On Some Geometric Constructions Related to Theta Characteristics

To J. Shalika, on his 60th birthday

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Joe Harris

The theory of quadratic forms over the field of 2 elements has many mathematical applications, from finite group theory to algebraic topology. Here we pursue a connection discovered by Mumford, relating theta characteristics on an algebraic curve to quadratic forms on the vector space of 2-torsion points in its Jacobian.

We develop the algebraic and combinatorial aspects of quadratic forms in the first three sections, then review some of the theory of theta characteristics in §4. The last three sections use this theory to investigate some classical geometric constructions on curves of genus 2 and 3.

Some of the material in sections 2 and 3 appears in the 19th century literature (cf. for example [C1], [C2] and [W]), and has been abstracted in expository articles (cf. [Sa]). Similarly, versions of the geometric constructions in sections 5-7 have appeared in several excellent modern expositions (cf. [G-H] and [D-O]). We wish to thank Igor Dolgachev, who guided us to much of the existing literature.

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1. Quadratic Forms

Throughout this paper, $k = \mathbb{Z}/2\mathbb{Z}$ is the field with 2 elements. Let V be a vector space of dimension $2g$ over k. We fix a nondegenerate, strictly alternating from $\langle , \rangle : V \otimes V \to k$. Thus $\langle v, v \rangle = 0$ for all $v \in V$, and the map $v \mapsto f_v(u) = \langle u, v \rangle$ gives an isomorphism from V to its dual space $\mathrm{Hom}(V, k)$.

The symplectic space $(V, \langle , \rangle)$ is uniquely determined up to isomorphism by its dimension $2g$. Let $Sp(V)$ be the group of all k-linear isomorphisms $T : V \to V$ which satisfy $\langle Tv, Tu \rangle = \langle v, u \rangle$ for all $v, u \in V$. The group $Sp(V)$ is generated by the transvections:

$$T_u(v) = v + \langle v, u \rangle u \tag{1.1}$$

where $u \neq 0$ in V, and these form a single conjugacy class of involutions in $Sp(V)$. The finite group $Sp(V)$ has order $2^{g^2}(2^{2g} - 1)(2^{2g-2} - 1) \cdots (2^2 - 1)$ ([A]).

A subspace $X \subset V$ is isotropic if $\langle x, x' \rangle = 0$ for all $x, x' \in X$. The maximal isotropic spaces all have dimension g, and may be completed to an isotropic decomposition of V:

$$V = X \oplus Y \tag{1.2}$$

with X and Y isotropic of dimension g. The pairing $\langle , \rangle$ on V puts the subspaces X and Y in duality. The isotropic decompositions (1.2) of V are all conjugate under $Sp(V)$, and the stability subgroup of a fixed decomposition is isomorphic to $GL(X)$ ([A]). If $\langle e_1, \ldots, e_g \rangle$ is a basis for X and $\langle f_1, \ldots, f_g \rangle$ is the dual basis of Y, the vectors $\langle e_1, \ldots, e_g; f_1, \ldots, f_g \rangle$ give a symplectic basis for V.

We say a function $q : V \to k$ is a quadratic form on V (relative to the fixed sympletic form $\langle , \rangle$) provided that

$$q(v + u) + q(v) + q(u) = \langle v, u \rangle \tag{1.3}$$

for all $v, u \in V$. If $V = X \oplus Y$ is any isotropic decomposition, the function

$$q_0(x + y) = \langle x, y \rangle \tag{1.4}$$

defines a quadratic form on V. In terms of a symplectic basis:

$$q_0\left(\sum_{i=1}^{g} \alpha_i e_i + \sum_{i=1}^{g} \beta_i f_i\right) = \sum_{i=1}^{g} \alpha_i \beta_i. \tag{1.5}$$

Let QV denote the set of all quadratic forms on V, relative to $\langle , \rangle$. Then QV is a principal homogeneous space for V: if $q \in QV$ and $v \in V$ we define the quadratic form $q + v$ by

$$q + v(u) = q(u) + \langle v, u \rangle. \tag{1.6}$$

Similarly, if q and q' are two elements of QV there is a unique vector $v = q + q'$ such that

$$\langle v, u \rangle = q(u) + q'(u). \tag{1.7}$$

This gives the disjoint union $W = V \cup QV$ the structure of a k-vector space of dimension $2g + 1$, which contains V as a subspace of codimension 1.

The group $Sp(V)$ acts on the set QV by the formula $q \mapsto Tq$, where

$$Tq(Tv) = q(v). \tag{1.8}$$

This gives a linear action of $Sp(V)$ on W, and we have an exact sequence of $Sp(V)$-modules

$$0 \to V \to W \to k \to 0. \tag{1.9}$$

We define the Arf invariant $a : QV \to k$ as follows. Let $\langle e_1, \ldots, e_g; f_1, \ldots, f_g \rangle$ be a symplectic basis of V. For $q \in QV$, let

$$a(q) = \sum q(e_i)q(f_i). \tag{1.10}$$

A priori, this depends on the symplectic basis chosen, but we have the following

Proposition 1.11. *The Arf invariant $a(q)$ does not depend on the symplectic basis chosen to define it. If q is defined by an isotropic decomposition, as in (1.4), then $a(q) = 0$.*

We have the formulas

$$a(Tq) = a(q) \qquad T \in Sp(V)$$

$$a(q + v) = a(q) + q(v) \qquad v \in V.$$

The group $Sp(V)$ has 2 orbits on QV, the $2^{g-1}(2^g + 1)$ even forms q with $a(q) = 0$, and the $2^{g-1}(2^g - 1)$ odd forms q with $a(q) = 1$.

Proof. Since $Sp(V)$ acts transitively on the collection of symplectic bases, and is generated by the transvections T_u, it suffices to check that

$$\sum q(e_i)q(f_i) = \sum q(T_u e_i)q(T_u f_i).$$

This is an amusing exercise, which uses the identity $\alpha^2 = \alpha$ in k.

If $q = q_0$ is defined using an isotropic decomposition (1.4), then $q = 0$ on the subspaces X and Y. Hence $a(q) = 0$.

The formula $a(Tq) = a(q)$ follows from the independence of basis. To prove that $a(q+v) = a(q)+q(v)$, extend $v = e_1$ to a symplectic basis of V and use (1.10) to calculate $a(q + v)$.

One shows, by induction on g, that the form q_0 defined by (1.4)–(1.5) has $2^{g-1}(2^g+1)$ zeroes on V. Since $a(q_0) = 0$, we have $a(q_0 + v) = q_0(v)$. Hence there are $2^{g-1}(2^g + 1)$ forms $q = q_0 + v$ with $a(q) = 0$, and $2^{g-1}(2^g - 1)$ forms q with $a(q) = 1$.

Now fix q, and consider the action of the involutions $T_u \in Sp(V)$. If $q(u) = 1$, we find that $T_u q = q$. If $q(u) = 0$, then $T_u q = q + u$. It follows that the group $Sp(V)$ acts transitively on the set of forms with either Arf invariant.

Corollary 1.12. *For $q \in QV$, the following conditions are all equivalent*

1) $a(q) = 0$
2) *q has $2^{g-1}(2^g + 1)$ zeroes on V.*
3) *There is an isotropic decomposition $V = X \oplus Y$ such that $q = 0$ on the subspaces X and Y.*

Corollary 1.13. *The stabilizer $O(V, q) \subset Sp(V)$ of a form $q \in QV$ has order*

$$2^{g^2-g+1}(2^{2g-2} - 1)(2^{2g-4} - 1) \cdots (2^2 - 1)(2^g - 1) \qquad \text{if} \quad a(q) = 0$$

$$2^{g^2-g+1}(2^{2g-2} - 1)(2^{2g-4} - 1) \cdots (2^2 - 1)(2^g + 1) \qquad \text{if} \quad a(q) = 1$$

The transvection T_u lies in $O(V, q)$ if $q(u) = 1$.

Corollary 1.14. *If $g \geq 2$, the sequence (1.9) of $Sp(V)$-modules $0 \to V \to W \to k \to 0$ is* **not** *split.*

2. Aronhold Sets

Recall that $W = V \cup QV$ is a k-vector space of dimension $2g+1$ over k. Let $S = \{q_1, q_2, \ldots, q_{2g+1}\}$ be a set of linearly independent vectors, all of which lie in the coset QV. Relative to this basis, any vector $w \in W$ has a unique expression $w = \sum \alpha_i q_i$ with $\alpha_i = 0, 1$ in $\mathbb{Z}$. We define $\#w = \sum \alpha_i$, so $0 \leq \#w \leq 2g+1$. If w lies in the coset QV, then $\#w$ is odd.

We say S is an Aronhold set provided that the Arf invariant of any element $q = \sum \alpha_i q_i$ in QV depends only on the residue class of the odd integer $\#q \pmod 4$. If S is an Aronhold set, we must have $a(q_1) = a(q_2) = \cdots = a(q_{2g+1})$, as these are the forms q with $\#q = 1$. Also, there is a unique form $q_S = \sum q_i$ with $\#q_S = 2g+1$, and this form must satisfy $a(q_S) \equiv a(q_i) + g \pmod 2$.

Proposition 2.1. *There exist Aronhold sets $S = \{q_1, \ldots, q_{2g+1}\}$ with*

$$a(q_i) = \begin{cases} 0 & g \equiv 0, 1 \pmod 4 \\ \\ 1 & g \equiv 2, 3 \pmod 4 \end{cases}$$

The group $Sp(V)$ acts transitively on the collection of Aronhold sets in W, and the stabilizer of S is the full symmetric group

$$\mathrm{Sym}(S) \hookrightarrow O(V, q_S) \hookrightarrow Sp(V).$$

Proof. Define the vector space N of dimension $2g+1$ over k, with basis $S = \{n_1, n_2, \ldots, n_{2g+1}\}$. Let M be the subspace $\{\sum \alpha_i n_i : \sum \alpha_2 \equiv 0\}$ of codimension 1. The bilinear form on N

$$\left\langle \sum \alpha_i n_i, \sum \beta_i n_i \right\rangle = \sum \alpha_i \beta_i$$

is strictly alternating and nondegenerate on M.

For $n = \sum \alpha_i n_i$ in N, with $\alpha_i = 0, 1$ in $\mathbb{Z}$, define $\#n = \sum \alpha_i$. Put

$$a(n_i) = \begin{cases} 0 & g \equiv 0, 1 \pmod 4 \\ \\ 1 & g \equiv 2, 3 \pmod 4 \end{cases}$$

For $n \in N - M$, $\#n$ is odd and we define

$$a(n) \equiv a(n_i) + \left(\frac{\#n - 1}{2} \right).$$

Clearly $a(n)$ depends only on $\#n \pmod 4$. We now show that, like the Arf invariant, $a(n)$ takes the value zero $2^{g-1}(2^g+1)$ times on $N - M$, and the value one on the remaining $2^{g-1}(2^g-1)$ elements.

Lemma 2.2

$$\sum_{k\equiv 1 \pmod 4} \binom{2g+1}{k} = \begin{cases} 2^{g-1}(2^g+1) & g \equiv 0,1 \pmod 4 \\ 2^{g-1}(2^g-1) & g \equiv 2,3 \pmod 4 \end{cases}$$

Proof. We have

$$\sum_{k \text{odd}} \binom{2g+1}{k} = 2^{2g}$$

If $i^2 = -1$ in $\mathbb{C}$, we also have

$$(1+i)^{2g+1} = (2i)^g(1+i) = 2^g(i^g + i^{g+1}) = \sum \binom{2g+1}{k} i^k$$

Taking the coefficient of i gives the identity

$$\sum_{k\equiv 1 \pmod 4} \binom{2g+1}{k} - \sum_{k\equiv 3 \pmod 4} \binom{2g+1}{k} = \begin{cases} 2^g & g \equiv 0,1 \pmod 4 \\ -2^g & g \equiv 2,3 \pmod 4 \end{cases}$$

Adding this to the first identity in the proof gives the desired formula.

For $n \in N - M$, we define the function $f_n : M \to k$ by the formula $f_n(m) = a(n+m) + a(n)$.

