} 4z\cos(y^2)\, dzdy\ .$$

Now we can evaluate the inner integral

$$\int_0^4 2y\cos(y^2)dy = \sin(y^2)|_0^4 = \sin(16)\ .$$

The final result is $\boxed{\sin(16)}$.

Problem 11) (10 points)

Let $\vec{F}(x,y,z) = \langle x+yz, xye^{-xz}, e^{-xz}\rangle$. Find

$$\int\int_S \vec{F}\cdot d\vec{S}\ ,$$

where S is the surface $z = 1-x^2-y^2, z \geq 0$ oriented so that the normal vector points upwards.

Solution:
We use the divergence theorem: let E be the solid enclosed by S and the disc D in the xy-plane oriented downwards. Then

$$\int\int_D \vec{F}\cdot d\vec{S} + \int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_E \mathrm{div}(\vec{F})\, dV\ .$$

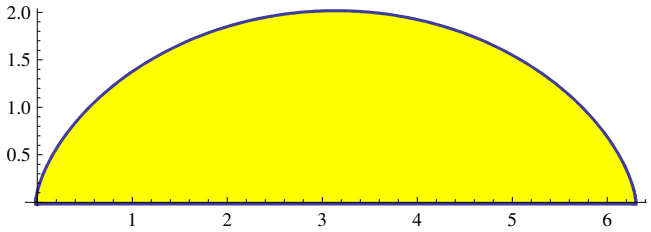
First the right hand side. The divergence of the vector field is 1.

$$\int\int\int_E \mathrm{div}(\vec{F})\, dxdydz = \int_0^{2pi}\int_0^1 (1-r^2)r\, drd\theta = 2\pi(\frac{r^2}{2} - \frac{r^4}{4})|_0^1\, d\theta = \frac{\pi}{2}\ .$$

On the xy plane, the field is $\langle x, xy, 1\rangle$. For the parametrization $\vec{r}(u,v) = \langle u, v, 0\rangle$ of the bottom surface, the normal vector is $\langle 0, 0, 1\rangle$. The flux through the bottom surface D is π if the surface is oriented upwards but it is $-\pi$ if the surface is oriented downwards. We solve for the unknown $\int\int_D \vec{F}\cdot d\vec{S}$ in the equation given by the divergence theorem and get $\boxed{3\pi/2}$.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t)\rangle$ with $0 \le t \le 2\pi$ and the x-axes.



Solution:
Use Greens theorem for the region D enclosed by the given curve and the x axes. We chose a vector field $\vec{F} = \langle -y, 0 \rangle$ which has curl 1. [While the computation can also be done with $\vec{F} = \langle 0, x \rangle$ for example, the first choice has the advantage that the field is zero on the x-axes.] Greens theorem tells that

$$-I + II = \int\int \text{curl}(\vec{F})\, dxdy = \text{area}(D) \; .$$

where $I = \int_C \vec{F}(\vec{r}(t))\, \vec{r}'(t)\, dt$ is the line integral along C and $II = \int_0^{2\pi} \vec{F}(t,0) \cdot \langle 1, 0 \rangle\, dt = 0$ is the line integral along the x axes from $(0,0)$ to $(2\pi, 0)$ with the curve $\vec{s}(t) = \langle t, 0 \rangle$. The minus sign for the first integral is because the curve $\vec{r}(t)$ goes in the clockwise direction. So,

$$\text{area}(D) = -I = -\int_C \vec{F}(\vec{r}(t))\, \vec{r}'(t)\, dt = -\int_0^{2\pi} -(1-\cos(t))^2 = 3\pi \; .$$

The area of the region is $\boxed{3\pi}$.

Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} \; ,$$

where $\vec{F}(x,y,z) = \langle xe^{y^2} z^3 + 2xyze^{x^2+z}, x + z^2 e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Solution:
Stokes theorem assures that the flux integral we are looking for is equal to the line integral along the boundary of the surface. The boundary is the ellipse $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$, $0 \le t \le 2\pi$. The vector field on the xy-plane $z = 0$ is

$$\vec{F}(x,y,0) = \langle 0, x, ye^{x^2} \rangle \; .$$

To compute the line integral of this vector field along the boundary curve, compute $\vec{r}'(t) = \langle -\sin(t), 2\cos(t), 0 \rangle$ and $\vec{F}(\vec{r}(t)) = \langle 0, \cos(t), 2\sin(t) e^{\sin^2(t)} \rangle$. The dot product of these two vectors is the function $2\cos^2(t)$, the power. Integrating this over $[0, 2\pi]$ gives $\boxed{2\pi}$. [P.S. The problem could also have been solved with the divergence theorem by computing the flux of $\text{curl}(\vec{F})$ through the bottom surface. This requires however to compute the curl of $\vec{F}$ and produces substantially more work.]

Problem 14) (10 points)

Let E be the rectangular solid $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$ and let S be the boundary of E. The surface S consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle \ ,$$

for which parameters a, b is the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S}$$

a global maximum?

Solution:
We have $\text{div}(F) = -2x - 4y - z + 12$. By the divergence theorem, we can compute the flux through S as a triple integral of $\text{div}(F)$ through E. This is

$$\int_0^a \int_0^b \int_0^1 -2x - 4y - z + 12 \, dzdydx = 23ab/2 - a^2b - 2ab^2 \ .$$

Extremizing this function gives 3 saddle points and one local maximum.

a	b	D	f_{aa}	Nature	$f(a,b)$
0	0	< 0	-	saddle	0
0	23/4	< 0	-	saddle	0
23/6	23/12	> 0	< 0	max	12167/432
23/2	0	< 0	-	saddle	0

The parameters $(a, b) = (23/6, 23/12)$ lead to a global maximum.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) [T] [F] The distance from $(1, 2, -1)$ to $(3, -2, 1)$ is $(-2, 4, -2)$.

Solution:
The distance is a number, not a vector.

2) [T] [F] The plane $y = 3$ is perpendicular to the xz plane.

Solution:
The plane is parallel to the xz plane.

3) [T] [F] All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.

Solution:
The function $u(x, y) = e^x y$ satisfies $u_x = u$ but $u_y = e^x$.

4) [T] [F] The best linear approximation at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$

Solution:
The linear approximation is a linear function in x, y, z. The correct linear approximation would be $L(x, y, z) = 3 + 3(x - 1) + 3(y - 1) + 3(z - 1)$.

5) [T] [F] If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y)\, dy \right) dx = \int_0^1 \left(\int_y^1 f(x, y)\, dx \right) dy$.

Solution:
The correct identity would be $\int_0^1 \left(\int_x^1 f(x, y)\, dy \right) dx = \int_0^1 \left(\int_0^y f(x, y)\, dx \right) dy$.

6) [T] [F] Let $C = \{(x, y) \mid x^2 + y^2 = 1 \}$ be the unit circle in the plane and $\vec{F}(x, y)$ a vector field satisfying $|\vec{F}| \leq 1$. Then $-2\pi \leq \int_C \vec{F} \cdot dr \leq 2\pi$.

Solution:
By definition: $\int \vec{F} \cdot dr = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ and so $|\int \vec{F} \cdot \vec{dr}| = \int_0^{2\pi} |\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)|\, dt \leq \int_0^{2\pi} |\vec{F}(\vec{r}(t))||\vec{r}\,'(t)|\, dt \leq \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = 2\pi$.

7) [T] [F] Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ always point in different directions.

Solution:
Take $\vec{a} = \langle 4, 2\rangle$ and $\vec{b} = \langle 2, 1\rangle$. Then $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ point in the same direction.

8) [T] [F] If all the second-order partial derivatives of $f(x, y)$ vanish at (x_0, y_0) then (x_0, y_0) is a critical point of f.

Solution:
Take $f(x, y) = x + x^3 + y^3$. Then $(0, 0)$ not a critical point even so all second-order partial derivatives are zero.

9) [T] [F] If $\vec{a}, \vec{b}$ are vectors, then $|\vec{a} \times \vec{b}|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Solution:
$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\,|\sin(\phi)|$. $|\vec{a}|$ is the length of the base of the parallelogram and $|\vec{b}|\,|\sin(\phi)|$ is the height of the parallelogram.

10) [T] [F] The distance between two points A, B in space is the length of the curve $\vec{r}(t) = A + t(B - A),\ t \in [0, 1]$.

Solution:
$|\vec{r}\,'(t)| = |B - A|$ and $\int_0^1 |r'(t)|\, dt = |B - A|$.

11) [T] [F] The function $f(x, y) = xy$ has no critical point.

Solution:
$(0, 0)$ is a critical point of f.

12) [T] [F] The length of a curve does not depend on the chosen parameterization.

Solution:
This is a consequence of the chain rule: if $r(s(t))$ is a new parameterization, then $\int_a^b |r(s(t))'|\, dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

13) [T] [F] There exists a non-zero function $f(x,y,z)$ and non-zero vector field $\vec{F}(x,y,z)$ so that $\vec{F} = \text{grad}(f)$ and $f = \text{div}\vec{F}$.

Solution:
We need a solution to graddiv$(F) = F$. Trying $F = \langle P, 0, 0\rangle$ we get $F = \langle e^x, 0, 0\rangle$.

14) [T] [F] For any numbers a, b satisfying $|a| \neq |b|$, the vector $\langle a-b, a+b\rangle$ is perpendicular to $\langle a+b, b-a\rangle$.

Solution:
The dot product is 0.

15) [T] [F] The line integral of $\vec{F}(x,y) = \langle -y, x\rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R.

Solution:
The curl of $\vec{F}$ is 2. The statement is a consequence of Greens theorem.

16) [T] [F] There is no surface for which both the parabola and the hyperbola appear as traces.

Solution:
The hyperbolic paraboloid is an example.

17) [T] [F] If $(u,v) \mapsto \vec{r}(u,v)$ is a parameterization for a surface, then $\vec{r}_u(u,v) + \vec{r}_v(u,v)$ is a vector which lies in the tangent plane to the surface.

Solution:
Both $\vec{r}_u(u,v)$ as well as as $\vec{r}_v(u,v)$ are tangent to the surface. Therefore, also the sum is tangent.

18) [T] [F] When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi)\, d\rho d\phi d\theta$.

Solution:
The correct factor would be $\rho^2 \sin(\phi)$.

19) [T] [F] A surface in space for which all normal vectors are parallel to each other must be part of a plane.

Solution:
Assume all normal vectors are parallel to $\vec{n} = \langle a, b, c\rangle$ and assume $\vec{x}_0$ is a point in the plane. Then $\vec{n}(\vec{x} - \vec{x}_0) = 0$. But this is the equation of a plane. (This TF question implicitly assumes the surface to be connected. Also a union of parallel planes has the property that all normal vectors are parallel.)

20) [T] [F] A vector field $\vec{F} = \langle P(x, y), Q(x, y)\rangle$ is conservative in the plane if and only if $P_y(x, y) = Q_x(x, y)$ for all points (x, y).

Solution:
This is a consequence of Green's theorem: the line integral along a closed curve is the double integral of $Q_x - P_y$ over the enclosed region and so zero. The reverse is easier to see: if a potential f satisfying $\nabla f = \langle f_x, f_y\rangle = \langle P, Q\rangle$ exists, then $Q_x - P_y = f_{yx} - f_{xy} = 0$ (Clairot).

Problem 2) (10 points)

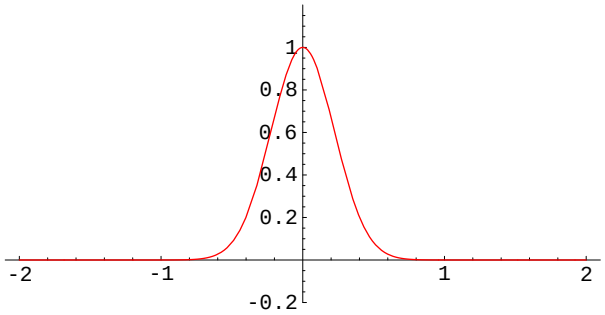
2 a) (5 points) Fill in names of the mathematicians: Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

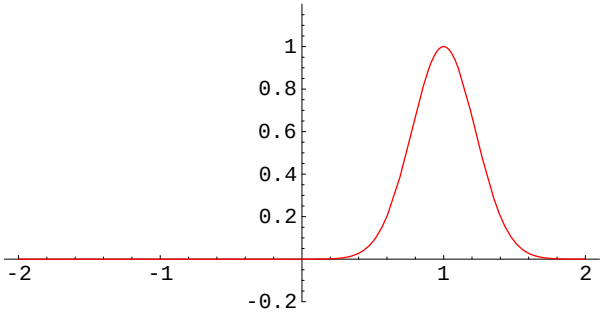
Formula	Name of the theorem
$\int_C \vec{F} \cdot \vec{dr} = \int\int_S \operatorname{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x,y) = f_{yx}(x,y)$	
$\int_C \vec{F} \cdot dr = \int\int_R \operatorname{curl}(\vec{F})\, dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int\int_S F \cdot dS = \int\int\int_E \operatorname{div}(F)\, dV$	
$\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$	

Solution:

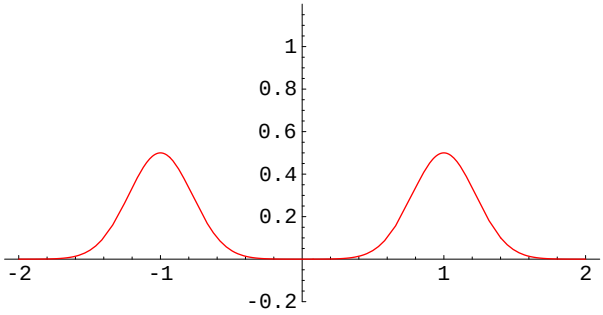
Formula	Name of the theorem
$\int_C \vec{F} \cdot \vec{dr} = \int\int_S \operatorname{curl}(\vec{F}) \cdot dS$	Stokes
$f_{xy}(x,y) = f_{yx}(x,y)$	Clairot
$\int_C \vec{F} \cdot dr = \int\int_R \operatorname{curl}(\vec{F})\, dxdy$	Green
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$	Fundamental theorem of line integrals
$\int\int_S F \cdot dS = \int\int\int_E \operatorname{div}(F)\, dV$	Gauss
$\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$	Fubini

2 b) (5 points) We have a function $u(t,x)$ which is a solution to a partial differential equation. In all cases, we have $u(0,x) = e^{-x^2}$. The picture to the right shows this function $u(0,x)$. Which partial differential equation is involved, when you see the function $u(1,x)$ as a graph?

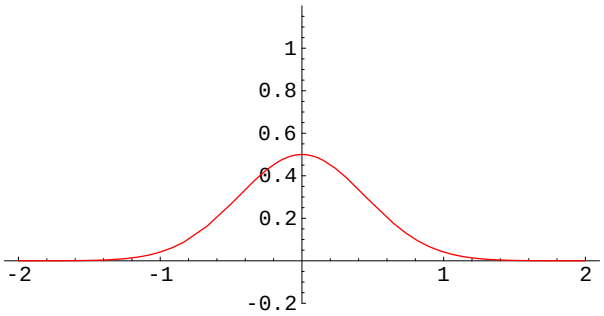




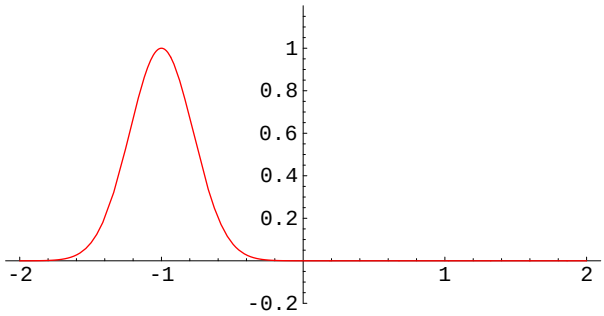
I



II



III



IV

Enter I,II,III,IV here	Equation
	$u_t(x,t) = u_x(x,t)$
	$u_t(x,t) = u_{xx}(x,t)$
	$u_{tt}(x,t) = u_{xx}(x,t)$
	$u_t(x,t) = -u_x(x,t)$

Solution:

Enter I,II,III,IV here	Equation
IV	$u_t(x,t) = u_x(x,t)$
III	$u_t(x,t) = u_{xx}(x,t)$
II	$u_{tt}(x,t) = u_{xx}(x,t)$
I	$u_t(x,t) = -u(x,t)$

Some more explanation:
Case $u_t = u_x$: If you look at the traces where t is fixed. This gives functions of one variable. $u_t = u_x$ means that the function will increase, where the slope u_x is positive and decrease, where the slope u_x is negative. This makes the graph move to the left. With $g(x) = u(0,x)$, the general solution is $u(t,x) = g(x+t)$ as you can check.
Case: $u_t = -u_x$: Dito, the graph moves to the right. With $g(x) = u(0,x)$, the general solution is $u(t,x) = g(x-t)$ as you can check.
Case: $u_t = u_{xx}$: The function will decrease where the second derivative u_{xx} is positive. So, the bump will become smaller. The tails of the bump have $u_x x < 0$, there, the function will increase.
Case: $u_{tt} = u_{xx}$: This is the wave equation. You best look at it physically. Pluck an infinite string in the middle. The disturbance will travel to both sides. You might also recall that with $g(x) = u(0,x)$ the function $u(t,x) = (g(x-t) + g(x+t))/2$ is the general solution.

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $O = (0, 0, 0)$ to Σ.

c) Find the distance from the point P to the line through Q, R.

d) Find the volume of the parallelepiped with vertices O, P, Q, R.

Solution:
a) Take the cross product between two vectors in the plane to get a normal vector $n = (a, b, c) = (1, 1, 2) \times (-1, 1, 4) = (2, -6, 2)$. The plane has the equation $ax + by + cz = d$, where d is obtained from plugging in one of the points. The answer is $\boxed{x - 3y + z = 2.}$.
b) The distance is $|(1, 0, 1) \cdot \vec{n}|/|n| = \boxed{2/\sqrt{11}}$.
c) The distance is $|\vec{PQ} \times \vec{QR}|/|\vec{QR}| = \sqrt{11/2}$.
d) The volume is $\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 5 \end{bmatrix} = 4$.

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y.

a) Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$.

b) Estimate $g(1.01, 0.99)$ using linear approximation.

Solution:
a) By the chain rule $g_x = -f_x/f_z = -yze^{xyz}/(1+xye^{xyz})$ and $g_y = -f_y/f_z = -xze^{xyz}/(1+xye^{xyz})$.
b) Note that $f(1, 1, 1) = 1 + e$ so that $(1, 1, 1)$ is on the surface and $g(1, 1) = 1$. From a) we know $g_x(1, 1) = -e/(1+e)$ and $g_y(1, 1) = -e/(1+e)$. By linearization, $g(1+a, 1+b) = g(1, 1)+g_x(1, 1)a+g_y(1, 1)b = 1-(a+b)e/(1+e)$. In our case, where $a = 0.01, b = -0.01$ we estimate $g(1.01, 0.99) = 1$.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\vec{r}(u,v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u,v) in the disc $D = \{u^2 + v^2 \leq 1\}$.

Solution:
$r_u = (1,0,u), r_v = (0,1,v)$ and $r_u \times r_v = (-u,-v,1)$.
The surface area is $\int\int_D \sqrt{1+u^2+v^2}\, dudv = \int_0^1 \int_0^{2\pi} \sqrt{1+r^2} r\, d\theta dr = 2\pi \int_0^1 \sqrt{1+r^2} r\, dr = 2\pi(1+r^2)^{3/2}/3|_0^1 = \boxed{2\pi(\sqrt{8}-1)/3}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x,y)$ which is the curl of $\vec{F}(x,y) = \langle -y^4/12 + y^3/6 - y, x^4/12 - x^3/6 \rangle$ on the disc $\{x^2 + y^2 \leq 4\}$.

a) Classify every critical point inside the disc $x^2 + y^2 < 4$.

b) Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of D.

Solution:
a) The function is $f(x,y) = x^3/3+y^3/3-x^2/2-y^2/2+1$. We have $\nabla f = (x^2-x, y^2-y)$. The critical points of f are $(0,0),(0,1),(1,0),(1,1)$.
The Hessian determinant (=discriminant) is $D = (2x-1)(2y-1)$, which is $1,-1,-1,1$. The point $(0,0)$ is a local maximum, the point $(1,1)$ is a local minimum and $(0,1),(1,0)$ are saddle points.
b) $g(x,y) = x^2+y^2-4$. The Lagrange equations are
$x^2 - x = \lambda 2x$
$y^2 - y = \lambda 2y$
$x^2+y^2-4 = 0$
which have solutions $\boxed{(-2,0),(0,-2),(0,2),(2,0),(-\sqrt{2},-\sqrt{2}),(\sqrt{2},\sqrt{2})}$. At these points, the function f takes values $-14/3, -14/3, 2/3, 2/3, -\sqrt{2}, -2-4\sqrt{2}/3, -2+4\sqrt{2}/3$.
c) To find the maximum and minimum, just compare the values at all the candidates obtained in a) and b):
$\boxed{\text{the minimum is at the two points } (-2,0),(0,-2)}$
where f takes the value $-14/3$.
$\boxed{\text{The global maximum is at } (0,0 \text{ because the value is there larger than } f(2,0) = f(0,2) = 2/3}$.

Problem 7) (10 points)

a) Given two nonzero vectors $\vec{u} = \langle a,b,c\rangle$ and $\vec{v} = \langle d,e,f\rangle$ in space, write down a formula for the cosine of the angle between them. Find a nonzero vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 3,2,1\rangle$. Describe geometrically the set of all $\vec{v}$, including zero, that are perpendicular to this vector $\vec{u}$.

b) Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P.

c) Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P.

d) Assume the level set of f is the graph of a function $g(x,y)$. Explain the relation between the gradient of g and the gradient of f. Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g?

Solution:
a) $\cos(\alpha) = ad + be + cf/(\sqrt{a^2+b^2+c^2}\sqrt{d^2+e^3+f^2})$. The vector $(1,-1,-1)$ is perpendicular to $(3,2,1)$. The set of points which are perpendicular to $(3,2,1)$ satisfies the equation $3x + 2y + z = 0$.
b) ∇f is orthogonal to every tangent vector to the level surface. ∇f is
orthogonal to the velocity vector of a curve $r(t)$ on the level surface.
c) Take two curves on the level surface which cross transversally. Because f is constant on the level surface, $f(r(t)) = c$ is constant or by the chain rule, $d/dtf(r(t)) = \nabla f(r(t)) \cdot r'(t) = 0$. This means that ∇f is orthogonal to the velocity vector $r'(t)$. By b), ∇f is orthogonal to the level surface.
d) In the special case $f(x,y,z) = g(x,y) - z$, we have $\nabla f = (\nabla g, -1)$. We see that projecting the vector ∇f onto the xy plane gives the vector ∇g.