Lemma 2.3. *The function f_n is a quadratic form on M associated to the symplectic form $\langle , \rangle$. The Arf invariant of f_n is $a(n)$.*

Proof. We must show that

$$f_n(m_1 + m_2) + f_n(m_1) + f_n(m_2) = \langle m_1, m_2 \rangle.$$

We first check this for the special case when $n = n_S = \sum n_i$ has $\# n = 2g+1$. In this case, we write f for the function f_n and observe that we have the simple formula

$$f(m) \equiv \frac{\# m}{2} \qquad \pmod 2.$$

Since $\#(m_1 + m_2) = \# m_1 + \# m_2 - 2\#(m_1 \cap m_2)$ with $\#(m_1 \cap m_2) = \sum \alpha_i(m_1)\alpha_i(m_2) \equiv \langle m_1, m_2 \rangle$, we have

$$f(m_1 + m_2) + f(m_1) + f(m_2) = \langle m_1, m_2 \rangle$$

as desired.

In general, $n = n_S + m'$ and $f_n(m) = f(m + m') + f(m')$. We must show that the four-term sum

$$f(m_1 + m_2 + m') + f(m_1 + m') + f(m_2 + m') + f(m')$$

is equal to $\langle m_1, m_2\rangle$. By the above identity for f, this sum is equal to

$$f(m_1+m_2)+\langle m', m_1+m_2\rangle + f(m_1)+\langle m', m_1\rangle + f(m_2)+\langle m', m_2\rangle.$$

Since $\langle m', m_1+m_2\rangle = \langle m', m_1\rangle + \langle m', m_2\rangle$, the sum is equal to $f(m_1+m_2)+f(m_1)+f(m_2)=\langle m_1, m_2\rangle$ as desired.

The Arf invariant of f_n clearly depends only on $\#n$ (mod 4), so it suffices to check that it is correct when $n=n_S$ has $\#n=2g+1$. An argument similar to Lemma 2.2 shows that the function f has $2^{g-1}(2^g+1)$ zeroes on M when $g\equiv 3,4 \pmod 4$ and $2^{g-1}(2^g-1)$ zeroes on M when $g\equiv 1,2 \pmod 4$. Hence the Arf invariant of the form f is equal to $a(n_S)\equiv a(n_i)+g(\text{mod } 2)$.

We now complete the proof of Proposition 2.1. Since M and V are both nondegenerate symplectic spaces of dimension $2g$, there is a linear isomorphism $T: M\to V$ which satisfies $\langle Tm_1, Tm_2\rangle = \langle m_1, m_2\rangle$ for $m_1, m_2\in M$. Via T, we may identify the elements n of $N-M$ with quadratic forms $q=T(n)$ on $V: q(Tm)=f_n(m)$. The fact that q is a quadratic form, with Arf invariant $a(n)\equiv a(n_i)+\binom{\frac{\#n-1}{2}}{}$, follows from Lemma 2.3.

The induced map $T: N\to W$ is a linear isomorphism, and the images $q_i = T(n_i)$ give an Aronhold set S in W. The transitivity of $Sp(V)$ on Aronhold sets, and the stability subgroup follow similarly. This completes the proof of Proposition 2.1.

We give some examples of Aronhold sets S for small values of g. When $g=1$, there are 3 even forms q on V. The group $Sp(V)$ is isomorphic to S_3 via its permutation representation on the set $S=\{q_1,q_2,q_3\}$ of even forms. This is the unique Aronhold set in W, and $q_S=q_1+q_2+q_3$ is the unique odd form on V.

When $g=2$ there are 6 odd forms q on V, and the group $Sp(V)$ is isomorphic to S_6 via its permutation representation on the set of odd forms. The Aronhold sets $S=\{q_1,q_2,\ldots,q_5\}$ are the 5-subsets of the set of odd forms, and q_S is the unique odd form not in S. The group $O(V,q_S)$ is isomorphic to the symmetric group $\mathrm{Sym}(S)=S_5$.

When $g=3$ there are 28 odd forms q on V. An Aronhold set $S=\{q_1,q_2,\ldots,q_7\}$ consists of 7 odd forms which give a basis for $W=V\cup QV$, such that

$$q_i+q_j+q_k \quad \text{is even} \quad i\neq j\neq k$$

$$q_i+q_j+q_k+q_\ell+q_m \quad \text{is odd} \quad i\neq j\neq k\neq \ell\neq m$$

$$q_S=\sum_{i=1}^{7} q_i \qquad \text{is even.}$$

The following simpler criterion is often useful.

Proposition 2.4. *Let $S=\{q_1,q_2,\ldots,q_7\}$ be a set of seven distinct odd forms on V, such that $q_i+q_j+q_k$ is even for all $i\neq j\neq k$. Then S is an Aronhold set in W.*

Proof. The hypothesis implies that the 21 vectors $v_{ij}=q_i+q_j$ are nonzero and distinct in V. For if $v_{ij}=v_{k,\ell}$, then $q_i+q_j+q_k+q_\ell=0$ in W, and the form $q_i+q_j+q_k=q_\ell$ would be odd.

If $k \neq i, j$ then $q_k(v_{ij}) = a(q_k) + a(q_i + q_j + q_k) = 1$. On the other hand, $q_i(v_{ij}) = q_j(v_{ij}) = a(q_i) + a(q_j) = 0$. Hence $q_S = \sum_{i=1}^{7} q_i$ takes the value 1 on the 21 vectors v_{ij}, and is not equal to any of the q_i. It follows that the vectors $\{q_i\}$ are linearly independent in W, so give a basis.

The forms $q_i + q_j + q_k$ $i \neq j \neq k$ are hence all distinct, and give $\binom{7}{3} = 35$ of the 36 even forms on V. It therefore suffices to prove that q_S is even.

Of the 21 forms $q_i + q_j + q_k + q_\ell + q_m$ with $i \neq j \neq k \neq \ell \neq m$, at least 20 must be odd. Assume $r = q_1 + q_2 + q_3 + q_4 + q_5$ is odd. Since $q_S = r + v_{67}$, $a(q_S) = a(r) + q_S(v_{67}) = 1 + 1 = 0$ as desired.

Corollary 2.5. *The 21 vectors $v_{ij} = q_i + q_j$ in V determine the Aronhold set $S = \{q_i\}$ in W.*

Proof. Indeed, there is a *unique* even form q $(= q_S)$ which takes the value 1 on the v_{ij}. Let $\{v_1, \ldots, v_7\}$ be the remaining vectors where $q(v_i) = 1$, and let $q_i = q + v_i$. These are the 7 forms of S.

The group $S_7 = \mathrm{Sym}(S)$ has index eight in the group $O(V, q_S)$, so for each even form q there are precisely 8 Aronhold sets S which satisfy $q_S = q$. (This gives $288 = 36.8$ Aronhold sets S in all.) The action of $O(V, q_S)$ on these 8 sets S gives an isomorphism $O(V, q_S) \simeq S_8$. Finally, each odd form q_1 appears in precisely two Aronhold sets S with $q_S = q$, so in 72 Aronhold sets in all.

If $\langle e_1, e_2, e_3; f_1, f_2, f_3 \rangle$ is a symplectic basis for V such that the even form q is given by

$$q\left(\sum \alpha_i e_i + \sum \beta_i f_i\right) = \sum \alpha_i \beta_i$$

then an explicit Aronhold set S with $q_S = q$ is given by

$$
\begin{aligned}
q_1 &= q + e_1 + f_1 + f_2\\
q_2 &= q + e_2 + f_2\\
q_3 &= q + e_3 + f_1 + f_2 + f_3\\
q_4 &= q + e_1 + e_2 + f_1 + f_3\\
q_5 &= q + e_1 + e_3 + f_2 + f_3\\
q_6 &= q + e_2 + e_3 + f_3\\
q_7 &= q + e_1 + e_2 + e_3 + f_1.
\end{aligned}
$$

Note. We have chosen the name Aronhold set for S in view of Aronhold's work on the 28 bitangents to a smooth plane quartic (the notion of an Aronhold set appears in Coble's book [C2, §26] as a "normal fundamental set"). The relationship between this work and the theory of quadratic forms on V when $g = 3$ will be explained in §4.

We define an *Aronhold basis* of W to be an ordered Aronhold set $\langle q_1, q_2, \ldots, q_{2g+1}\rangle$ in QV. The group $Sp(V)$ acts simply-transitively on Aronhold bases of W, and on symplectic bases $\langle e_1, \ldots, e_g; f_1, \ldots, f_g\rangle$ of V. We may identify these principal homogeneous spaces for $Sp(V)$ by associating to each Aronhold basis the symplectic basis:

$$
\begin{array}{ll}
e_1 = q_1 + q_2 & f_1 = q_1 + q_{2g+1}\\[1em]
e_2 = q_3 + q_4 & f_2 = q_1 + q_2 + q_3 + q_{2g+1}\\[1em]
\vdots & \vdots\\[1em]
e_g = q_{2g-1} + q_{2g} & f_g = q_1 + q_2 + \cdots + q_{2g-1} + q_{2g+1}
\end{array}
$$

3. A Bipartite Graph

Many of the combinatorial questions involving quadratic forms on V can be studied by a consideration of a certain bipartite graph Γ. The two sets of vertices of Γ correspond respectively to the elements of V and QV. The two vertices v and q are joined by an edge if $a(q+v)=1$.

The group $Sp(V) \propto W$ acts as automorphism of Γ. The subgroup W permutes the vertices simply-transitively, and $Sp(V)$ preserves the vertex $v=0$. The subgroup $O(V,q)$ of $Sp(V)$ preserves the two vertices $v=0$ and q.

Let q be a fixed *even* quadratic form, and let $\sigma = \sigma_q$ be the involution of Γ given by translation by q. Since w is not connected to $w+q$, σ fixes no edge of Γ, and the quotient graph $\Delta = \Gamma/\langle\sigma\rangle$ has no loops.

The combinatorial graph Δ has 2^{2g} vertices, indexed by $v \in V$. The vertices v and u are connected by an edge if $a(q+v+u)=1$. Therefore, Δ is regular with valency

$$2^{g-1}(2^g-1) = \#\{v : q(v)=1\}.$$

The group $O(V,q) \propto V$ acts as automorphism of the quotient graph Δ. The subgroup V permutes the vertices simply-transitively, and $O(V,q)$ preserves the vertex $v=0$. It acts on the complete subgraph $\Delta(0)$, which is, by definition, the induced graph on the star of $v=0$ in Δ.

We give some examples of Δ and $\Delta(0)$ for small g. When $g=1$, Δ is the graph

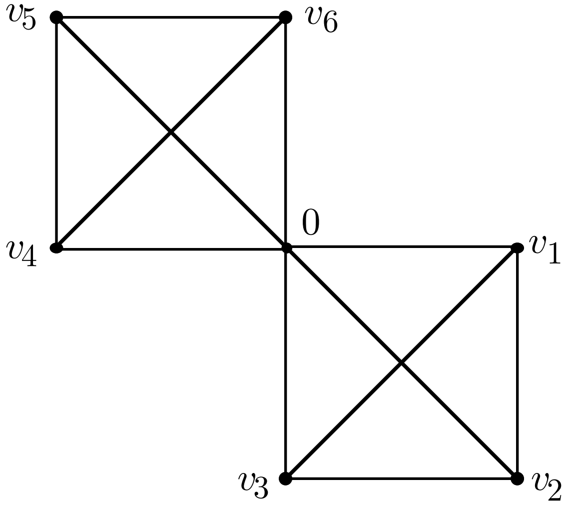
where v is the unique vector in V with $q(v)=1$. In this case, $O(V,q)$ is the group S_2 of order 2. In this case, $O(V,q)$ is the group S_2 of order 2.