Problem 8) (10 points)

Let R be the region inside the circle $x^2 + y^2 = 4$ and above the line $y = \sqrt{3}$. Evaluate

$$\int\int_R \frac{y}{x^2+y^2}\, dA\ .$$

Solution:
The key is to set up the integral in polar coordinates:

$$\int_{\pi/3}^{2\pi/3} \int_{\sqrt{3}/\sin(\theta)}^{2} \frac{r\sin(\theta)}{r^2}\, r dr d\theta\ .$$

This gives $\int_{\pi/3}^{2\pi/3}(2\sin(\theta) - \sqrt{3})\, d\theta = (-2\cos(\theta) - \sqrt{3}\theta)|_{\pi/3}^{2\pi/3} = -2(-1/2 - 1/2) - \sqrt{3}\pi/3 =$ $2 - \sqrt{3}\pi/3$.

Problem 9) (10 points)

A region W in $\mathbf{R}^3$ is given by the relations

$$x^2 + y^2 \leq z^2 \leq 3(x^2+y^2)$$
$$1 \leq x^2 + y^2 + z^2 \leq 4$$
$$x \geq 0$$

1. Sketch the region W.

2. Find the volume of the region W.

Solution:

1. First inequalities: the region is sandwiched between two cones. Second inequalities: the region is sandwiched between two spheres.
2. Use spherical coordinates:

$$\int_1^2 \int_{\pi/6}^{\pi/4} \int_{-\pi/2}^{\pi/2} \rho^2 \sin(\phi)\ d\theta d\phi d\rho + \int_1^2 \int_{3\pi/4}^{5\pi/6} \int_{-\pi/2}^{\pi/2} \rho^2 \sin(\phi)\ d\theta d\phi d\rho$$

Both integrals are the same so that we have to compute one and multiply in the end by 2. The answer is the product of the integrals $\int_1^2 \rho^2\ d\rho = 7/3$, $\int_{-\pi/2}^{\pi/2}\ d\theta = \pi$ and $\int_{\pi/6}^{\pi/4} \sin(\phi)\ d\phi = (\sqrt{3}-\sqrt{2})/2$ which is $(7/3)\pi(\sqrt{3}-\sqrt{2})/2$. The final answer is $\boxed{(7/3)\pi(\sqrt{3}-\sqrt{2})}$.

Problem 10) (10 points)

Consider the vector field

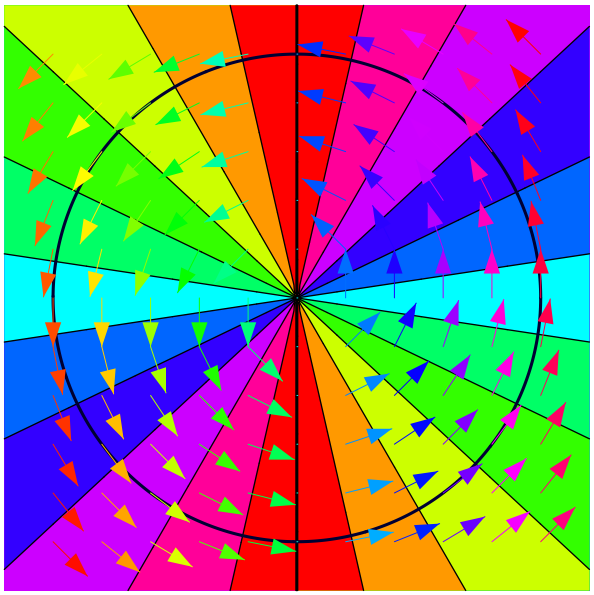
$$\vec{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$$

defined everywhere in the plane $\mathbf{R}^2$ except at the origin.

a) Let C be any closed curve which bounds a region D. Assume that $(0,0)$ is not contained in D and does not lie on C. Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0\ .$$

b) Let C be the unit circle oriented counterclockwise. What is $\int_C \vec{F} \cdot d\vec{r}$? Explain why your answer shows that there is no function f for which $\vec{F}(x,y) = \nabla f(x,y)$ everywhere.



Solution:
a) F is conservative in the region $\mathbf{R}^2 \backslash \{(0,0)\}$ because with $\vec{F} = \langle P, Q \rangle$ we have $Q_x - P_y = 0$ and Greens theorem assures that the line integral along any closed curve is zero, provided the region does not contain 0.
b) A direct calculation with $r(t) = (x(t), y(t)) = (\cos(t), \sin(t))$ gives $\int_0^{2\pi} 1 \; dt = 2\pi$. If there would exist a potential function f defined all over the plane, then the line integral along any closed curve would be zero. This would contradict the result for the curve along the unit circle.

Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function $f(x, y, z) = xyz$ over the solid in space described by the inequalities $0 \leq z \leq \sqrt{(1 - x^2 - y^2)}, x^2 + y^2 \leq 1, x - y \geq 0, y \geq 0$.

Solution:
Euclidean: $\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz \; dx \; dy$.
Cylindrical: $\int_0^1 \int_0^{\pi/4} \int_0^{\sqrt{1-r^2}} r^3 \cos(\theta) \sin(\theta) z \; dz \; d\theta \; dr$.
Spherical: $\int_0^1 \int_0^{\pi/4} \int_0^{\pi/2} \rho^3 \cos(\phi) \sin^2(\phi) \cos(\theta) \sin(\theta) \rho^2 \sin(\phi) \; d\phi d\theta \; d\rho$.

Problem 12) (10 points)

Let $\vec{F}(x,y)$ be a vector field in the plane given by the formula

$$\vec{F}(x,y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4+1}} \rangle \ .$$

If C is the path which goes from from $(-1,0)$ to $(1,0)$ along the semi circle $x^2+y^2=1$, $y \geq 0$, evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
The curl of F is -2. The negative of line integral along the curve C_1 from $(-1,0)$ to $(1,0)$ plus the line integral along the curve $C_2 : r(t) = (t,0)$ for $t \in [-1,1]$ is by Green's theorem $\int\int_D (-2)\ dxdy = -\pi$.

$$-\int_{C_1} \vec{F}\ dr + \int_{C_2} \vec{F}\ dr = \int\int_G \text{curl}(F)\ dxdy = -\pi$$

The line integral along the line segment C_2 is $\int_{-1}^{1} t^2\ dt = \boxed{2/3}$.

$$-\int_{C_1} \vec{F}\ dr + (2/3) = -\pi$$

Therefore, the final result for $\int_{C_1} \vec{F}\ dr$ is $\boxed{2/3 + \pi}$.

Problem 13) (10 points)

In appropriate units, the charge density $\sigma(x,y,z)$ in a region in space is given by $\sigma = \nabla \cdot \vec{E} = \text{div}(\vec{E})$, where $\vec{E}$ is the electric field. Consider the cube of side lengths 1 given by $0 \leq x,y,z \leq 1$. What is the total charge in this cube if

$$\vec{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3+y^3+z^3), z(1-z)e^{\sqrt{x+y}} \rangle \ .$$

(The total charge is the integral of the charge density over the cube.)

Solution:
On the x-faces, we know that $\vec{F} = \langle 0, Q, R \rangle$. The flux through the x-faces (normal to the x-axes) is 0. Similarly, the fluxes through the other sides is zero. By the divergence theorem, the triple integral on the unit cube is $\boxed{0}$.

Problem 14) (10 points)

a) By calculating the integral $\int\int_S \vec{F} \cdot d\vec{S}$ directly, find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z\rangle$ through the sphere $x^2+y^2+z^2=9$, where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z\rangle$ through the sphere $x^2+y^2+z^2 = 9$ using the divergence theorem.

c) Explain in words without invoking any integral theorem, why the flux integral of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z\rangle$ through any sphere with positive radius centered at $(0,0,0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

Solution:
a) We parametrize the sphere as usual and get $\vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\theta,\phi) = \rho^2 \sin(\phi)\vec{r}(\theta,\phi)$ so that $\vec{F}(\vec{r}(\theta,\phi)) \cdot \vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi) = \rho^3 \cos^2(\phi)\sin(\phi) + \rho^3 \cos(\theta)\sin^2(\phi)$. When we integrate the second summand over θ, we get zero. We are left with

$$\int_0^{2\pi}\int_0^{\pi} 27\cos^2(\phi)\sin(\phi)\, d\phi d\theta == 27 \cdot 2\pi \cdot (-\frac{\cos^3(\phi)}{3}|_0^{\pi}) =$$

$\boxed{36\pi}$. b) The divergence of $\vec{F}$ is 1. The integral of the divergence over the sphere of radius R is $4\pi R^3/3 = \boxed{36\pi}$.
c) The flux integral of $\vec{F}(x,y,z) = \langle 0,0,x\rangle$ is zero by symmetry: there is the same flux on upper and lower hemisphere with opposite sign. The flux integral of $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ is positive because everywhere on the sphere, the normal vector and the field form an acute angle.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
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11		10
12		10
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14		10
Total:		150

Problem 1) True/False questions (20 points)

1) [T] [F] For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v})$ is parallel to $\vec{w}$.

Solution:
Take $\vec{v} = \langle 1, 0, 0 \rangle, \vec{w} = \langle 0, 1, 0 \rangle$ so that $\vec{v} \times \vec{w} = \langle 0, 0, 1 \rangle$ and $(\vec{v} \times \vec{w}) \times \vec{v}) = \langle 0, 1, 0 \rangle$ and $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v}) = \langle 0, 0, 1 \rangle$.

2) [T] [F] The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

Solution:
Take $\vec{v} = \vec{w}$, then the right hand side is the zero vector while the left hand side is not zero in general (for example if $\vec{u} = \vec{i}, \vec{v} = \vec{j}$).

3) [T] [F] If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

Solution:
One can see that with the formula $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$ which shows that the acceleration $\vec{r}\,''(t)$ is in the velocity direction at all times. One can also see it intuitively or with the definition $\kappa(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$. If curve is not part of a line, then $\vec{T}\,'$ has to change which means that κ is not zero somewhere.

4) [T] [F] The curve $\vec{r}(t) = (1-t)A + tB, t \in [0, 1]$ connects the point A with the point B.

Solution:
The curve is a parameterization of a line and for $t = 0$, one has $\vec{r}(0) = A$ and for $t = 1$ one has $\vec{r}(1) = B$.

5) [T] [F] For every c, the function $u(x, t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

Solution:
Just differentiate.

6) [T] [F] The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1 + \cos^2(t)}\, dt$.

Solution:
The speed at time t is $|\vec{r}'(t)| = \sqrt{1 + \cos^2(t)}$.

7) [T] [F] Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

Solution:
While this would be true for $g(x, y) = f(y)$, where the constraint is a straight line parallel to the y axis, it is false in general.

8) [T] [F] The function $f(x, y, z) = x^2 - y^2 - z^2$ decreases in the direction $(2, -2, -2)/\sqrt{8}$ at the point $(1, 1, 1)$.

Solution:
It **increases** in that direction.

9) [T] [F] Assume $\vec{F}$ is a vector field satisfying $|\vec{F}(x, y, z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0, 1]$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is less or equal than the arc length of C.

Solution:
$|\vec{F} \cdot \vec{r}'| \leq |\vec{F}||\vec{r}'| \leq |\vec{r}'|$

10) [T] [F] Let $\vec{F}$ be a vector field which coincides with the unit normal vector $\vec{N}$ for each point on a curve C. Then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:
The vector field is orthogonal to the tangent vector to the curve.

11) [T] [F] If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a, b, c)$, where a, b, c are constants.

Solution:
One can also have $\vec{F} = \vec{G} + \text{grad}(f)$ which are vectorfields with the same curl.

12) [T] [F] If a nonempty quadric surface $g(x,y,z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.

Solution:
If one or two of the constants a, b, c are negative, we have a hyperboloid which all can not be contained into a finite space. If all three are negative, then the surface is empty.

13) [T] [F] If $\operatorname{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z), then $\operatorname{curl}(\vec{F}) = (0,0,0)$ for all (x,y,z).

Solution:
There are counter examples: take $(-y, x, 0)$ for example.

14) [T] [F] If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

Solution:
Otherwise, it is would be a cone (or for $\alpha = 0$ or $\alpha = \pi$ a half line).

15) [T] [F] The divergence of the gradient of any $f(x,y,z)$ is always zero.

Solution:
$\operatorname{div}(\operatorname{grad}(f)) = \Delta f$ is the Laplacian of f.

16) [T] [F] For every vector field $\vec{F}$ the identity $\operatorname{grad}(\operatorname{div}(\vec{F})) = \vec{0}$ holds.

Solution:
$F = (x^2, y^2, z^2)$ has $\operatorname{div}(F) = 2x + 2y + 2z$ which has a nonzero gradient $\nabla f = \langle 2, 2, 2 \rangle$.

17) [T] [F] For every function f, one has $\operatorname{div}(\operatorname{curl}(\operatorname{grad}(f))) = 0$.

Solution:
Both because $\operatorname{div}(\operatorname{curl}(F) = 0$ and $\operatorname{curl}(\operatorname{grad}(f) = 0$.

18) [T] [F] If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

Solution:
While it is true that the flux of curl(F) vanishes through every closed surface, this is not true for $\vec{F}$ itself. Take for example $F = (x, y, z)$.

19) [T] [F] The flux of the vector field $\vec{F}(x, y, z) = (y + z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

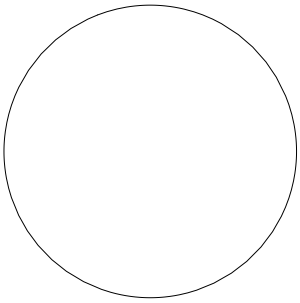
Solution:
By the divergence theorem, the flux through the boundary is $\int\int\int_E \text{div}(\vec{F})\ dV$ but $\text{div}(\vec{F}) = 0$. So the flux is zero.

20) [T] [F] For every function $f(x, y, z)$, there exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = f$.

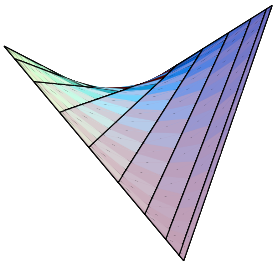
Solution:
In order to solve $P_x + Q_y + R_z = f$ just take $\vec{F} = (0, 0, \int_0^z f(x, y, w)\ dw)$.

Problem 2) (10 points)

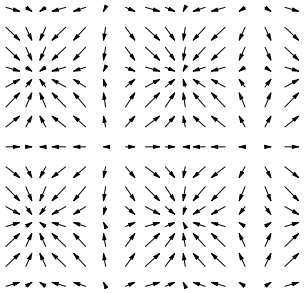
Problem 2a) (5 points) Match the equations with the objects. No justifications are needed.



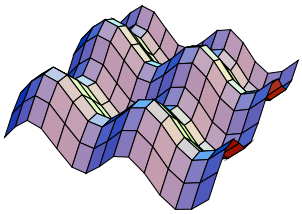
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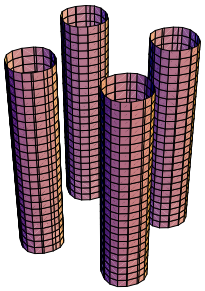
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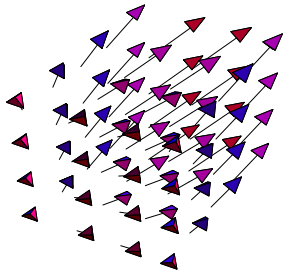
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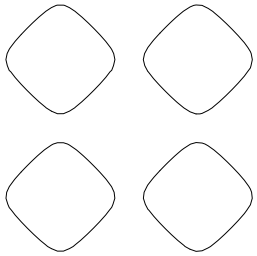
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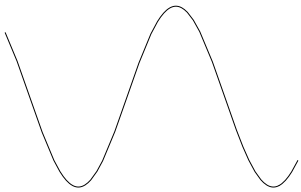
V



VI



VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x,y,z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$
	$\vec{r}(u,v) = \langle \cos(u), \sin(v), \cos(u)\sin(v) \rangle$
	$\vec{F}(x,y,z) = \langle \cos(x), \sin(x), 1 \rangle$
	$z = f(x,y) = \cos(x) + \sin(y)$
	$g(x,y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x,y) = \langle \cos(x), \sin(y) \rangle$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
V	$g(x,y,z) = \cos(x) + \sin(y) = 1$
VIII	$y = \cos(x) - \sin(x)$
I	$\vec{r}(t) = (\cos(t), \sin(t))$
II	$\vec{r}(u,v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
VI	$\vec{F}(x,y,z) = (\cos(x), \sin(x), 1)$
IV	$z = f(x,y) = \cos(x) + \sin(y)$
VII	$g(x,y) = \cos(x) - \sin(y) = 1$
III	$\vec{F}(x,y) = (\cos(x), \sin(x))$

Problem 2b) (5 points) Mark with a cross in the column below "irrotational" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x,y,z) = (0,0,0)$ for all points (x,y,z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z)). No justifications are needed.

Vectorfield	irrotational $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x,y,z) = \langle -5, 5, 3\rangle$		
$\vec{F}(x,y,z) = \langle x, y, z\rangle$		
$\vec{F}(x,y,z) = \langle -y, x, z\rangle$		
$\vec{F}(x,y,z) = \langle x^2 + y^2, xyz, x - y + z\rangle$		
$\vec{F}(x,y,z) = \langle x - 2yz, y - 2zx, z - 2xy\rangle$		

Solution:

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x,y,z) = \langle -5, 5, 3\rangle$	X	X
$\vec{F}(x,y,z) = \langle x, y, z\rangle$	X	
$\vec{F}(x,y,z) = \langle -y, x, z\rangle$		
$\vec{F}(x,y,z) = \langle x^2 + y^2, xyz, x - y + z\rangle$		
$\vec{F}(x,y,z) = \langle x - 2yz, y - 2zx, z - 2xy\rangle$		

Problem 3) (10 points)

a) (2 points) What is the area of the triangle A, B, P, where $A = (1, 1, 1), B = (1, 2, 3)$ and $P = (3, 2, 4)$?

b) (2 points) Find the distance between the point the point P and the line L passing through the points A with B.

Let E be a general parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

c) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

d) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

e) (3 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Solution:
a) The area is half of the cross product of $\vec{AB}$ and $\vec{AP}$ which is $(0, 1, 2) \times (2, 1, 3)$ which is $|(1, 4, -2)|$ which is $\sqrt{21}$. The triangle has the area $\sqrt{21}/2$.
b) The distance formula is $|\vec{AB} \times \vec{AP}|/|\vec{AB}| = |(1, 4, -2)|/|(0, 1, 2)| = \sqrt{\frac{21}{5}}$.
c) first diagonal $\boxed{u + v}$, second diagonal $\boxed{u - v}$.
d) $(u + v) \times (u - v) = 2v \times u = \boxed{\text{2 times area of of parallelogram}}$.
e) $(u + v) \cdot (u - v) = |u|^2 - |v|^2 = 0$, so that $\boxed{|u| = |v|}$.

Problem 4) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x, y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2} .$$

There is a lake in that area as you can see in the photo.

a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

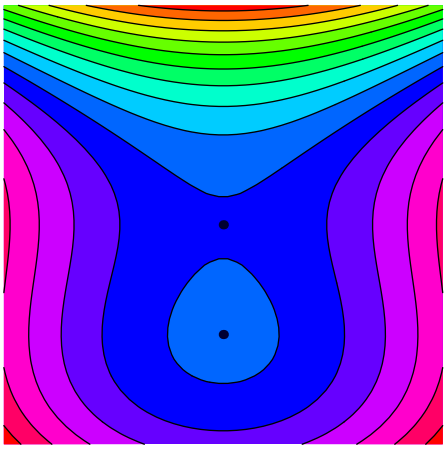
b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \, \vec{dr}$.



Photo of the lake in the Swiss alps near the Simplon mountain pass.

Solution:
a) The gradient is $\nabla f(x,y) = \langle x-1, -y-y^2 \rangle$. This gradient vanishes if $x = 1$ and $y = -1$ or $y = 0$. So, there are two critical points $(1,-1), (1,0)$. The Hessian matrix is $H(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1-2y \end{bmatrix}$.

point	discriminant	f_{xx}	nature
$(1,-1)$	D= 1	1	min
$(1,0)$	D= -1	1	saddle



b) By the fundamental theorem of line integrals, the line integral between the two points is the difference of the potentials which is $f(1,-1) - f(1,0) = (-1+1/3-1/2+1/2) - (-1+1/2) = -1/6$. Also the answer $1/6$ is correct of course, since we did not specify the direction.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Solution:
The volume of the box is $8xyz$. The Lagrange equations are

$$\begin{aligned} 8yz &= \lambda x/2 \\ 8xz &= \lambda 2y/9 \\ 8xy &= \lambda 2z/25 \end{aligned}$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1 = 0$$

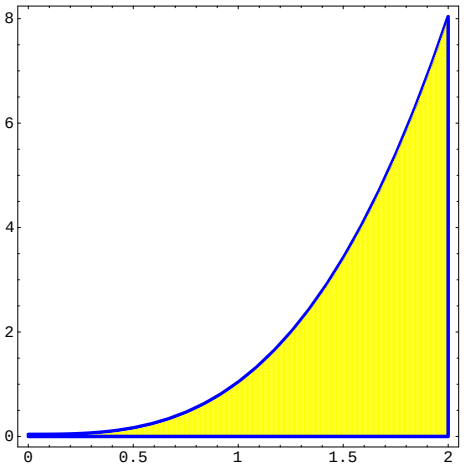
We can solve this by solving the first three equations for λ and expressing y, z by x, plugging this into the fourth equation. An other way to solve this is to multiply the first equation with x, the second with y and third with z.
The solution is $\boxed{x = 2/\sqrt{3}, y = \sqrt{3}, z = 5/\sqrt{3}}$. The maximal volume is $8xyz = 80/\sqrt{3}$.

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\, dxdy.$$

Solution:
This type II integral can not be computed as it is. We write it as a type I integral: from the boundary relation $x = y^{1/3}$ we obtain $y = x^3$ and $y = 8$ corresponds to $x = 2$:



$$\int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8}\, dydx$$
$$\int_0^2 \frac{x^9}{3} \frac{e^{x^2}}{x^8}\, dx$$
$$\int_0^2 \frac{x}{3} e^{x^2}\, dx$$
$$e^{x^2}/6|_0^2 = (e^4 - 1)/6$$

The result is $\boxed{\frac{e^4-1}{6}}$.