When $g=2$, Δ has 16 vertices and 48 edges. There is a unique partition $\{v_1,v_2,v_3\}$, $\{v_4,v_5,v_6\}$ of the six vectors v with $q(v)=1$ into two 3-element subsets, such that each element in the first 3-subset is orthogonal to each element in the second 3-subset. Indeed, if $\langle e_1,e_2;f_1,f_2\rangle$ is a symplectic basis for V and $q(\alpha_1e_1+\alpha_2e_2+\beta_1f_1+\beta_2f_2)=\alpha_1\beta_1+\alpha_2\beta_2$, we have the partition

$$\begin{cases} v_1 = e_1+f_1 \\ v_2 = e_2+f_2+e_1 \\ v_3 = e_2+f_2+f_1 \end{cases}$$

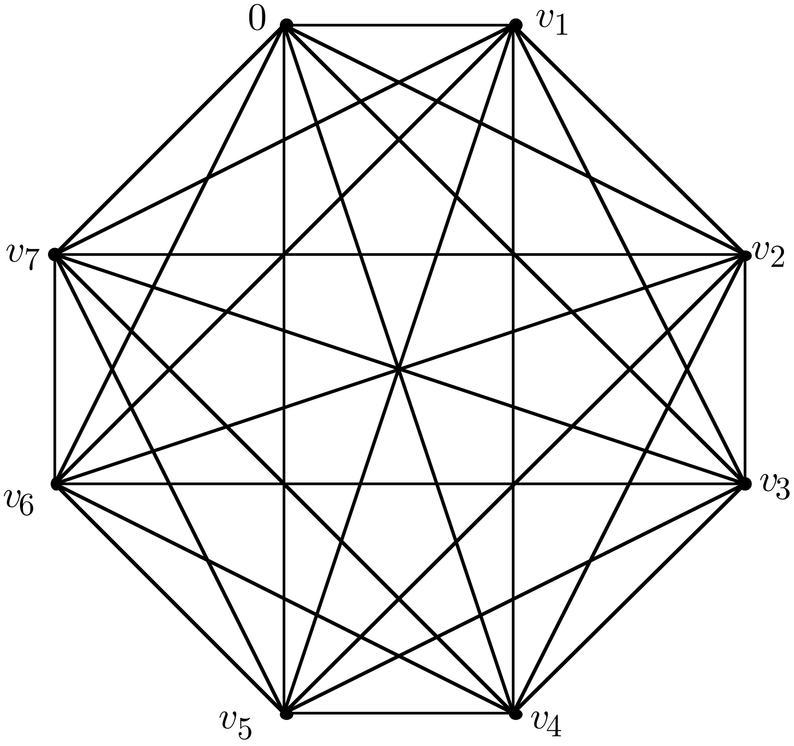
$$\begin{cases} v_4 = e_2+f_2 \\ v_5 = e_1+f_1+e_2 \\ v_6 = e_1+f_1+f_2 \end{cases}$$

The subgraph $\Delta(0)$ is given by



and the group $O(V,q)$ is isomorphic to the subgroup $(S_3 \times S_3) \cdot 2$ of $S_6 = Sp(V)$.

When $g = 3$ the graph Δ has $64 = 2^6$ vertices and $896 = 2^7 \cdot 7$ edges. There are 28 vectors v with $q(v) = 1$. Let $S = \{q_1, q_2, \ldots, q_7\}$ be an Aronhold set with $q = q_S = \sum q_i$, and write $q_i = q + v_i$. Since $q(v_i) = a(q_i) = 1$, the seven vertices v_i lie in $\Delta(0)$. Since $q(v_i + v_j) = a\left(\sum_{k \neq i,j} q_k\right) = 1$, the vertices v_i and v_j are connected by an edge in $\Delta(0)$. Hence $\Delta(0)$ contains the complete graph Δ_S on the eight vertices $\{0, v_1, v_0, \ldots, v_7\}$:



There are 8 choices for S with $q_S = q$. If $S' \neq S$, the complete graphs Δ_S and $\Delta_{S'}$ meet along a single edge with vertex $v = 0$ in $\Delta(0)$. Of the 28 vertices $v \neq 0$ in $\Delta(0)$, each is connected to 12 vertices $w \neq 0$. The group $O(V, q) = S_8$ acts on $\Delta(0)$; the subgroup S_7 stabilizing Δ_S permutes the 7 vertices $\{v_1, v_2, \ldots, v_7\}$, as well as the 7 remaining complete graphs $\Delta_{S'}$.

4. Theta Characteristics

Following Mumford [M], we recall how the theory of theta characteristics is linked to the theory of quadratic forms over $\mathbb{Z}/2\mathbb{Z}$.

Let C be a complete, nonsingular algebraic curve of genus g, defined over an algebraically closed field of characteristic $\neq 2$. Let $\operatorname{Pic}(C)$ be the group of divisor classes on C, and $\operatorname{Pic}^n(C)$ the divisor classes of degree n. If $d \in \operatorname{Pic}(C)$, we write $\mathcal{L}(d)$ for the corresponding line bundle on C, and $h^0(d)$ for the dimension of the space of sections $H^0(C, \mathcal{L}(d))$.

Let

$$V = \{v \in \operatorname{Pic}^0(C) : 2v = 0\} \tag{4.1}$$

be the classes killed by 2. This is a vector space of dimension $2g$ over $k = \mathbb{Z}/2\mathbb{Z}$. It has a nondegenerate symplectic form defined by the Weil pairing. If d and e are divisors with disjoint support in the classes of v and u, and $2d = \operatorname{div}(f)$, $2e = \operatorname{div}(g)$, then

$$(-1)^{\langle v,u\rangle} = f(e)/g(d).$$

Let κ be the canonical class in $\operatorname{Pic}^{2g-2}(C)$. The set of theta characteristics

$$QV = \{q \in \operatorname{Pic}^{g-1}(C) : 2q = \kappa\} \tag{4.2}$$

is a principal homogeneous space for V which can be identified with the set of quadratic forms on V. The form q is given by the formula

$$q(v) \equiv h^0(q+v) + h^0(q) \pmod 2. \tag{3.3}$$

The Arf invariant of q is given by

$$a(q) \equiv h^0(q) \pmod 2. \tag{3.4}$$

The vector space $W = V \cup QV$ appears as a subgroup of order 2^{2g+1} in $\operatorname{Pic}(C)/\mathbb{Z}\kappa$, when $g \neq 1$.

We consider some examples in low genus. When $g = 0$, there is a single (even) characteristic q. In this case $C \simeq \mathbb{P}^1$, $\mathcal{L}(q) \simeq \mathcal{O}(-1)$, and $h^0(q) = 0$.

When $g = 1$, C is an elliptic curve and $\kappa = 0$. In this case, $QV = V$. There are three even characteristics $q = v \neq 0$ with $h^0(q) = 0$, and one odd characteristic with $h^0(q) = 1$.

When $g = 2$, the canonical series $|\kappa|$ gives a two-fold covering $C \to \mathbb{P}^1$. Let τ be the hyperelliptic involution of C and let $\{p_1, p_2, \ldots, p_6\}$ be the fixed points of τ. The six odd characteristics q correspond to the line bundles $\mathcal{L}(p_i)$ and satisfy $h^0(q) = 1$. The 10 even characteristics q correspond to the line bundles $\mathcal{L}(p_i + p_j - p_k)$, where $\{p_i, p_j, p_k\}$ is a 3-subset of the 6 fixed points, well-defined up to complementation. They satisfy $h^0(q) = 0$.

When $g = 3$, the canonical series $|\kappa|$ gives a morphism $\pi : C \to \mathbb{P}^2$. If C is hyperelliptic, this map is 2-to-1 onto a smooth conic $D \subset \mathbb{P}^2$. In this case, there is a distinguished even theta characteristic q_C with $h^0(q_C) = 2$; we have $\mathcal{L}(q_C) = \pi^*\mathcal{O}_D(1)$. Let τ be the hyperelliptic involution of C, and let $\{p_1, p_2, \dots, p_8\}$ be its 8 fixed points. The 28 odd theta characteristics q correspond to the line bundles $\mathcal{L}(p_i + p_j)$, and satisfy $h^0(q) = 1$.

If C is not hyperelliptic, the canonical series embeds C as a smooth quartic in $\mathbb{P}^2$. The 28 odd theta characteristics q correspond to the 28 bitangent lines to this quartic; the corresponding line bundles are $\mathcal{L}(p + r)$ where p and r are the two points of double tangency.

Classically, an Aronhold set S consisted of 7 bitangents $\{(p_1, r_1), \dots, (p_7, r_7)\}$ on $C \subseteq \mathbb{P}^2$, with the property that no 6 points of the form $\{p_i, r_i, p_j, r_j, p_k, r_k\}$ $i \neq j \neq k$ were on the intersection of C with a conic. Equivalently, S is a collection of 7 odd theta characteristics $\{q_1, \dots, q_7\}$ such that $q_i + q_j + q_k - \kappa$ is an even characteristics, for $i \neq j \neq k$. The remaining 21 odd characteristics have the form $q_i + q_j + q_k + q_\ell + q_m - 2\kappa$, so the remaining 21 bitangents to C can be obtained as the residual intersection of C with a cubic passing through 10 points of the form $\{p_i, r_i, p_j, r_j, p_k, r_k, p_\ell, r_\ell, p_m, r_m\}$.

The Aronhold set $S = \{q_1, \dots, q_7\}$ determines an even characteristic $q_s = q_1 + q_2 + \cdots + q_7 - 3\kappa$. Since C is not hyperelliptic, $h^0(q_S) = 0$. The linear series $|\kappa + q_S|$ embeds C as a nonsingular sextic curve in $\mathbb{P}^3$, and Hesse showed that the 28 lines $\overline{pr}$ in $\mathbb{P}^3$ given by the odd theta characteristics meet in 8 distinct points $Y = \{y_1, y_2, \dots, y_8\}$. The scheme Y is the base locus of a net of quadrics $\{Q_\lambda\}_{\lambda \in \mathbb{P}^2}$ in $\mathbb{P}^3$, whose discriminant locus defines the smooth quartic curve C in $\mathbb{P}^2$:

$$C = \{\lambda \in \mathbb{P}^2 : \det Q_\lambda = 0\}.$$

The lines $\overline{pr}$ in $\mathbb{P}^3$ give the complete graph on the 8 points of $Y = \bigcap\limits_\lambda Q_\lambda$, and the Aronhold set S is a complete fan from one of the vertices. We will recover and extend these results in §6.

5. Curves of Genus 2

Let C be a nonsingular, complete curve of genus 2 defined over an algebraically closed field of characteristic $\neq 2$. Let $A = \mathrm{Pic}^0(C)$ be the Jacobian of C, and let κ be the canonical class in $\mathrm{Pic}^2(C)$. We have the symplectic space $V = \{v \in A : 2v = 0\}$ of dimension 4 over $\mathbb{Z}/2\mathbb{Z}$, and identify QV with the set of 16 theta characteristics $\{q \in \mathrm{Pic}^1(C) : 2q = \kappa\}$ as in §4.

In this section, we will show how C gives rise to a $K3$-surface X, together with a very ample line bundle $\mathcal{L}$ of degree 8 on X. We will also see how to associate to each even theta characteristic q on C an Enriques involution $\sigma = \sigma_q$ of X fixing $\mathcal{L}$, and will study a configuration of 16 rational curves on the quotient Enriques surface $Y = X/\langle\sigma_q\rangle$, whose dual graph is the combinatorial graph Δ described in §3. The subgraph $\Delta(0)$ will be used to produce an elliptic fibration $f : Y \to \mathbb{P}^1$ with two double fibres, each of Kodaira type I_3.

Our treatment follows [GH, Ch. 6]. Let τ be the hyperelliptic involution of C, which induces the automorphism -1 on A. Besides the 16 points of V on A, which are fixed by -1, we have the 16 curves C_q of genus 2 corresponding to classes in QV.

$$C_q = \{(p) - q : p \in C\}. \tag{5.1}$$

The involution -1 of A fixes C_q, and induces its hyperelliptic involution. If $q \neq q'$, then C_q and $C_{q'}$ meet in 2 points of V, and the point $v \in V$ lies on the curve C_q if $a(q+v) = 1$.

The class of the divisor $E = 2 \cdot (C_q)$ in $\mathrm{Pic}(A)$ is independent of the choice of $q \in QV$; this is the linear series usually denoted $|2\Theta|$. The line bundle $\mathcal{L}(E)$ is ample, and satisfies $\mathcal{L}(E)^2 = 8$, $h^0(\mathcal{L}(E)) = 4$. The sections of $\mathcal{L}(E)$ are all fixed by -1, and give a projective embedding of the Kummer surface $K = A/\langle -1\rangle$.

$$K \hookrightarrow \mathbb{P}^3 \tag{5.2}$$

where the image is a hypersurface of degree 4. This quartic has 16 ordinary double points, at the images of elements in V. The curves C_q on A give 16 rational curves $C_q/\langle\tau_q\rangle$ on K, which map to conics in the 16 reducible hyperplane sections.

Let X be the blow-up of K at the 16 double points V. This is abstractly a $K3$ surface, with 32 obvious rational curves

$$\text{(5.3)}\qquad \begin{array}{c} \text{the 16 proper transforms } D_q \text{ of the curves } C_q/\langle\tau_q\rangle \\ \\ \text{the 16 exceptional divisors } D_V \end{array}$$

These all satisfy $D^2 = -2$, and the dual graph of the configuration is the bipartite graph Γ:

$$D_v \cdot D_q = \begin{cases} 1 & \text{if } a(q+v) = 1 \\ 0 & \text{if } a(q+v) = 0 \end{cases} \tag{5.4}$$

Next, we show that the finite group $W = V \cup QV \cong (\mathbb{Z}/2\mathbb{Z})^5$ acts as automorphisms of X, and permutes the 32 curves D_w simply-transitively. The involutions σ_v associated to points $v \neq 0$ in V are induced by the translations $a \mapsto a + v$ on A. They have 8 fixed points on X—the images of the 16 points a of order 4 on A which satisfy $2a = v$.

The involutions σ_q associated to forms $q \in QV$ are more subtle to define, as they act only on X (not on A or K). We specify σ_q by insisting that it permutes the curves D_w in the obvious manner: $\sigma_q(D_w) = D_{w+q}$. This gives an involution of the subgroup of $NS(X)$ spanned by the curves D_w, which is free of rank 17, and extends uniquely to an involution of the Hodge structure on $H^2(X)$, acting trivially on the image of $H^2(A)$. By the Torelli theorem for K3 surfaces, σ_q arises from a unique automorphism of X.

Let B be the blow-up of A at the 16 points $v \in V$. Then -1 lifts to an involution of B, and $B/\langle -1\rangle = X$. The branch divisor of the 2-fold cover $B \to X$ is equal to $\sum_v (D_v)$, so this class is divisible by 2 in $\mathrm{Pic}(X) = NS(X)$. Since $\mathrm{Pic}(X)$ is torsion-free, the class $\frac{1}{2}\sum(D_v)$ is well-defined in $\mathrm{Pic}(X)$.

For $q \in QV$, define the class

$$H_q = 4(D_q) + 2\sum_{a(q+v)=1}(D_v) - \frac{1}{2}\sum_v (D_v) \tag{5.5}$$

in $\mathrm{Pic}(X)$. Then

$$H_q \cdot D_w = 1 \qquad w \in W = V \cup QV. \tag{5.6}$$

Hence $H = H_q$ is independent of the choice of q, and fixed under the action of W on $\mathrm{Pic}(X)$. The associated line bundle $\mathcal{L} = \mathcal{L}(H)$ satisfies

$$\mathcal{L} \cdot \mathcal{L} = 8, \qquad h^0(\mathcal{L}) = 6. \tag{5.7}$$

This is the very ample class in $\mathrm{Pic}(X)$ determined by C.

The sections of $\mathcal{L}$ give a projective embedding

$$X \hookrightarrow \mathbb{P}^5 \tag{5.8}$$

where the image has degree 8. The 32 rational curves D_w are mapped to lines in $\mathbb{P}^5$, which lie on the 80 reducible hyperplane sections (each of which contains 4 lines D_v and 4 lines D_q). [GH, Ch. 6].

To understand the projective representation of W on $\mathbb{P}^5 = \mathbb{P}(H^0(X, \mathcal{L}))$, we introduce a central extension U of W which acts linearly on $H^0(X, \mathcal{L})$. Let U be a 6-dimensional vector space over $k = \mathbb{Z}/2\mathbb{Z}$, with basis $\langle u_1, u_2, \ldots, u_6\rangle$, and let $\langle q_1, q_2, \ldots, q_6\rangle$ be the odd quadratic forms on V. We have an exact sequence of $Sp(V) = S_6$–modules

$$\begin{gathered} 0 \to k \to U \to W \to 0 \\ u_i \mapsto q_i \\ 1 \mapsto \sum u_i. \end{gathered} \tag{5.9}$$

The projective representation of W lifts uniquely to a linear representation of U on $H^0(X, \mathcal{L})$, such that the space of sections decomposes as the direct sum of the six lines

$L_i = \{s \in H^0(X, \mathcal{L}) : u_i(s) = -s\}$. Using this decomposition, we obtain eigencoordinates $\langle x_1, x_2, \ldots, x_6\rangle$ on $\mathbb{P}^5$ such that X appears as the intersection of 3-diagonal quadrics [GH, 768–769]:

$$\sum \alpha_i x_i^2 = \sum \beta_i x_i^2 = \sum \gamma_i x_i^2 = 0.$$

If $q = q_i$ is odd, σ_q fixes a hyperplane section of X, which is a canonical curve of genus 5. If q is even, σ_q is given by the action of $u_i \cdot u_j \cdot u_k$, where $\{q_i, q_j, q_k\}$ is the 3-subset of the odd forms well-determined by q up to complementation. In this case, σ_q is fixed-point free on X.

We summarize the results obtained so far.

Proposition 5.10. *Associated to a curve C of genus 2 is a $K3$ surface X, together with a very ample line bundle $\mathcal{L}$ on X which satisfies $\mathcal{L} \cdot \mathcal{L} = 8$.*

The surface X contains 32 rational curves D_w corresponding to elements $w \in W$. The dual graph of the configuration of these curves is the bipartite graph Γ. The group W acts as automorphisms of X, fixes the class $\mathcal{L}$ in $\operatorname{Pic}(X)$, and permutes the curves D_w simply-transitively.

The sections of $\mathcal{L}$ give a projective embedding $X \hookrightarrow \mathbb{P}^5$ in which W acts as the group $\langle \pm 1\rangle^6 / \langle -1, -1, \ldots, -1\rangle$ of sign changes on the 6 coordinates. The surface X is the complete intersection of 3 diagonal quadrics, and the curves D_w are mapped to lines in $\mathbb{P}^5$.

If q is an even theta characteristic on C, the involution σ_q involves 3 sign changes on $\mathbb{P}^5$ and is fixed-point free on X.

Now let q be a fixed even characteristic on C, and write $\sigma = \sigma_q$ for the corresponding Enriques involution of X. The quotient $Y = X/\langle\sigma\rangle$ is an Enriques surface, with 16 rational curves D_v (the image of either D_v or D_{v+q}) indexed by $v \in V$. The group V acts as automorphisms of Y and permutes the curves D_v simply-transitively. The dual graph of the configuration of these curves is the graph $\Delta = \Gamma/\langle\sigma\rangle$ studied in §3.

$$\begin{cases} \qquad D_v^2 = 2 \\ D_v \cdot D_u = \begin{cases} 1 & \text{if} \quad a(q+v+u) = 1 \\ 0 & \text{if} \quad a(q+v+u) = 0 \end{cases} \end{cases}$$

Let $T = \{q_1, q_2, q_3\}$ and $T' = \{q_4, q_5, q_6\}$ be the two 3-subsets of odd characteristics determined by q. We have $q_1 + q_2 + q_3 = q_4 + q_5 + q_6 = q$ in W. The embedding $X \hookrightarrow \mathbb{P}^5 = \mathbb{P}(V)$ gives rise to two coverings:

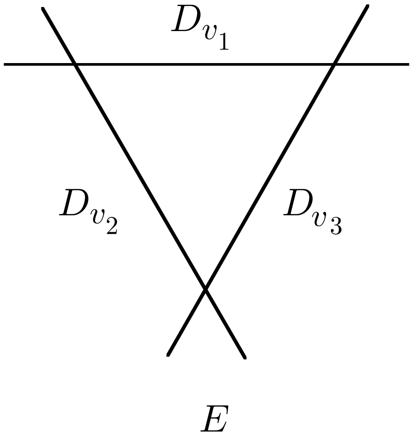
$$\pi_T : Y \to \mathbb{P}^2 = \mathbb{P}(V_{123})$$
$$\pi_{T'} : Y \to \mathbb{P}^2 = \mathbb{P}(V_{456})$$

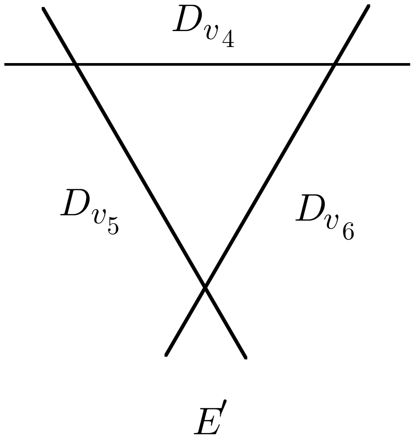
Each has degree 4, and the line bundle $\pi_T^* \mathcal{O}(1) \otimes \pi_{T'}^* \mathcal{O}(1)^{-1}$ is isomorphic to the canonical bundle Ω_Y^2 of Y.

Write $q_i = q + v_i$, and consider the two divisors

$$E = D_{v_1} + D_{v_2} + D_{v_3}$$
$$E' = D_{v_4} + D_{v_5} + D_{v_6}$$

on Y. By our picture of $\Delta(0)$ in §3, these give disjoint configurations of -2 curves on Y:





Hence $E^2 = (E')^2 = E \cdot E'$. Again, the difference $(E) - (E')$ represents the canonical class K_Y. Since this class has order 2 in $\mathrm{Pic}(Y)$ there is a function $f : Y \to \mathbb{P}^1$ with $\mathrm{div}(f) = 2(E) - 2(E')$. The map f is an elliptic fibration of Y, with 2 double fibres $2E$ and $2E'$. (Both E and E' are generalized elliptic curves, of Kodaira type I_3). Of the remaining curves D_v on Y, the nine curves with $v \neq 0$ are bisections of this fibration, and the curve D_0 is a six-fold section.

6. Curves of Genus 3

In this section C is a nonsingular, complete curve of genus 3 over an algebraically closed field of characteristic $\neq 2$, which is not hyperelliptic. Let q be an even theta characteristic on C, and let $K = 2q$ be the canonical class. We study the sesquicanonical embedding of C into $\mathbb{P}^3$ given by the very ample linear series $|3q| = |K + q|$ of degree 6.

To understand the geometry of the image B of C in $\mathbb{P}^3$, which is a sextic space curve, we have to arrive at it from another point of view, that of symmetric determinantal representations of the plane quartic.

To start with, suppose we are given a *net of quadrics* in $\mathbb{P}^3$, which we define to be an inclusion

$$\varphi : V \hookrightarrow \mathrm{Sym}^2 W^*$$

of a three-dimensional vector space V into the space of symmetric bilinear forms on a four-dimensional vector space W, up to the action of $\mathrm{Aut}(V)$ and $\mathrm{Aut}(W)$ on the left and right. We can immediately associate to such an object the locus

$$C = \{[v] \in \mathbb{P}V : \varphi(v) \text{ is singular}\} \subset \mathbb{P}V \cong \mathbb{P}^2.$$

Given that this is the zero locus of the determinant of a symmetric 4×4 matrix of linear forms on $\mathbb{P}V$, we expect it to be a quartic curve; we will call the net *typical* if it is a smooth plane quartic.

It is natural to ask whether every plane quartic curve C may be realized in this fashion, and if in turn C determines the net. The answers to these two questions are, respectively, "yes" and "not quite". In fact, we need only a little extra structure beyond the specification of C to specify the net; and once we have said what that is, we will readily prove that we have a one-to-one correspondence between nets and curves with this structure.

Now, suppose our net is typical. It follows then that the rank of $\varphi(v)$ is never less than 3: the variety in $\mathrm{Sym}^2 W^*$ of quadrics of rank 3 or less is singular along the locus of quadrics of rank strictly less than 3. From this we see that we can further associate to our net a line bundle on C: the line bundle M whose fiber at any point $p = [v]$ on C is the kernel of $\varphi(v)$, viewed as a map from W to W^*. In a similar way, we can associate another line bundle N on C, whose fiber at p is the cokernel of $\varphi(v)$.