Problem 7) (10 points)

In this problem we evaluate $\int\int_D \frac{(x-y)^4}{(x+y)^4}\,dxdy$, where D is the triangular region bounded by the x and y axis and the line $x+y=1$.

a) (3 points) Find the region R in the uv-plane which is transformed into D by the change of variables $u=x-y, v=x+y$. (It is enough to draw a carefully labeled picture of R.)

b) (3 points) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $(x,y)=(\frac{u+v}{2},\frac{v-u}{2})$.

c) (4 points) Evaluate $\int\int_D \frac{(x-y)^4}{(x+y)^4}\,dxdy$ using the above defined change of variables.

Hint. The general topic of change of variables does not appear this semester. You can solve the problem nevertheless, when given the formula $\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ for the integration factor (analogous to r when changing to polar coordinates, or $\rho^2\sin(\phi)$ when going to spherical coordinates). The integral in c) becomes then $\int\int_R u^4/v^4\,dudv$. The region R is the triangle bounded by the edges $(0,0),(1,1),(-1,1)$.

Solution:
a) Take R with the edges $(0,0),(1,1)$ and $(-1,1)$.
The region R is the red triangle shown in the picture. You get that triangle by mapping the points of the triangle D with the map $T(x,y)=(x-y,x+y)$. (This is analogue to the coordinate change $T(r,\theta) = (r\cos(theta), r\sin(\theta))$.)

$$(0,0)\Rightarrow(0,0)$$

$$(1,0)\Rightarrow(1,1)$$

$$(0,1)\Rightarrow(-1,1)$$



These are the corners of the red triangle.
b) The Jacobian is $x_u y_v - x_v y_u = (1/4+1/4) = 1/2$.
(The Jacobian in the case of polar coordinates is r.)
c) The integral is best evaluated as a type II integral:
$\frac{1}{2}\int_0^1\int_{-v}^{v} u^4/v^4\,dudv = \frac{1}{2}\int_0^1 2v/5 =\ dv = \boxed{1/10}$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x,y) = -(x^4-8x^2+y^2+1)$.
b) (3 points) Classify the critical points.
c) (2 points) Locate the local and absolute maxima of f.
d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Solution:
a) $\boxed{(\pm 2, 0) \text{ and } (0, 0)}$.

b) $\boxed{(-2, 0) \text{ is a local maximum with value } 15}$.

$\boxed{(0, 0) \text{ is a saddle with value } -1}$.

$\boxed{(2, 0) \text{ is a maximum with value } 15}$.

c) $\boxed{\text{The local maxima are } (\pm 2, 0)}$. They are also the absolute maxima because f decays at infinity.

d) To calculate the tangent plane at the maximum, write the graph of f as a level surface $g(x, y, z) = z - f(x, y)$. The gradient of g is orthogonal to the surface. We have $\nabla g = (0, 0, 1)$ so that the tangent plane has the equation $z = d = const$. Plugging in the point $(\pm 2, 0, 15)$ shows that $\boxed{z = 15}$ is the equation for the tangent plane for both maxima.

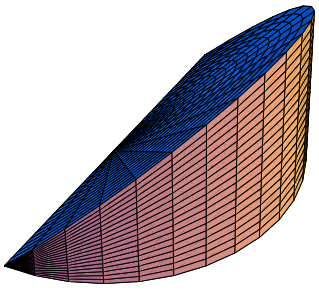
Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Solution:
Use polar coordinates and note that the wedge is above the right side of the unit disc:

$$\int_0^2 \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta)\, d\theta dr = 16/3$$

The solution is $\boxed{16/3}$.



Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = \langle t, \sin t, t^2 \cos t \rangle$ for $0 \leq t \leq \pi$. Let $f(x, y, z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals. The result is $f(r(\pi)) - f(r(0)) = f(\pi, 0, -\pi^2) - f(0, 0, 0) = \pi^4 e^{\pi} + \pi^2 - 0 = \boxed{\pi^4 e^{\pi} + \pi^2}$.

Problem 11) (10 points)

A cylindrical building $x^2 + (y-1)^2 = 1$ is intersected with the paraboloid $z = 4 - x^2 - y^2$.

a) Parametrize the intersection curve and set up an integral for its arc length.

b) Find a parametrization of the surface obtained by intersecting the paraboloid with the solid cylinder $x^2 + (y-1)^2 \leq 1$ and set up an integral for its surface area.

Solution:
a) $\vec{r}(t) = \langle \cos(t), \sin(t) + 1, 4 - \cos^2(t) - (1 + \sin(t))^2 \rangle$.
Write down $\int_0^{2\pi} |r'(t)| \, dt = \int_0^{2\pi} \sqrt{3 + 2\cos(2t)} \, dt$.
b) $\vec{r}(u, v) = \langle u, v, 4 - u^2 - v^2 \rangle$.
So, the integral is

$$\int \int_{u^2 + (v-1)^2 \leq 1} \sqrt{1 + 4u^2 + 4v^2} \, dudv = \int_{-1}^{1} \int_{-\sqrt{1-u^2}+1}^{\sqrt{1-u^2}+1} \sqrt{1 + 4u^2 + 4v^2} \, dvdu$$

Problem 12) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x, y) = \langle y^2, x^2 \rangle$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0, 0)$, $(1, 0)$ and $(1, 1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Green's theorem.

Solution:
The problem asks to do this in the clockwise direction. We do it in the counterclockwise direction and change then the sign.
a) $\int_0^1 \vec{F}(t,0)\cdot(1,0)\,dt + \int_0^1 \vec{F}(1,t)\cdot(0,1)\,dt + \int_0^1 \vec{F}(1-t,1-t)\cdot(-1,-1)\,dt = 0+1-2/3 = 1/3$.
So, the result for the clockwise direction is $\boxed{-1/3}$.
b) The curl of F is $2x-2y$.

$$\int_0^1 \int_0^x (2x-2y)dy\,dx = \int_0^1 2x^2 - x^2\,dx = 1/3$$

So, the result for the clockwise direction is $\boxed{-1/3}$.

Problem 13) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x,y,z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 1-\cos(t)-\sin(t)\rangle$ with $t \in [0, 2\pi]$.

Solution:
The curve is contained in the graph of the function $f(x,y) = 1-x-y$. That surface is parameterized by $\vec{r}(u,v) = \langle u, v, 1-u-v\rangle$ and has the normal vector $\vec{r}_u \times \vec{r}_v = \langle 1,0,-1\rangle \times \langle 0,1,-1\rangle = \langle 1,1,1\rangle$. The curl of F is $\langle 0,0,3x^2+3y^2\rangle$ so that $\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v) = 3(x^2+y^2)$. The surface is parameterized over the region $R = \{u^2+v^2 \le 1\}$ and $\int\int_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^{2\pi} 3r^3\,d\theta dr = \boxed{\frac{3\pi}{2}}$.

Problem 14) (10 points)

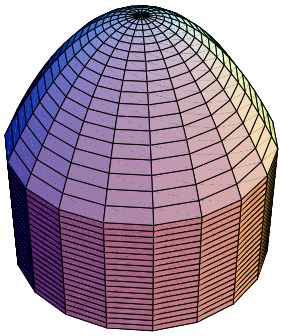
Let S be the graph of the function $f(x,y) = 2-x^2-y^2$ which lies above the disk $\{(x,y) \mid x^2+y^2 \le 1\}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int\int_S \vec{F}\cdot d\vec{S}$ of the vector field

$$\vec{F} = \langle -4x + \frac{x^2+y^2-1}{1+3y^2}, 3y, 7-z-\frac{2xz}{1+3y^2}\rangle$$

through S using the divergence theorem.

Solution:

We apply the divergence theorem to the region $E = \{0 \le z \le f(x,y),\ x^2+y^2 \le 1\ \}$. Using $\mathrm{div}(F) = -2$, we get



$$\begin{aligned}\int\int\int \mathrm{div}(F)\, dV &= \int_0^1\int_0^{2\pi}\int_0^{2-r^2} (-2)\, r\, dr d\theta dz \\ &= (-2)\int_0^1\int_0^{2\pi} (2-r^2)\, r\, d\theta dr \\ &= (-2)(2\pi)(2/2-1/4) = -3\pi\ .\end{aligned}$$

By the divergence theorem, this is the flux of $\vec{F}$ through the boundary of E which consists of the surface S, the cylinder $S_1 : r(u,v) = (\cos(u), \sin(u), v)$ with normal vector $r_u \times r_v = (-\sin(u), \cos(u), 0) \times (0,0,1) = (\cos(u), \sin(u), 0)$ plus the flux through the floor $S_2 : \vec{r}(u,v) = (v\sin(u), v\cos(u), 0)$ with normal vector $r_u \times r_v = (0,0,-v)$. The flux through S_1 is

$$\begin{aligned}\int\int_{S_1} \vec{F}\cdot dS &= \int_0^1\int_0^{2\pi} \vec{F}(\cos(u),\sin(u),v)\cdot(\cos(u),\sin(u),0)\, dudv \\ &= \int_0^1\int_0^{2\pi} (-4\cos^2(u) + 3\sin^2(u))\, dudv = -\pi\ .\end{aligned}$$

The flux through S_2 is

$$\begin{aligned}\int\int_{S_2} \vec{F}\cdot dS &= \int_0^1\int_0^{2\pi} F(v\sin(u), v\cos(u), 0)\cdot(0,0,-v)\, dudv \\ &= \int_0^1\int_0^{2\pi} (-7v)\, dudv = -7\pi\ .\end{aligned}$$

By the divergence theorem, $\int\int_S \vec{F}\cdot dS + \int\int_{S_1} \vec{F}\cdot dS + \int\int_{S_2} \vec{F}\cdot dS = -3\pi$ so that $\int\int_S \vec{F}\cdot dS = -3\pi + \pi + 7\pi = \boxed{5\pi}$.

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) [T] [F] The projection vector $\text{proj}_{\vec{v}}(\vec{w})$ is parallel to $\vec{w}$.

Solution:
It is parallel to $\vec{v}$.

2) [T] [F] Any parametrized surface S is a graph of a function $f(x, y)$.

Solution:
A counter example is the sphere.

3) [T] [F] If the directional derivatives $D_{\vec{v}}(f)(1, 1)$ and $D_{\vec{w}}(f)(1, 1)$ are both 0 for $\vec{v} = \langle 1, 1\rangle/\sqrt{2}$ and $\vec{w} = \langle 1, -1\rangle/\sqrt{2}$, then $(1, 1)$ is a critical point.

Solution:
Indeed $\nabla f(1, 1)$ must be perpendicular to $\vec{v}$ and $\vec{w}$ and so be the zero vector.

4) [T] [F] The linearization $L(x, y)$ of $f(x, y) = x + y + 4$ at $(0, 0)$ satisfies $L(x, y) = f(x, y)$.

Solution:
The linearization of any linear function at $(0, 0)$ is the function itself.

5) [T] [F] For any function $f(x, y)$ of two variables, the line integral of the vector field $\vec{F} = \nabla f$ on a level curve $\{f = c\}$ is always zero.

Solution:
The gradient is perpendicular to the velocity vector.