To be more precise, our net φ may be viewed as giving a morphism of locally free sheaves on $\mathbb{P}V$:

$$\psi : \mathcal{O}_{\mathbb{P}V} \otimes W \to \mathcal{O}_{\mathbb{P}V}(1) \otimes W^*.$$

The cokernel of this morphism of sheaves on $\mathbb{P}V$ is supported on C; in fact, it is just the line bundle N. We thus have an exact sequence of sheaves on $\mathbb{P}^2$:

$$0 \to \mathcal{O}_{\mathbb{P}V} \otimes W \to \mathcal{O}_{\mathbb{P}V}(1) \otimes W^* \to N \to 0. \tag{6.1}$$

When we restrict to C—that is, tensor with $\mathcal{O}_C$—this does not remain exact; rather we get a four-term exact sequence

$$0 \to M \to \mathcal{O}_C \otimes W \to \mathcal{O}_C(1) \otimes W^* \to N \to 0. \tag{6.2}$$

From this we see in particular that

$$c_1(N) - c_1(M) = c_1(\mathcal{O}_C(1) \otimes W^*),$$

i.e.,

$$N \otimes M^* \cong \mathcal{O}_C(4).$$

At the same time, the symmetry of the map $\tilde{\varphi}$ gives another relation between N and M. Dualizing the four-term sequence of locally-free sheaves on C, we get a sequence

$$0 \to N^* \to \mathcal{O}_C(-1) \otimes W \to \mathcal{O}_C \otimes W^* \to M^* \to 0$$

where the map in the middle is simply the transpose of $\tilde{\varphi}$. But this is simply the map $\tilde{\varphi}$ again, tensored with (the identity map on) $\mathcal{O}_C(-1)$; in other words, tensoring this with $\mathcal{O}_C(1)$ we have the same sequence as before, and thus deduce that

$$M^*(1) \cong N \qquad \text{and (equivalently)} \quad N^*(1) \cong M.$$

Combining this with our earlier relation, we see that

$$N \otimes N \cong N \otimes M^*(1) \cong \mathcal{O}_C(5)$$

and correspondingly

$$M^* \otimes M^* \cong \mathcal{O}_C(3),$$

in other words, the line bundle $L = M^*$ is sesquicanonical, and can be written as

$$L \cong K_C \otimes \Theta$$

for some theta-characteristic Θ on C. Moreover, tensoring the exact sequence (1) above with $\mathcal{O}_{\mathbb{P}V}(2)$ we arrive at

$$0 \to \mathcal{O}_{\mathbb{P}V}(-2) \otimes W \to \mathcal{O}_{\mathbb{P}V}(-1) \otimes W^* \to N(-2) \to 0$$

and we may deduce, given that the left-hand term has no cohomology whatsoever, that

$$h^0(C, \Theta) = h^0(C, N(-2)) = 0,$$

i.e., that Θ is an even theta-characteristic. We have thus described a map

$$\left\{\begin{matrix}\text{nets of quadrics in} \\ \mathbb{P}^3\end{matrix}\right\} \xrightarrow{\alpha} \left\{\begin{matrix}\text{smooth nonhyperelliptic curves } C \\ \text{of genus 3 with even theta-characteristics } \Theta\end{matrix}\right\}$$

Lemma 6.3. *The map α is a bijection.*

Proof. We have to exhibit an inverse, that is, associate to a pair (C, Θ) a net of quadrics.

We will give a relatively concrete approach. We start with an abstract curve C and an even theta-characteristic Θ on C; by way of notation, we will denote by C again its canonical image in $\mathbb{P}V \cong \mathbb{P}^2$, and by B the image of C in $\mathbb{P}W \cong \mathbb{P}^3$ under the map $\varphi = \varphi_L$ associated to the line bundle $L = K_C \otimes \Theta$ on C.

Choose a basis $\sigma_1, \ldots, \sigma_4$ for $H^0(C, L)$. For every pair i, j the product $\sigma_i \sigma_j$ will be a section of

$$2 \cdot L = 2(K_C + \Theta) = 3 \cdot K_C = \mathcal{O}_C(3),$$

and so will be the restriction to C of a unique cubic polynomial $F_{ij}(X)$ on $\mathbb{P}^2$ (viewed as a section of $\mathcal{O}_{\mathbb{P}^2}(3)$).

Now, consider the (symmetric) 4×4 matrix

$$\Phi = \begin{pmatrix} F_{1,1} & F_{1,2} & F_{1,3} & F_{1,4} \\ F_{2,1} & F_{2,2} & F_{2,3} & F_{2,4} \\ F_{3,1} & F_{3,2} & F_{3,3} & F_{3,4} \\ F_{4,1} & F_{4,2} & F_{4,3} & F_{4,4} \end{pmatrix}$$

On C, this matrix has rank one—that is, every 2×2 minor vanishes on C; every 3×3 minor vanishes to order 2 on C and the determinant of Φ vanishes to order 3. In fact, since the determinant of Φ is a homogeneous polynomial of degree 12 it follows from the last that it must be (up to scalars) simply the cube of the quartic polynomial $G(X)$ defining C.

Now let Ψ_0 be the matrix of cofactors of Φ . By what we have just said, every entry of Ψ is divisible by $G(X)^2$; set

$$\Psi = \frac{\Psi_0}{G(X)^2}.$$

Ψ is then a symmetric matrix of linear forms on $\mathbb{P}^2$. Now, since

$$\Phi \cdot \Psi_0 = \det(\Phi) \cdot I$$

we have

$$\det(\Phi) \cdot \det(\Psi_0) = G(X)^3 \cdot \det(\Psi_0) = G(X)^{12},$$

that is, the determinant of Ψ_0 is $G(X)^9$ and hence

$$\det(\Psi) = G(X).$$

We thus arrive at a symmetric 4×4 matrix Ψ of linear forms on $\mathbb{P}^2$—that is, a net of quadrics in $\mathbb{P}^3$—whose discriminant curve is C. To complete the proof of the Lemma, then it remains to see that the theta characteristic on C associated to this net is indeed Θ.

To do this, we go back initially to the matrix Φ. The restriction of this matrix to C we may view as a map of vector bundles

$$\omega : \mathcal{O}_C(-3)^{\oplus 4} \to \mathcal{O}_C{}^{\oplus 4}$$

having rank 1 everywhere. Moreover, at each point $p \in C$ the image of this map is simply the one-dimensional subspace of $\mathbb{C}^4$ corresponding to the point $p \in B$, thus the image of φ is the line bundle L^{-1}.

In the same way, we may view the matrix Ψ of cofactors (after dividing out by $G(X)^2$) as a vector bundle map

$$\psi : \mathcal{O}_C{}^{\oplus 4} \to \mathcal{O}_C(1)^{\oplus 4}$$

having rank 3 everywhere on C; and the kernle of this map, at each point $p \in C$, will be simply the subspace of $\mathbb{C}^4$ spanned by any of the rows of Φ, that is to say, again the one-dimensional subspace corresponding to the point p on B. Thus

$$\operatorname{Ker}(\psi) = L^{-1} = -K_C - \Theta$$

and so the theta characteristic associated to our net is indeed Θ.

Now suppose we are given a smooth non-hyperelliptic curve C of genus 3 and an even theta characteristic Θ on C. Let $V = H^0(C, K_C)^*$ and $W = H^0(C, K_C \otimes \Theta)^*$, and let $\varphi : V \to \operatorname{Sym}^2 W^*$ be the associated net of quadrics. By way of notation, we will denote by C again its canonical image in $\mathbb{P}V \cong \mathbb{P}^2$, and by B its image in $\mathbb{P}W \cong \mathbb{P}^3$.

The association to the pair (C, Θ) of the net of quadrics does two things. First of all, it allows us to realize the curve $B \subset \mathbb{P}W$ directly: B is simply the locus of singular points of the singular quadrics Q_P in the net; as we will see, this description will be instrumental in describing the geometry of B. Secondly, we see that the ambient space $\mathbb{P}W = \mathbb{P}^3$ of the curve B carries additional structure: in particular, it contains 8 distinguished points, the base points of the net.

Lemma 6.4. *A net $\{Q_P\}$ of quadrics in $\mathbb{P}^3$ is typical if and only if the base locus of the net is zero-dimensional, reduced and in linear general position, that is, it consists of 8 distinct points, no four coplanar.*

Proof. The proof is based on one simple observation: in the space $\mathbb{P}^9$ of all quadrics in $\mathbb{P}^3$, the locus Σ of singular quadrics is smooth exactly along the open subset of quadrics of rank 3, that is, cones Q over smooth plane conics; and the projective tangent space $\mathbb{T}_Q(\Sigma) \subset \mathbb{P}^9$ to Σ at such a point$Q$ is simply the hyperplane in $\mathbb{P}^9$ of quadrics containing the vertex Q_{sing} of Q. It follows that a net of quadrics is typial if and only if it satisfies the two conditions

i. it contains no quadrics of rank 2 or less; and
ii. no singular point of any quadric of the net is a base point of the net.

Now, suppose first that the tangent space to the intersection $\Gamma = \cap Q_P$ at a point r is positive-dimensional; let $v \in T_r(\mathbb{P}^3)$ be a tangent vector to Γ at this point. Then not only do all the quadrics Q_p contain the point r, the partial derivatives of their defining equations in the direction v all vanish; it follows that at least one of them, say Q_0, is singular at r. But by our initial remark, this means the net is not typical.

Similar, suppose that the base locus of the net contains four coplanar points $p_1, \ldots, p_4 \in H$. If three of them are colinear the base locus of the pencil will be positive-dimensional, and hence the net cannot be typical by the above; so we may assume this is not the case. It follows that there are only two conics in H containing $p_1, \ldots, p_4$, that is, the restriction map from our net of quadrics in $\mathbb{P}^3$ to H must have a kernel. Thus at least one of the quadrics in our net must contain H, and hence have rank at most 2; so our net cannot be typical.

The reverse implication likewise follows immediately from our remark. Given any quadric Q_0 in our net, we can write the base locus of the net as a complete intersection $\Gamma = Q_0 \cap Q_1 \cap Q_2$; if Γ consists of 8 distinct points it follows that Q_0 must be smooth at each of them. Similarly, if Γ does not contain a planar subscheme of degree 4, no union of two planes can contain it, so no quadric in the net can have rank less than 4, and our net must be typical. This completes the proof of the lemma.

□

We have thus seen that to a sesquicanonical curve $B \subset \mathbb{P}^3$ of genus 3 we may associate a configuration of 8 points in $\mathbb{P}^3$, in linear general position, and (by the initial remark in the proof of the lemma) disjoint from B. What is the relationship between these 8 points and the curve? It turns out to be a beautiful one. Briefly, it is this: the 28 lines joining the 8 points pairwise—which a priori need not meet the curve $B \subset \mathbb{P}^3$ at all—all turn out to be bisecants to the curve B; and *the pairs of points of incidence of these lines with B are exactly the odd theta-characteristics of B.*

It will take us the next few pages to establish these facts. To start with, let p and q be two of the 8 points, and consider first the line $L = \overline{pq}$ containing them. Since L contains two base points of the net, the net cuts out on L a fixed divisor; thus the kernel of the restriction map of our net to the line must be two-dimensional; or in other words, the net contains a pencil of quadrics containing L. Let $M \subset \mathbb{P}V$ be the line corresponding to this pencil. Now, what does a pencil of quadrics in $\mathbb{P}^3$ containing a line L look like? The answer is that if no element of the pencil has rank 2 or less it has as base; locus the union of the line L with a twisted cubic curve T meeting L twice or tangent to it once. To see this, observe first that the base locus Ψ of the pencil must be one-dimensional, if no element of the pencil is reducible; in particular, the pencil will consist of all the quadrics containing Ψ. Now let T be the curve residual to L in this intersection. Let Q be a general quadric of the net, so that $L \subset Q$ is a curve of type (1,0) on Q and T is correspondingly a curve of type (1,2). Either T is irreducible, in which case it is a twisted cubic curve meeting L in a total of $(T \cdot L)_Q = 2$ points counting multiplicity; or it is reducible, in which case it contains a planar curve of degree 2 and we see that the pencil contains a reducible member.