6) [T] [F] If $\vec{F}$ is a vector field of unit vectors defined in $1/2 \leq x^2 + y^2 \leq 2$ and $\vec{F}$ is tangent to the unit circle C, then $\int_C \vec{F} \cdot d\vec{r}$ is either equal to 2π or -2π.

Solution:
$\vec{r}'$ is parallel to $\vec{F}$ so that $\vec{F} \cdot \vec{r}'$ is equal to 1 or -1.

7) [T] [F] If a curve C intersects a surface S at a right angle, then at the point of intersection, the tangent vector to the curve is parallel to the normal vector of the surface.

Solution:
This is clear once you know what the question means.

8) [T] [F] The curvature of the curve $\vec{r}(t) = \langle \cos(3t), \sin(6t) \rangle$ at the point $\vec{r}(0)$ is smaller than the curvature of the curve $\vec{r}(t) = \langle \cos(30t), \sin(60t) \rangle$ at the point $\vec{r}(0)$.

Solution:
The curvature is independent of the parametrization of the curve.

9) [T] [F] At every point (x, y, z) on the hyperboloid $x^2 + 2y^2 - z^2 = 1$, the vector $\langle x, 2y, -z \rangle$ is normal to the hyperboloid.

Solution:
Look at the gradient. It is parallel to the vector.

10) [T] [F] The set $\{\phi = \pi/2, \theta = \pi\}$ in spherical coordinates is the negative x axis.

Solution:
$\phi = \pi/2$ forces us to be on the xy-plane. $\theta = \pi$ is the negative x axis

11) [T] [F] The integral $\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi) \; d\phi \; d\theta \; d\rho$ is equal to the volume of the unit ball.

Solution:
If $\sin^2(\phi)$ would be $\sin(\phi)$, then it would be the integral in spherical coordinates.

12) [T] [F] Four points A, B, C, D are located in a single common plane if $(B - A) \cdot ((C - A) \times (D - A)) = 0$.

Solution:
This is the volume of the parallelepiped with corners A, B, C, D. If the volume is zero, then the parallelepiped is flat and the points in a plane.

13) [T] [F] If a function $f(x, y)$ has a local maximum at $(0, 0)$, then the discriminant D is negative.

Solution:
False, we also can have $D = 0$ like for $f(x, y) = 1 - x^4 - y^4$.

14) [T] [F] The integral $\int_0^x \int_y^1 f(x, y)\, dx\, dy$ represents a double integral over a bounded region in the plane.

Solution:
The integral is not properly defined. There can be no variable in the most outer integral.

15) [T] [F] The following identity is true: $\int_0^3 \int_0^{2x} x^2\, dydx = \int_0^6 \int_{y/2}^3 x^2\, dx\, dy$

Solution:
Make a picture and draw the triangle.

16) [T] [F] The integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ over the surface S of a cube is zero for all vector fields $\vec{F}$.

Solution:
By the divergence theorem, this flux integral is equal to the triple integral of $\text{div}(\vec{F})$ over the cube. But since $\text{div}(\text{curl}(\vec{F})) = 0$, this integral is zero.

17) [T] [F] A vector field $\vec{F}$ defined on three space which is incompressible ($\text{div}(\vec{F}) = 0$) and irrotational ($\text{curl}(\vec{F}) = 0$) can be written as $\vec{F} = \nabla f$ with $\Delta f = \nabla^2 f = 0$.

Solution:
Every gradient field $F = \nabla f$, for which $\Delta(f) = 0$, is also incompressible.

18) [T] [F] If a vector field $\vec{F}$ is defined at all points of three-space except the origin, and $\text{curl}(\vec{F}) = \vec{0}$ everywhere, then the line integral of $\vec{F}$ around the circle $x^2 + y^2 = 1$ in the xy-plane is equal to zero.

Solution:
The circle is the boundary of a hemisphere which is contained in the region, where $\vec{F}$ is defined.

19) [T] [F] The identity $\text{curl}(\text{grad}(\text{div}(\vec{F}))) = \vec{0}$ is true for all vector fields $\vec{F}(x, y, z)$.

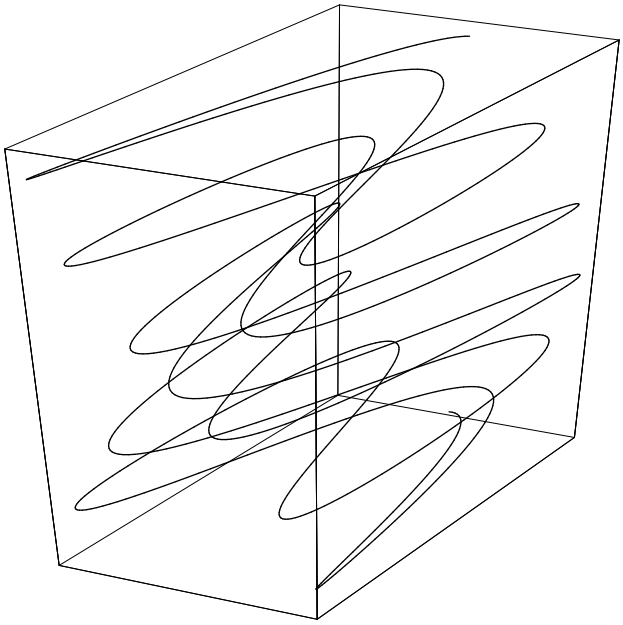
Solution:
Already $\text{curl}(\text{grad}(f)) = \vec{0}$.

20) [T] [F] If $\vec{F} = \text{curl}(\vec{G})$, where $\vec{G} = \langle e^{e^x}, 5^x z^5, \sin y \rangle$, then $\text{div}(\vec{F}(x, y, z)) > 0$ for all (x, y, z).

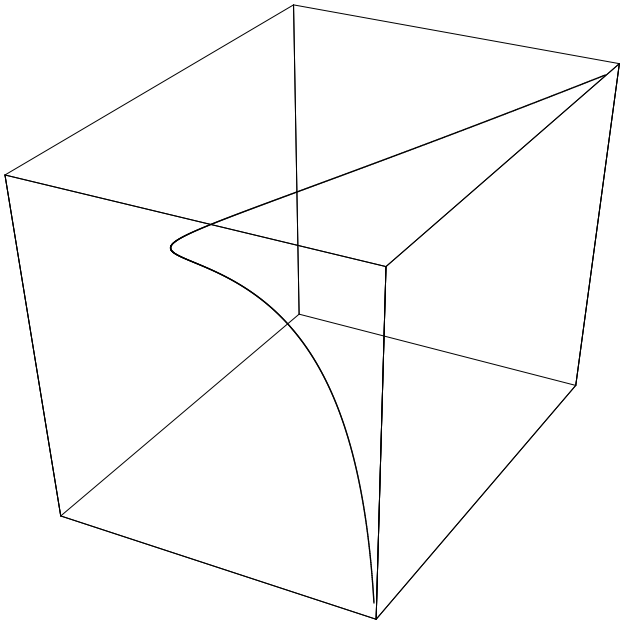
Solution:
$\text{div}(\text{curl}(\vec{F})) = 0$.

Problem 2) (10 points)

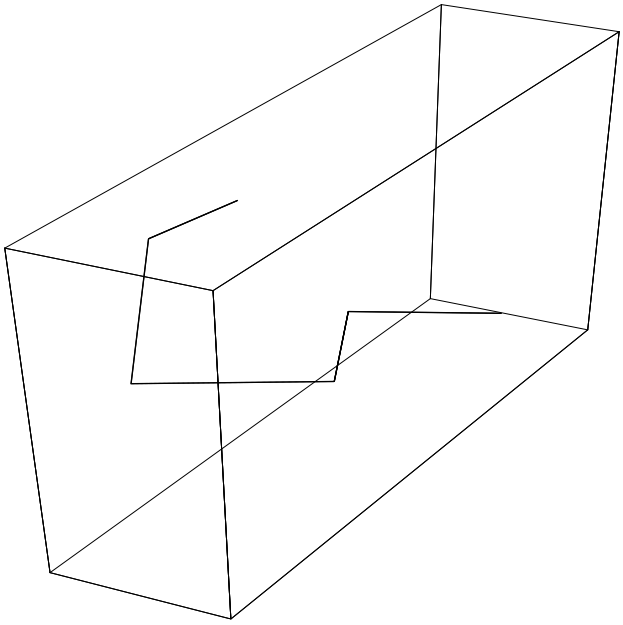
Match the equations with the space curves. No justifications are needed.



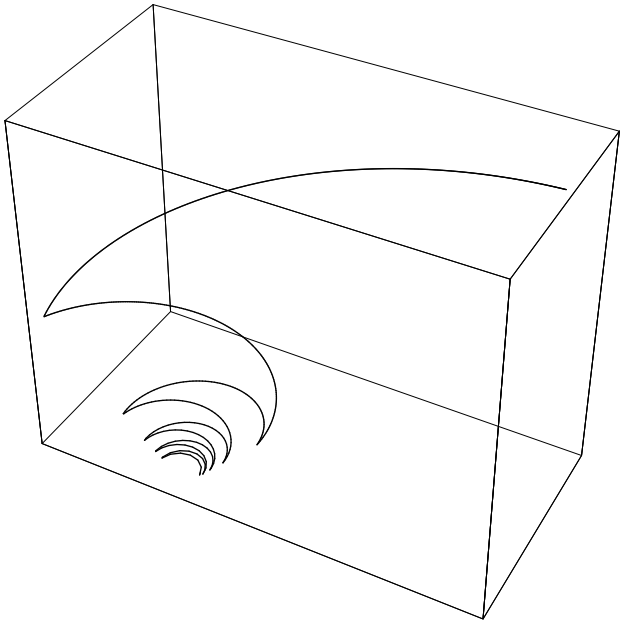
I



II



III



IV

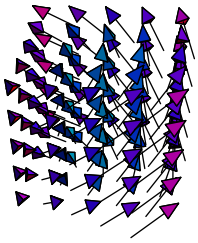
Enter I,II,III,IV here	Equation
	$\vec{r}(t) = \langle t^2, t^3 - t, t \rangle$
	$\vec{r}(t) = \langle \lvert 1 - \lvert t \rvert \rvert, \lvert t - \lvert t - 1 \rvert \rvert, t \rangle$
	$\vec{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$
	$\vec{r}(t) = \langle t\sin(1/t), t\lvert\cos(1/t)\rvert, t \rangle$

Solution:

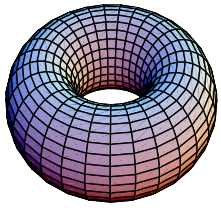
Enter I,II,III,IV here	Equation
II	$\vec{r}(t) = \langle t^2, t^3 - t, t \rangle$
III	$\vec{r}(t) = \langle \|1 - \|t\|\|, \|t - \|t - 1\|\|, t \rangle$
I	$\vec{r}(t) = \langle 2\sin(5t), \cos(11t), t \rangle$
IV	$\vec{r}(t) = \langle t\sin(1/t), t\|\cos(1/t)\|, t \rangle$

Problem 3) (10 points)

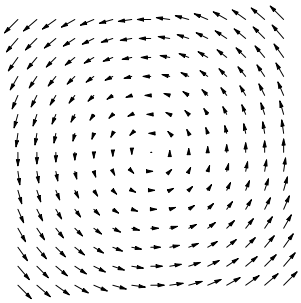
Match the equations with the objects. No justifications are needed.