From this picture it is easy to see what the singular elements of the pencil must be. For one thing, all the lines of a quadric cone pass through its vertex; so if the union $T \cup L$ lies on a quadric cone, of course the vertex r of that cone must lie on L. At the same time, projection of T from r must be a conic, so r must lie on T as well; thus r must be one of the points of intersection of T with L. One thing that follows from this is that the line L must have been a chord to B, meeting B in two points a and b.

Next, since the pencil has only one or two singular elements rather than the expected four, the line $M \subset \mathbb{P}^2$ must be in special position with respect to the curve $C \subset \mathbb{P}^2$ of singular elements of the net: specifically, it can meet C in at most two points. In fact, we can see directly that it must be a bitangent (a hyperflex is considered a bitangent). We can do this in two ways: by the same sort of argument as given above, we can argue that since the base locus of the pencil is singular at the vertex of each singular element of the pencil, the curve C must be tangent to M everywhere they meet, that is, the intersection $M \cap C$ is everywhere nonreduced. Alternatively, we can observe that since we have a marked ruling on each quadric of the pencil (namely, the one containing L) the discriminant of the pencil can vanish only to even order.

The conclusion in any case is this: the line $L = \overline{pq}$ is a chord to B; the line M is a bitangent line to C; and the two points of intersection of M with C are the points at which the line L meets B. Moreover, since there are 28 lines joining the eight base points of the net pairwise, the converse is also true: for any odd theta-characteristic $\Xi = \mathcal{O}_C(a+b)$, the line $\overline{ab}$ will contain two of the base points of the pencil.

We thus have a correspondence between the odd theta characteristics on C and the pairs of base points of the net. In particular, labelling the eight base points of the net $p_1, \dots, p_8$ is equivalent to labelling of the 28 odd theta characteristics on C.

Let us take a moment out and consider in some more detail this correspondence between the odd theta characteristics on C and the pairs of base points of the net, that is, edges of the complete octagon with vertices $p_1, \dots, p_8$. For this purpose, we will for $1 \leq i$, $j \leq 8$ identify the theta characteristic cut by the line $p_i p_j$ by $E_{i,j}$. Moreover, so as not to confuse the notions of equality in $\mathrm{Pic}(C)$ and equality in the group W in $\mathrm{Pic}(C)/\mathbb{Z}K_C$ associated to the curve C in the preceding chapter, we will denote by q_0 and $q_{i,j}$ the elements of W corresponding to Θ and $E_{i,j}$.

We may make one preliminary observation: since, as a net of quadrics varies among all typical nets, the monodromy action on the base points is the full symmetric group on 8 letters. It follows that for any subset of these edges, the sum of the corresponding theta characteristics will be zero or nonzero (if it is an even sum) and odd or even (it is an odd sum) depending only on the configuration of the corresponding edges. We will try to say in some cases which it is.

Now, our first observation is simple: for any triple of base points of the net, the sum of the corresponding divisors $E_{i,j}$, $E_{i,k}$ and $E_{j,k}$ is a hyperplane section of the curve B, so that in $\mathrm{Pic}(C)$ we have

$$E_{i,j} + E_{i,k} + E_{j,k} = K_C + \Theta$$

and correspondingly in the group W we have

$$q_{i,j} + q_{i,k} + q_{j,k} = q_0. \tag{$*_{i,j,k}$}$$

Next, since the theta characteristics $q_{i,j}$ are all distinct, the pairwise sums $q_{i,j} + q_{k,\ell}$ are all nonzero. The distinction is this: if two theta characteristics correspond to incident lines $p_i p_j$ and $p_j p_k$, then since $E_{i,j} + E_{j,k} + \Theta = K_C + E_{i,k}$, we have

$$q_0(q_{i,j} + q_{j,k}) = 1,$$

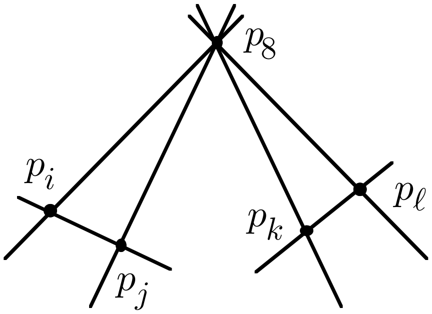
i.e., the sum of the theta characteristics corresponding to two incident lines is not a zero of the quadratic form q_0 on V. Conversely, the sum of the theta characteristics corresponding to two disjoint lines is not a zero of the quadratic form q_0: $q_0(q_{i,j} + q_{k,\ell}) = 1$ would mean that the divisor

$$E_{i,j} + E_{k,\ell} + \Theta - K_C = K_C + \Theta - E_{i,j} - E_{k,\ell}$$

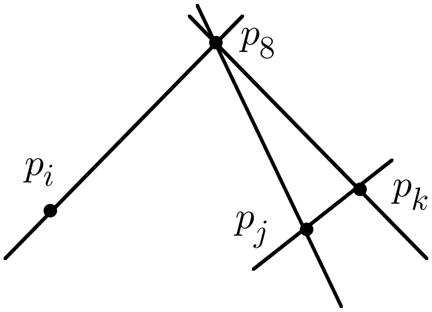
was effective; but since the complete linear series $|K_C + \Theta|$ is cut on B by planes, this is not the case. (In fact, by monodromy considerations, the 216 pairwise sums of theta characteristics corresponding to incident lines must be evenly distributed among the 28 vectors $v \in V$ such that $q_0(v) = 1$, that is, each must occur 6 times; and the 210 pairwise sums of theta characteristics corresponding to skew lines must include all 35 zeroes of the form q_0 other than 0, each occurring 8 times.)

Now, fix a single base point p_i, say p_8, and for each i, j consider the vector $v_{i,j} = q_{i,8} + q_{j,8}$. We claim that the pairwise sums of the vectors $v_{i,j}$ are all distinct, that is, the four-fold sum $q_{i,8} + q_{j,8} + q_{k,8} + q_{\ell,8} \neq 0$ for any i, j, k, ℓ distinct. To see this, simply observe that by the equalities $(*_{i,j,8})$ and $(*_{k,\ell,8})$ above, $v_{i,j} = q_{i,8} + q_{j,8} = q_0 + q_{i,j}$ and likewise $v_{k,\ell} = q_0 + q_{k,\ell}$; thus

$$v_{i,j} + v_{k,\ell} = q_{i,8} + q_{j,8} + q_{k,8} + q_{\ell,8} = q_{i,j} + q_{k,\ell} \neq 0$$



It also follows from this that *the sums of theta characteristic corresponding to three concurrent lines are all even;* given any distinct i, j, k between 1 and 7, we can write $q_{i,8} + q_{j,8} + q_{k,8} = q_{i,8} + q_{j,k} + q_0$, so that the parity of $q_{i,8} + q_{j,8} + q_{k,8}$ is $q_0(q_{i,8} + q_{j,k}) = 0$.



In the same way, we can determine the parity of the sum of any three theta characteristics in terms of the configuration of the corresponding lines: if we have theta characteristics $q_{i,j}$ and $q_{j,k}$ corresponding to two incident lines and one $q_{\ell,m}$ corresponding to a line skew to both, we may add the relation $(\ast_{i,j,k})$ to deduce that

$$q_{i,j} + q_{j,k} + q_{\ell,m} = q_{j,k} + q_{\ell,m} + q_0$$

is even. Similarly, if we have three theta characteristics $q_{i,j}$, $q_{j,k}$ and $q_{k,\ell}$ corresponding to lines forming a chain, we may add the relations $(\ast_{i,j,k})$ and $(\ast_{i,k,\ell})$ to deduce that

$$q_{i,j} + q_{j,k} + q_{k,\ell} = q_{i,\ell}$$

is odd. The last case, that of three theta characteristics corresponding to mutually skew lines, yields an even theta characteristic, as will be seen in a moment.

To conclude, we consider further sums of theta characteristics corresponding to concurrent lines. First, fix one base point p_8 as before, and consider the sum of all seven theta characteristics $E_{i,8}$ To describe this, note that projection from the point p_8 maps $B \subset \mathbb{P}^3$ to a plane curve $\bar{B}$. Being of degree 6, the curve $\bar{B}$ has arithmetic genus $\binom{5}{2} = 10$; since it has geometric genus 3, we would expect $\bar{B}$ to have $10 - 3 = 7$ nodes. In fact we can see them all: they are exactly the images of the lines $\overline{pp_i}$. By adjunction, the canonical series on B will be cut out by the series of plane cubics passing through the nodes of $\bar{B}$; in other words, we have a linear equivalence

$$K_B = \mathcal{O}_B(3)(-E_{1,8} - \ldots - E_{7,8}).$$

But $\mathcal{O}_B(1) = K_B \otimes \Theta$, and this amounts to saying that

$$E_{1,8} + \ldots + E_{7,8} = 3K_C + 3\Theta - K_C = 3K_C + \Theta;$$

in other words,

$$q_{1,8} + \ldots + q_{7,8} = q_0.$$

Next, for any $i_1, \ldots, i_5 \subset (1, \ldots, 7)$, consider the corresponding subset $E_{i_1,8}, \ldots, E_{i_5,8}$ of five of the seven theta characteristics E_i. Since any five of the lines pp_i lie on a quadric (the cone over the conic containing the images of the p_i under projection from p), we see that $h^0(\mathcal{O}_B(2)(-E_{i_1} - \ldots - E_{i_5})) > 0$; this translates into the assertion that

$$\begin{aligned} 0 &\leq 2(K_C + \Theta) - E_{i_1,8} - \ldots - E_{i_5,8} \\ &= E_{i_1,8} + \ldots + E_{i_5,8} - 2K_C \end{aligned} ,$$

that is, the sum $E_{i_18} + \ldots + E_{i_8} - 2K_C$ *is an odd theta characteristic* Ω. In fact, we can say which one it is: if $i, j \in \{1, \ldots, 7\}$ are the two indices not included in the subset $\{i_1, \ldots, i_5\}$, we can add the relation $(\ast_{i,j,8}$ to the relation $q_{1,8} + \ldots + q_{7,8} = q_0$ above to onclude that

$$q_{i_1,8} + \ldots + q_{i_5,8} = q_{i,j}.$$

Note that the assertions verified above imply (somewhat redundantly, in fact) that the theta characteristics corresponding to the seven lines through one of the base points of the net *form an Aronhold set with associated even theta characterristic* Θ. That is, the seven lines through a base point give an Aronhold set $q_1, \ldots, q_7$ and $q_S = \sum_{i=1}^{7} q_i = 3K$ is the class of Θ. We may thus refine the correspondence α above further, to arrive at bijections:

$$\left\{ \begin{array}{c} \text{typical nets of quadrics} \\ \text{in } \mathbb{P}^3 \text{ with choice of} \\ \text{one base point} \end{array} \right\} \overset{\alpha'}{\longleftrightarrow} \left\{ \begin{array}{c} \text{smooth nonhyperelliptic} \\ \text{curves } C \text{ of genus } 3 \\ \text{with Aronhold set} \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{typical nets of quadrics} \\ \text{in } \mathbb{P}^3 \text{ with labeled} \\ \text{base points} \end{array} \right\} \overset{\alpha''}{\longleftrightarrow} \left\{ \begin{array}{c} \text{smooth nonhyperelliptic} \\ \text{curves } C \text{ of genus } 3 \\ \text{with full level 2 structure} \end{array} \right\}$$

7. Del Pezzo Surfaces

Now that we have established this correspondence, let us take it one step further and consider how a general net of quadrics in $\mathbb{P}^3$ may be specified. Of course, a net is specified by its base points, but it is not the case conversely that any 8 points in $\mathbb{P}^3$ form the base of a net of quadrics. What is true, however, is that seven points in linear general position do impose independent conditions on quadrics, so that there will be exactly a net of quadrics containing them. We accordingly make the

Definition. *We will say that a collection $\Phi = \{p_1, \dots, p_7\}$ of seven points in $\mathbb{P}^3$ is typical if they are in linear general position, and the net of quadrics containing them is typical.*

Note that there are two ways that a configuration of seven points in linear general position may fail to be typical in this sense. First of all, the base locus Γ of the net of quadrics containing them may be positive-dimensional. Now, if this is the case, the degree of Γ can be at most 3; so it must contain either a line, a plane conic, or a twisted cubic curve. In fact, it cannot contain a conic, since the restriction map from our net to the plane containing the conic would have to have a kernel, i.e., the net would have to include reducible quadrics and seven points in linear general position cannot lie on the union of two planes. Similarly, Γ cannot contain a line, by the same argument applied to the plane spanned by that line and any of the points p_i not lying on it. Thus, Γ can be positive-dimensional only if it contains a twisted cubic curve X, in which case the net is simply the net of quadrics containing X and $\Gamma = X$; in fact this will occur if and only if the points $p_1, \dots, p_7$ lie on a twisted cubic curve.

If indeed the base locus of the net determined by Φ is zero-dimensional, the points $p_1, \dots, p_7$ may still fail to be typical if that base locus is nonreduced, that is, contains one of the points p_i multiply. By what we have said, this will in turn be the case only if some member of the net is singular at the point p_i; which is in turn equivalent to saying that the projection of the remaining 6 points $\{p_j : j \neq i\}$ from p_i lie on a conic. Since, if all seven points lie on a twisted cubic, the projection from any one of the points of the remaining six lie on the conic that is the image under projection of the twisted curbic, we have the

Lemma 7.1. *A collection $\{p_1, \dots, p_7\}$ of seven points in $\mathbb{P}^3$ in linear general position is typical if and only if the projection of any six from the remaining point does not lie on a conic.*

Now that we have characterized typical collections of 7 points in $\mathbb{P}^3$, we can extend our correspondence. Tautologously, an unordered collection Φ of 7 points in $\mathbb{P}^3$ determines a typical net, and of course determines also one base point of that net, namely the one not in Φ. Similarly, an ordered collection Ψ of 7 points in $\mathbb{P}^3$ determines a typical net together with an ordering of the base points of that net. We thus have two further correspondences, expressed in the

Theorem 7.2. *We have bijections*

$$\left\{\begin{array}{c}\text{smooth nonhyperelliptic}\\ \text{curves } C \text{ of genus 3}\\ \text{with Aronhold set}\end{array}\right\} \xleftrightarrow{\beta'} \left\{\begin{array}{c}\text{unordered collections of}\\ \text{7 typical points in } \mathbb{P}^3,\\ \text{modulo } PGL_4\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\text{smooth nonhyperelliptic}\\ \text{curves } C \text{ of genus 3}\\ \text{with full level 2 structure}\end{array}\right\} \xleftrightarrow{\beta''} \left\{\begin{array}{c}\text{ordered collections of 7}\\ \text{typical points in } \mathbb{P}^3,\\ \text{modulo } PGL_4\end{array}\right\}$$

Moreover, in terms of the structure of coarse moduli space on all four sets, these two maps are isomorphisms of varieties.

Note in particular that the moduli space of ordered collections of 7 typical points in $\mathbb{P}^3$ is rational: there is a unique element of PGL_4 sending the first 5 of the seven points to the standard points [1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1] and [1,1,1,1] so that this moduli space is in fact isomorphic to an open subset of $\mathbb{P}^3 \times \mathbb{P}^3$. We thus have the

Corollary 7.3. *The moduli space* $\mathcal{M}_3[2]$ *of curves of genus 3 with full level 2 structure is rational . In particular, the locus of curves* C *of genus 3 defined over* $\mathbb{Q}$*, all of whose line bundles of order 2 and all of whose theta characteristics are defined over* $\mathbb{Q}$*, is Zariski dense in the moduli space of curves of genus 3.*

For the next step, suppose we are given a typical collection $\Phi = \{p_1, \dots, p_7\}$ of 7 points in $\mathbb{P}^3$, either ordered or not. We can associate to this the base locus Γ of the net of quadrics they determine, which is a collection of 8 points $p_1, \dots, p_8$, any seven of which are typical. Now, consider in turn the projection of the original collection Φ from the eight point p_8 to the plane. This will be a configuration Ψ of 7 points in the plane, no three of which will be collinear and no six of which will lie on a conic, since any subset of seven of the eight points $p_1, \dots, p_8$ is typical. We thus make yet another

Definition. *We will say that a collection* $\Psi = \{q_1, \dots, q_7\} \subset \mathbb{P}^2$ *of seven points in the plane is* **typical** *if no three are collinear and no six lie on a conic and we observe that we have a natural map* γ *from the space* $\mathcal{C}_{3,7}$ *of typical 7-tuples of points in* $\mathbb{P}^3$ *modulo* PGL_4 *to the space* $\mathcal{C}_{2,7}$ *of typical 7-tuples of points in* $\mathbb{P}^2$ *modulo* PGL_3*, defined simply by sending a typical 7-tuple* $p_1, \dots, p_7$ *to the projection of* $p_1, \dots, p_7$ *from the eighth point of intersection of the quadrics containing them.*

Now, it may seem at first glance that, in projecting seven points $p_1, \dots, p_7 \in \mathbb{P}^3$ to $\mathbb{P}^2$ from the eighth base point of the net they determine we are necessarily losing information. Remarkably (to us, anyway) this is not the case; in fact, we claim that γ *is an isomorphism.*

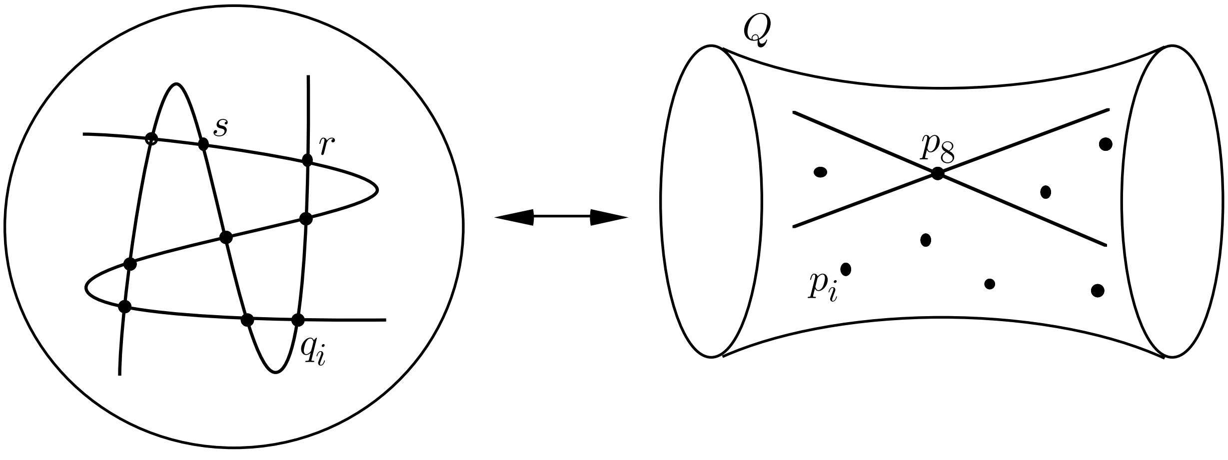
To prove this, we will exhibit an explicit inverse map $\gamma' : \mathcal{C}_{2,7} \to \mathcal{C}_{3,7}$. This goes as follows: suppose we are given a typical configuration $\Psi = \{q_1, \dots, q_7\} \subset \mathbb{P}^2$ of seven points in the plane. Such a collection of points imposes independent conditions on cubic curves in the plane, so that it lies exactly on a net of cubics; and moreover this net has no other base points beyond the q_i. Now, choose a general pencil $\mathcal{D} = \{C_\lambda\}$ of cubics in this net; it will have nine base points, consisting of $q_1, \ldots, q_7$ and two further points r and s. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^3$ be the rational map (regular on $\mathbb{P}^2 - \{r, s\}$) given by the web of conics through r and s, and let $p_i = \varphi(q_i) \in \mathbb{P}^3$, $i = 1, \ldots, 7$. We will then set

$$\gamma'(\Psi) = \Gamma = \{p_1, \ldots, p_7\} \subset \mathbb{P}^3.$$

We have then the

Lemma 7.4. *γ' is a well-defined map from $\mathcal{C}_{2.7}$ to $\mathcal{C}_{3,7}$ (that is, up to PGL_4 the configuration $\{p_1, \ldots, p_7\} \subset \mathbb{P}^3$ does not depend on the choice of pencil), and γ and γ' are inverse isomorphisms.*

Proof. It will turn out to be simpler to establish a more refined bijection. We have seen in the definition of γ' that the data of a typical configuration $\Psi \subset \mathbb{P}^2$ of seven points in the plane, together with a choice of $\mathcal{D}$ of pencil of cubics in the net $|\mathcal{I}_\Psi(3)|$ containing Ψ, determines a configuration $\Gamma \subset \mathbb{P}^3$ of seven points in space together with a quadric in the net $|\mathcal{I}_\Gamma(2)|$ containing Γ, namely, the image $Q = \varphi(\mathbb{P}^2)$.



Note also that φ blows up the points r and s and collapses the line $L = \overline{rs}$ joining them to a point, which we will call p_8; the rulings of the quadric Q correspond to the pencils of lines through r and s. At the same time, φ carries cubics through r and s—and in particular the cubics in our pencil— into quartic curves in $\mathbb{P}^3$; more specifically, since C_λ meets the general line through r or s in two other points, this will be a curve of type (2,2) on Q. Also since every cubic C_λ in the pencil meets L at one point beyond r and s, their images $E_\lambda \subset Q$ will all pass through the point p_8. The curves E_λ thus form a pencil on Q with base points $p_1, \ldots, p_8$; and this pencil is the restriction to Q of a net of quadrics in $P3$ with these base points. In particular, $p_1, \ldots, p_8$; form the base of a net of quadrics in $\mathbb{P}^3$, so that $\gamma(\gamma'(\Psi)) = \pi_{p_8}(\Gamma) = \Psi$.

Conversely, suppose we are given a typical configuration $\Gamma = \{p_1, \ldots, p_7\} \subset \mathbb{P}^3$ together with a general quadric Q in the net $|\mathcal{I}_\Psi(3)|$; let $p_8 \in Q$ be the eighth base point of the net and M and $N \subset Q$ the two lines of Q passing through the point p_8. Now, the net of quadrics through Γ cuts on Q a pencil $\mathcal{E} = \{E_\lambda\}$ of curves of type (2,2) on Q (in particular, quartic curves of arithmetic genus 1), whose base locus consists of $p_1, \ldots, p_8$. When we project from p_8, the curves E_λ are mapped isomorphically into a pencil $\mathcal{D} = \{C_\lambda\}$ of plane cubic curves, whose base points will include the images $q_i = \pi(p_i)$, $i = 1, \ldots, 7$. The other two base points of $\mathcal{D}$ will be the images r and $s \in \mathbb{P}^2$ of the two lines L and M: since each curve E_λ meets M and N in one point other than p_8, their images C_λ will all contain the points r and s.

In sum, we see that we have a bijection

$$\left\{\begin{array}{c}(\Gamma, Q) : \Gamma \text{ is a typical}\\ \text{configuration of 7}\\ \text{points in } \mathbb{P}^3, \text{ and } Q \supset \Gamma\\ \text{a quadric, modulo } PGL_4\end{array}\right\} \overset{\tilde{\gamma}}{\longleftrightarrow} \left\{\begin{array}{c}\Psi, \mathcal{D} : \Psi \text{ is a typical}\\ \text{configuration of 7}\\ \text{points in } \mathbb{P}^2; \mathcal{D} \subset |\mathcal{I}_\Psi(3)|\\ \text{a pencil, modulo } PGL_3\end{array}\right\}$$

that induces the bijections

$$\left\{\begin{array}{c}\text{ordered collections of 7}\\ \text{typical points in } \mathbb{P}^3\\ \text{modulo } PGL_4\end{array}\right\} \begin{array}{c}\xrightarrow{\gamma}\\ \xleftarrow[\gamma']{}\end{array} \left\{\begin{array}{c}\text{ordered collections of}\\ \text{7 typical points in } \mathbb{P}^2,\\ \text{modulo } PGL_3\end{array}\right\}$$

We should mention in passing that γ and γ' represent a very special case of the *Gale transform*; see for example [E-P].

Now that we have arrived at configuration of seven points in the plane, what can we do with those? One answer from classical algebraic geometry is immediate: we can blow them up to obtain a quadric del Pezzo surface. This is what we will do next, after a short interlude to discuss del Pezzo surfaces in general.