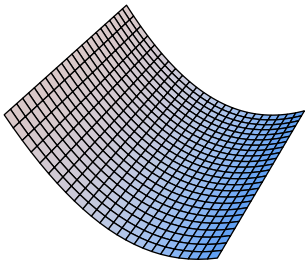
I



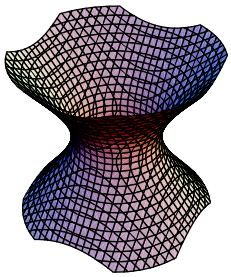
II



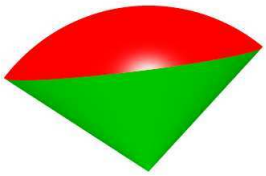
III



IV



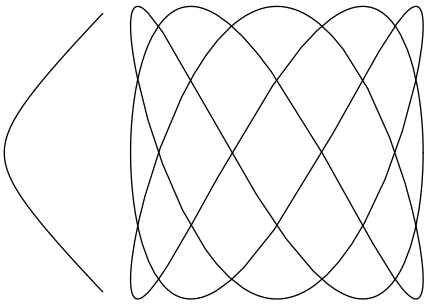
V



VI



VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$
	$x^2 + y^2 - z^2 = 1$
	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
	$x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0$
	$z = f(x,y) = x^2 - y$
	$g(x,y) = x^2 - y^2 = 1$
	$\vec{F}(x,y) = \langle -y, x \rangle$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
II	$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
VIII	$\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$
V	$x^2 + y^2 - z^2 = 1$
I	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
VI	$x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq z^2, z \geq 0$
IV	$z = f(x,y) = x^2 - y$
VII	$g(x,y) = x^2 - y^2 = 1$
III	$\vec{F}(x,y) = (-y, x)$

Problem 4) (10 points)

a) Find an equation for the plane Σ passing through the points $\vec{r}(0), \vec{r}(1), \vec{r}(2)$, where $\vec{r}(t) = \langle t^2, t^4, t \rangle$.

b) Find the distance between the point $\vec{r}(-1)$ and the plane Σ found in a).

Solution:
a) $r(0) = (0,0,0), r(1) = (1,1,1), r(2) = (4,16,2)$. The normal vector is $n = \langle a,b,c\rangle = \langle 1,1,1\rangle \times \langle 4,16,2\rangle = \langle -14,2,12\rangle$. The plane has the equation $ax+by+cz = d$, where d is obtained from plugging in one of the points. The answer is $\boxed{-14x+2y+12z=0.}$.
b) $Q = (0,0,0)$ is a point on the plane. The point $P = \vec{r}(-1) = (1,1,-1)$ has distance $|\vec{PQ}\cdot\vec{n}|/|\vec{n}| = |\langle 1,1,-1\rangle \cdot \langle -14,2,12\rangle|/\sqrt{196+4+144} = \boxed{24/\sqrt{344}}$ from the plane.

Problem 5) (10 points)

A vector field $\vec{F}(x,y)$ in the plane is given by $\vec{F}(x,y) = \langle x^2+5, y^2-1\rangle$. Find all the critical points of $|\vec{F}(x,y)|$ and classify them. At which point or points is $|\vec{F}(x,y)|$ minimal?

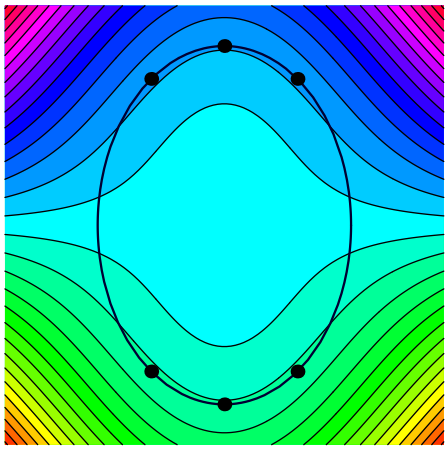
Solution:
Instead of extremizing the length, we can instead extremize the square of the length. This has the advantage of not involving square roots.
Extremize $f(x,y) = x^4+10x^2+25+y^4-2y^2+1$. The gradient is $\nabla f(x,y) = \langle 4x^3+20x, 4y^3-4y\rangle = \langle 4x(x^2+5), 4y(y^2-1)\rangle$. We have $\nabla f(x,y) = \langle 0,0\rangle$ for the critical points $(0,-1)$, $(0,0)$, and $(0,1)$. We have $f_{xx} = 20$ at all three points and $D(0,0) = -80, D(0,\pm 1) = 160$. The point $(0,0)$ is a saddle point and the two critical points $(0,1),(0,-1)$ are the global minima.

Problem 6) (10 points)

A house is situated at the point $(0,0)$ in the middle of a mountainous region. The altitude at each point (x,y) is given by the equation $f(x,y) = 4x^2y+y^3$. There is a pathway in the shape of an ellipse around the house, on which the (x,y) coordinates satisfy $2x^2+y^2 = 6$. Find the highest and lowest points in the closed region bounded by the path.

Solution:
The gradient of f is $\nabla f = \langle 8xy, 4x^2 + 3y^2 \rangle$. From the second equation, the only critical point for the function without constraints is $x = y = 0$. At this critical point, the function f has value 0. If $g(x, y) = 2x^2 + y^2$, then $\nabla g = \langle 4x, 2y \rangle$.
$\boxed{(0, 0) \text{ is a critical point with } f(0, 0) = 0.}$
For local maxima and minima on the ellipse, the equation $\nabla f = \lambda \nabla g$ must hold, so

$$\langle 8xy, 4x^2 + 3y^2 \rangle = \langle 4x\lambda, 2y\lambda \rangle.$$



Equating the first coordinates gives $8xy = 4x\lambda$, so $x = 0$ or $\lambda = 2y$. In the first case, $y^2 = 6$, so $y = \pm\sqrt{6}$. Therefore the function f has value $\pm 6\sqrt{6}$ in this case. Otherwise, $\lambda = 2y$, and so equating the second coordinates, $2y\lambda = 2y \cdot 2y = 4y^2 = 4x^2 + 3y^2$. Hence $y^2 = 4x^2$. At this point, one can work out $y = \pm 2x$. In any event, $2x^2 + y^2 = 6x^2$, so $6x^2 = 6$. Therefore $x = \pm 1$ and $y = \pm 2$.
In conclusion, there are 6 constraint extrema on the boundary $(0, \sqrt{6}), (0, -\sqrt{6}), (1, 2), (1, -2), (-1, 2), (-1, -2)$
The maximal value of f is $+16$ and obtained at $\boxed{\text{the points } (1, 2) \text{ and } (-1, 2)}$. The minimal value of f is obtained at $\boxed{\text{the points } (1, -2) \text{ and } (-1, -2).}$

Problem 7) (10 points)

We are given a function $f(x, y)$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$ as well as the following data points. Evaluate $\frac{\partial^2 f}{\partial \theta^2}$ at the point $r = 2,\ \theta = \frac{\pi}{2}$.

(x, y)	$(0, 2)$	$(2, 0)$	$(\pi, 2)$	$(2, \pi)$	$(0, 0)$
$f(x, y)$	2004	2005	2002	2003	2006
$f_x(x, y)$	3	4	6	2	0
$f_y(x, y)$	2	3	4	5	0
$f_{xx}(x, y)$	6	5	4	0	0
$f_{xy}(x, y)$	0	1	0	2	0
$f_{yy}(x, y)$	2	0	2	2	0

Solution:
The first derivative is

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta f_x + r\cos\theta f_y\end{aligned}$$

So when you take the 2nd derivative you have to use the product rule and the chain rule. The product rule gives you the derivative:

$$\frac{\partial^2 f}{\partial \theta^2} = -r\cos\theta f_x - r\sin\theta\frac{\partial}{\partial\theta}(f_x) - r\sin\theta f_y + r\cos\theta\frac{\partial}{\partial\theta}(f_y)$$

The chain rule is used to find:

$$\frac{\partial}{\partial\theta}(f_x) = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial\theta} = f_{xx} * (-r\sin\theta) + f_{xy} * (r\cos\theta)$$

$$\frac{\partial}{\partial\theta}(f_y) = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial\theta} = f_{yx} * (-r\sin\theta) + f_{yy} * (r\cos\theta)$$

So then plugging these in you get:

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta^2} &= -r\cos\theta f_x - r\sin\theta(f_{xx} * (-r\sin\theta) + f_{xy} * (r\cos\theta)) - r\sin\theta f_y + r\cos\theta(f_{yx} * (-r\sin\theta) + f_y \\ &= -r\cos\theta f_x + r^2\sin^2\theta f_{xx} - r^2\sin\theta\cos\theta f_{xy} - r\sin\theta f_y - r^2\sin\theta\cos\theta f_{yx} + r^2\cos^2\theta f_{yy} \\ &= -r\cos\theta f_x - r\sin\theta f_y + r^2\sin^2\theta f_{xx} - 2r^2\sin\theta\cos\theta f_{yx} + r^2\cos^2\theta f_{yy}\end{aligned}$$

The point $r = 2, \theta = \pi/2$ is in Cartesian coordinates the $(x, y) = (0, 2)$, so that we have to look at the first column of the table to look up the derivatives of f. Because $\cos(\pi/2) = 0$, two terms of the above formula vanish and the answer is

$$\frac{\partial^2 f}{\partial \theta^2} = 4f_{xx}(0, 2) - 2f_y(0, 2) = 24 - 4 = 20\ .$$

Problem 8) (10 points)

a) (4 points) Where does the tangent plane at $(1, 1, 1)$ to the surface $z = e^{x-y}$ intersect the z axis?

b) (4 points) Estimate $f(x, y, z) = 1 + \log(1 + x + 2y + z) + 2\sqrt{1 + z}$ at the point $(0.02, -0.001, 0.01)$.

c) (2 points) $f(x, y, z) = 0$ defines z as a function $g(x, y)$ of x and y. Find the partial derivative $g_x(x, y)$ at the point $(x, y) = (0, 0)$.

Solution:
a) The tangent plane has the equation $ax + by + cz = d$, where $\langle a, b, c\rangle = \nabla g(1, 1, 1)$, where $g(x, y, z) = z - e^{x-y}$. Because $\nabla g(x, y, z) = \langle -e^{x-y}, e^{x-y}, 1\rangle$, we have $\nabla g(1, 1, 1) = \langle -1, 1, 1\rangle$. The plane has the equation $-x + y + z = d$. The number d can be obtained by plugging in the point $(1, 1, 1)$. Therefore the plane is $-x + y + z = 1$. This plane intersects the z axes at $\boxed{z = 1}$.
b) The linearization of $1 + \log(1 + x + 2y + z) + 2\sqrt{1 + z}$ at $(0, 0, 0)$ is $L(x, y, z) = 3 + x + 2y + 2z$. Now $L(0.02, -0.001, 0.01) = 3 + 0.02 - 2 \cdot 0.001 + 2 * 0.01$ $\boxed{= 3.038}$.
c) The implicit computation formula is $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$. We have $f_z = (1/(1+x+2y+z)+1/\sqrt{1 + z})$ and $f_x = 1/(1+x+2y+z)$ so that $g_x(0, 0) = -1/(\sqrt{1 + z}+1)$. The value of z would have to be computed numerically.

Problem 9) (10 points)

For each of the following quantities, set up a double or triple integral using any coordinate system you like. You do not have to evaluate the integrals, but the bounds of each single integral must be specified explicitly.

1. (3 points) The volume of the tetrahedron with vertices $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$.
2. (4 points) The surface area of the piece of the paraboloid $z = x^2 + y^2$ lying in the region $z = x^2 + y^2$, where $0 \leq z \leq 1$.
3. (3 points) The volume of the solid bounded by the planes $z = -1$, $z = 1$ and the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
(1) The tetrahedron is bounded by the xy, yz, zx-planes and the plane $z = 3 - x - y$. The triple integral would be: $\int_0^3 \int_0^{3-x} \int_0^{3-x-y} \, dz \; dy \; dx$. Evaluating the inner integral $\int_0^3 \int_0^{3-x} (3 - x - y) \, dy dx$.
(2) The parameterization $\vec{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r^2 \rangle$ gives

$$\begin{aligned} |\vec{r}_r \times \vec{r}_\theta| &= |\langle \cos(\theta), \sin(\theta), 2r \rangle \times \langle -r\sin(\theta), r\cos(\theta), 0 \rangle| \\ &= |\langle \, -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle| = \sqrt{4r^4 + r^2} \, . \end{aligned}$$

The surface area integral is $\int_0^{2\pi} \int_0^1 \sqrt{4r^4 + r^2} \; dr \; d\theta$ or $\int_0^1 \int_0^{2\pi} \sqrt{4r^4 + r^2} \; d\theta \; dr$.
Parameterization $\vec{r}(x, y) = \langle \; x, \; y, \; x^2 + y^2 \; \rangle$ gives

$$|\vec{r}_x \times \vec{r}_y| = |\langle 1, 0, 2x \; \rangle \times \langle 0, 1, 2y \; \rangle| = |\langle -2x, -2y, 1 \rangle| = \sqrt{4x^2 + 4y^2 + 1} \, ,$$

and the integral $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{4x^2 + 4y^2 + 1} \, dxdy$ or $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2 + 4y^2 + 1} \, dydx$.
(3) Using cylindrical coordinates, the integral is $\int_{-1}^1 \int_0^{\sqrt{1+z^2}} \int_0^{2\pi} r \;\; d\theta dr \;\; dz$ which is $2\pi \cdot \int_{-1}^1 \int_0^{\sqrt{1+z^2}} \; r dr \; dz$. If the order of integration is changed, then - still in cylindrical coordinates - we have

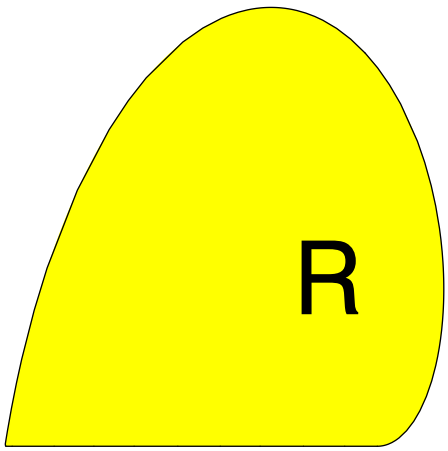
$$2\pi \Big(\int_0^1 \int_{-1}^1 r dz \; dr \; + 2 \int_1^{\sqrt{2}} \int_{\sqrt{r^2-1}}^1 \; r dz \; dr \Big) \, .$$

When using spherical coordinates, one would have to split up the integral into two parts and setting up the integral is harder.

Problem 10) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}(\pi - \sqrt{x^2 + y^2})} \; dx \; dy \, .$$



Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = drd\theta$ does not work. The integral $\int_0^\pi \int_0^\theta \frac{\cos(r)}{r(\pi-r)} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^\pi \int_r^\pi \frac{\cos(r)}{r(\pi - r)} r \; d\theta \; dr$$

Evaluating the inner integral gives $\int_0^\pi \cos(r) \; dr = \boxed{0}$.

Problem 11) (10 points)

A car drives up a freeway ramp C which is parametrized by

$$\vec{r}(t) = \langle \cos(t), 2\sin(t), t \rangle, \qquad 0 \leq t \leq 3\pi \; .$$

a) (3 points) Set up an integral which gives the length of the ramp. You do need not need to evaluate it.

b) (3 points) Find the unit tangent vector $\vec{T}$ to the curve at the point where $t = 0$.

c) (4 points) Suppose the wind pattern in the area is such that the wind exerts a force $\vec{F} = \langle 4x^2, y, 0 \rangle$ on the car at the position (x, y, z). What is the total work gain as the drives up the ramp? In other words, what is the line integral $\int_C \vec{F} \; \vec{dr}$.

Solution:
a) $\vec{r}\,'(t) = \langle -\sin(t), 2\cos(t), 1 \rangle$ and $|\vec{r}\,'| = \sqrt{\sin^2(t) + 4\cos^2(t) + 1}$. The integral is

$$\int_0^{3\pi} \sqrt{\sin^2(t) + 4\cos^2(t) + 1} \; dt \; .$$

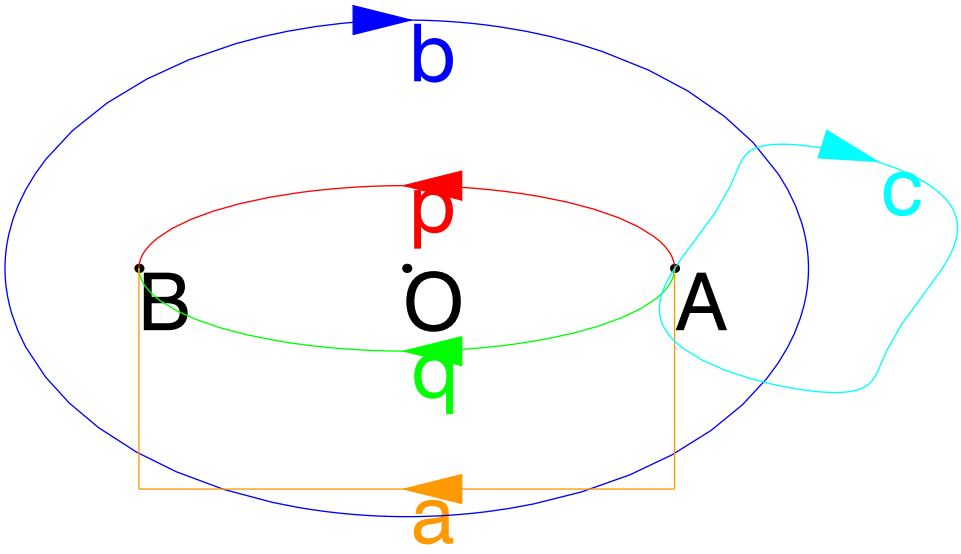
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$$\int_0^{3\pi} \sqrt{2 + 3\cos^2(t)} \; dt \; .$$

b) $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)| = \langle -\sin(t), 2\cos(t), 1 \rangle / \sqrt{\sin^2(t) + 4\cos^2(t) + 1}$. At $t = 0$, we have $\vec{T}(t) = \langle 0, 2, 1 \rangle / \sqrt{5}$.
c) The line integral is $\int_0^{3\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$. Note that $\vec{F}$ is a gradient field with potential function $f(x, y, z) = 4x^3/3 + y^2/2$ so that the line integral is $f(\vec{r}(3\pi)) - f(\vec{r}(0)$ by the fundamental theorem of line integrals. Now $f(\vec{r}(3\pi)) = f(-1, 0, 3\pi) = -4/3$ and $f(\vec{r}(0)) = f(1, 0, 0) = 4/3$, so that the result is $\boxed{-8/3}$.

Problem 12) (10 points)

Suppose $\vec{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. Suppose the line integral of $\vec{F}$ along the path p from A to B is 5 and the line integral of $\vec{F}$ along the path q from A to B is -4. Find the line integral of $\vec{F}$ along the following three paths:



a) (3 points) The path a from A to B going clockwise below the origin.

b) (4 points) The closed path b encircling the origin in a clockwise direction.

c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Solution:
a) The result is the same for the path a and the path q. The vector field is conservative in the lower half plane. The result is $\boxed{-4}$. b) The line integral is the same as the difference of the line integral along q and the line integral along p which is $-4-5 = \boxed{-9}$. The path $q - p$ encircles the origin in the same direction than the path b. Because the curl is 0 in the region enclosed by these two curves, Greens theorem assures that the line integrals are the same.
c) The vector field $\vec{F}$ is conservative in the right half plane. By the fundamental theorem of line integrals or using the closed loop property, the result is $\boxed{0}$.

Problem 13) (10 points)

Let S be the surface which bounds the region enclosed by the paraboloid $z = x^2 + y^2 - 1$ and the xy plane. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle x + e^{\sin(z)}, z, -y \rangle$.

a) (5 points) Find the flux of $\vec{F}$ through the surface S.

b) (5 points) Find the flux of $\vec{F}$ through the part of the surface S that belongs to the paraboloid, oriented so that the normal vector points downwards.

Solution:
a) The flux out of the whole surface can be determined by the Divergence Theorem. The flux over the whole surface is the integral of the divergence over the solid region it encloses. The divergence of the vector field is 1. So the flux is the volume of the solid region, namely the integral $\int_D 1 - x^2 - y^2 \; dydx$ over the unit disc D. Converting to polar coordinates, this is $\int_0^{2\pi} \int_0^1 (1 - r^2) r \; dr \; d\theta = \int_0^{2\pi} \int_0^1 r - r^3 \; dr \; d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{4} \; d\theta = \int_0^{2\pi} \frac{1}{4} \; d\theta = \frac{\pi}{2}$. So the flux out of the whole surface is $\boxed{\pi/2}$.
b) There the outward unit normal vector to the top surface which is $\vec{k}$, so the flux is

$$\int_D -y \; dA = \int_0^{2\pi} \int_0^1 -r \sin\theta \; r \; dr \; d\theta = \int_0^{2\pi} -\frac{1}{3} \sin\theta \; d\theta = 0 \; .$$

Therefore the flux out of the "roof" D is $\boxed{0}$. The flux through the "floor" the paraboloid part is $\pi/2 - 0 = \boxed{\pi/2}$.

Problem 14) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \langle 4z + \cos(\cos x), y^2, x + 2y \rangle$.

a) (4 points) Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$, for $0 \leq t \leq 2\pi$. Find the line integral of $\vec{F}$ along C.

b) (6 points) Let S be the hemisphere of the unit sphere defined by $y \leq 0$. Find the flux of the curl of $\vec{F}$ out of S. In other words, find

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \; .$$

For part b), the surface S is oriented so that the normal vector has a positive y-component.

Solution:
a) The curl of the vector field is $\langle 2, 3, 0\rangle$. The parameterization describes the circle $x^2 + z^2 = 1$, where $y = 0$. The curve starts at $(1, 0, 0)$ and rotates towards back towards $(0, 0, 1)$. By Stokes theorem, the line integral can be computed as the flux of curl(F) through the unit disk D in the xz plane which has the normal vector $r_u \times r_v = -\vec{j}$ and $\text{curl}(\vec{F})(x, y, z) \cdot (r_u \times r_v) = -3$. The flux is

$$\int\int_D -3 \; dx \; dz = -3\pi \; .$$

b) The boundary of S is the curve C, oriented differently as in part a). Therefore, the answer is 3π by **Stokes theorem**. Note that the sign is different because the orientation of the curve C and the surface S do not match.