For our present purposes, it will make sense to define a *del Pezzo* surface to be simply one whose anticanonical bundle is ample. It is then a classical result that any such surface S is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at $m \leq 8$ points p_i, no three collinear and no six on a conic. In the latter case, we will denote by ℓ the pullback to S, via the blow-up map, of the class of a line in $\mathbb{P}^2$. Note that this is not intrinsic to the abstract surface S, but will (as we will see) depend on the representation $\pi : S \to \mathbb{P}^2$ of S as a blow-up of $\mathbb{P}^2$. We will denote by E_i the exceptional divisor lying over the point p_i, and its class by e_i; so that, for example, the anticanonical class is given by

$$H = -K_S \sim 3\ell - \sum e_i.$$

Given that the self-intersection of ℓ on S is 1, as it is on $\mathbb{P}^2$, the self-intersection of e_i is -1, and $(\ell \cdot e_i) = 0$ for each i, we see that the self-intersection of $-K_S$ is $9 - m$; we will call this quantity the *degree* of S and denote it d. In that vein, we will call a *line* of S any curve with intersection number 1 with H. We can easily list the lines on S: they are

1. the exceptional divisor E_i;

2. the proper transforms of the lines joining two of the points p_i;
3. the proper transforms of the conics containing five of the points p_i;
4. the proper transforms of the cubics double at one of the points p_i and containing six others;
5. the proper transforms of the quartics double at three of the points p_i and containing five others; and
6. the proper transforms of the quintics double at six of the points p_i and containing two others.

Note that the last two are possible only when $m = 8$, the last three only when $m \geq 7$, and so on.

It is not hard to see directly that the points p_i impose independent conditions on cubic curves in the plane, so that the dimension of the anticanonical series will be exactly $9 - m = d$. Now, in case $d \geq 3$, this series will in fact be very ample, giving an embedding of S in $\mathbb{P}^d$ (the images of these maps are what is often referred to as del Pezzo surfaces, i.e., the definition of del Pezzo may in some sources require the anticanonical series to be very ample rather than merely ample).

The principal case of interest to us at present, however, is the case of $m = 7$, that is, del Pezzos of degree 2. In this case, the anticanonical series gives a regular map $\varphi : S \rightarrow \mathbb{P}^2$, expressing S as a double cover of the plane. It is not hard to see what the branch divisor $C \subset \mathbb{P}^2$ must be, in any of several ways. For one thing, the inverse image E of a general line $L \subset \mathbb{P}^2$ will be a smooth curve of genus

$$\frac{(-K_S \cdot (-K_S + K_S))}{2} + 1 = 1;$$

inasmuch as the map φ expresses E as a double cover of a line branched at its points of intersection with C, we conclude that C must be a quartic curve. C moreover must be smooth since S is. Alternatively, we can apply the Riemann-Hurwitz formula: if $R \subset S$ is the ramification divisor, the canonical line bundle of S is given by

$$\begin{aligned} K_S &= \varphi^*(K_{\mathbb{P}^2})(R) \\ &\sim 3 \cdot K_S + R, \end{aligned}$$

we conclude that $R \sim -2K_S$, and in particular that the degree of the image $C = \varphi(R)$ is

$$(R \cdot -K_S) = 2(K_S \cdot K_S) = 4.$$

Thus, a del Pezzo of degree 2 is a double cover of $\mathbb{P}^2$ branched along a smooth quartic. Conversely, if $\varphi : S \rightarrow \mathbb{P}^2$ is a double cover branched along a quartic, the canonical bundle of S is given by Riemann-Hurwitz as

$$K_S = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(-1));$$

and since the map φ is finite we conclude that $-K_S$ is indeed ample, so S is a del Pezzo surface.

Note that the expression $\varphi : S \to \mathbb{P}^2$ of S as a double cover defines a regular involution ψ on S exchanging the sheets of this map; ψ may also be viewed as a birational involution of the plane $\mathbb{P}^2$ of which S is the blow-up. (Here and elsewhere there is some potential confusion between the two copies of $\mathbb{P}^2$ floating around, inasmuch as φ defines a rational map $\mathbb{P}^2 \to \mathbb{P}^2$ of degree 2.) This involution is known as the *Geyser involution*, and has the following alternative description. Given a general point $q \in \mathbb{P}^2$, the cubic curves passing through the eight points $p_1, \ldots, p_7$ and q will form a pencil, and this pencil will have one other base point $r \in \mathbb{P}^2$; the involution is the one sending q to r.

Next, we would like to discuss the lines on S. We already know them in terms of the description of S as a blow-up of $\mathbb{P}^2$; they are

1. the 7 exceptional divisors E_1;
2. the 21 proper transforms of the lines joining two of the points p_i;
3. the 21 proper transforms of the conics containing five of the points p_i;
4. the 7 proper transforms of the cubics double at one of the points p_i and containing six others.

We would now like to describe their images under the double cover φ. This is in fact easy: since by definition a line of S is a curve on S mapped by φ one-to-one onto a line in $\mathbb{P}^2$, the inverse images in S of the image in $\mathbb{P}^2$ of any line on S must consist of exactly two lines on S. In particular, the 56 lines of S map onto exactly 28 lines in $\mathbb{P}^2$. Moreover, inasmuch as the inverse images of these 28 lines in $\mathbb{P}^2$ are reducible, they cannot have any points of odd intersection multiplicity with the branch divisor C of φ; thus the 28 images of the lines of S must be exactly the 28 bitangent lines to the curve C.

Note finally that labeling all 56 lines of S (equivalently, choosing a set of generators for $\mathrm{Pic}(S)$) amounts to labeling the 28 bitangents to C, so that a quadric del Pezzo surface together with a set of generators for its Picard group gives us the data of curve of genus 3 with full level two structure. If we do not identify the elements of $\mathrm{Pic}(S)$, but only the expression $\Pi : S \to \mathbb{P}^2$ of S as a blow-up of $\mathbb{P}^2$ at 7 points (that is, equivalently, specify only the unordered set $\{E_1, \ldots, E_7\}$ of exceptional divisors), we arrive at a curve C of genus 3, together with the specification of an Aronhold set. And finally, if we specify only the abstract surface S, and not its representation as a blow-up of the plane, we find the curve C, but no further level 2 structure on it.

We have now come, as promised, all the way around. To recap the journey:

We start with a *smooth, nonhyperelliptic curve of genus* 3, *with full level* 2 *structure.* This structure, as we have seen in preceding sections, is equivalent to specifying an *ordered Aronhold set* $\Theta_1, \ldots, \Theta_7$ *of odd theta characteristics on* C.

We may then associate to this data a typical net φ of quadrics in $\mathbb{P}^3$ with base points labeled $p_1, \ldots, p_8$. The discriminant curve of this net—that is, the set of singular quadrics in the net—will be $C \subset \mathbb{P}^2$; and the locus in $\mathbb{P}^3$ of vertices of singular quadrics in the net will be the image $B \subset \mathbb{P}^3$ of the curve C, embedded by the linear series $K_C + \Theta$ where Θ is the even theta characteristic associated to our Aronhold set (in other words, $\Theta = \Sigma\Theta_i$, in the group W). The lines joining the points p_i pairwise will cut on B the 28 odd theta characteristics, with our Aronhold set $\Theta_1, \dots, \Theta_7$ cut by the seven lines $p_i p_8$ through the point p_8.

Now, a typical net φ of quadrics is determined by any seven of its base points; thus we may associate to φ simply the typical configuration $p_1, \dots, p_7$ of seven ordered points in $\mathbb{P}^3$.

Next, we may project the configuration $p_1, \dots, p_7$ from p_8 to obtain a typical configuration of seven points $q_1, \dots, q_7$ in the plane. Again, it is far from clear at first that this is equivalent data, but it is, as may be seen either from the identification of $\{q_1, \dots, q_7\} \subset \mathbb{P}^2$ with the Gale transform of $\{p_1, \dots, p_7\} \subset \mathbb{P}^3$ or the explicit (if complicated) inverse given by the cycle of associations here.

Now, a typical configuration $\Gamma = \{q_1, \dots, q_7\} \subset \mathbb{P}^2$ of seven ordered points in the plane determines a del Pezzo surface S, together with a standard basis for the Picard group of S, namely, we take $S = \mathrm{Bl}_\Gamma(\mathbb{P}^2)$ to be the blow-up of the plane at $q_1, \dots, q_7$ and the generators of $\mathrm{Pic}(S)$ the pullback of the class of a line in $\mathbb{P}^2$ and classes of the seven exceptional divisors E_i.

Finally, a del Pezzo surface determines a smooth plane quartic, namely the branch divisor $C \subset \mathbb{P}^2$ of the anticanonical map $\varphi_{-K} : S \to \mathbb{P}^2$. Moreover, the seven generators in a normalized set correspond to seven exceptional divisors, whose images in $\mathbb{P}^2$ will be bitangent lines to C forming an Aronhold set. Thus we arrive once more at a smooth nonhyperelliptic curve C with an ordered Aronhold set, that is to say, full level 2 structure. We may represent the various stages in this cycle in a diagram:

$$\begin{tikzcd}
 & \begin{Bmatrix} \text{typical nets of quadrics} \\ \text{in } \mathbb{P}^3 \text{ with marked} \\ \text{base points} \end{Bmatrix} \arrow[dr, shift left] \arrow[ddl, shift left] & \\
 & & \begin{Bmatrix} \text{ordered collections of} \\ 7 \text{ typical points in } \mathbb{P}^3, \\ \text{modulo } PGL_4 \end{Bmatrix} \arrow[ul, shift left] \arrow[dd, shift left] \\
\begin{Bmatrix} \text{smooth nonhyperelliptic} \\ \text{curves } C \text{ of genus } 3 \\ \text{with full level 2 structure} \end{Bmatrix} \arrow[uur, shift left] \arrow[ddr, shift left] & & \\
 & & \begin{Bmatrix} \text{ordered collections of} \\ 7 \text{ typical points in } \mathbb{P}^2, \\ \text{modulo } PGL_3 \end{Bmatrix} \arrow[uu, shift left] \arrow[dl, shift left] \\
 & \begin{Bmatrix} \text{quadric del Pezzo} \\ \text{surfaces } S \text{ with} \\ \text{marked Picard group} \end{Bmatrix} \arrow[ur, shift left] \arrow[uul, shift left] &
\end{tikzcd}$$

Alternatively, we may go through an analogous cycle of objects and associations starting with the specification on C only of an unordered Aronhold set. In this case we get a typical net of quadrics in $\mathbb{P}^3$ with one distinguished base point, but no ordering of the remaining seven; this gives us in turn an unordered collection of 7 typical points in $\mathbb{P}3$, then an unordered collection of 7 typical points in $\mathbb{P}^2$, then a quadric del Pezzo surface with a choice of regular birational map $\pi : S \to \mathbb{P}^2$ (equivalently, a divisor class L with $L^2 = 1$ and $L \cdot K_S = -3$). Finally, the branch divisor of the anticanonical map of S is a plane quartic curve, on which the seven exceptional divisors of the map π cut an unordered Aronhold set of odd theta characteristics. In other words, we have the analogous diagram

$$\begin{tikzcd}
 & \left\{\begin{matrix}\text{typical nets of quadrics}\\ \text{in } \mathbb{P}3 \text{ with one}\\ \text{marked base point}\end{matrix}\right\} \arrow[dr, shift left] \arrow[ddl, shift left] & \\
 & & \left\{\begin{matrix}\text{unordered collections of}\\ 7 \text{ typical points in } \mathbb{P}3,\\ \text{modulo } PGL_4\end{matrix}\right\} \arrow[ul, shift left] \arrow[dd, shift left] \\
\left\{\begin{matrix}\text{smooth nonhyperelliptic}\\ \text{curves } C \text{ of genus } 3\\ \text{with Aronhold set}\end{matrix}\right\} \arrow[uur, shift left] \arrow[ddr, shift left] & & \\
 & & \left\{\begin{matrix}\text{unordered collections of}\\ 7 \text{ typical points in } \mathbb{P}^2,\\ \text{modulo } PGL_3\end{matrix}\right\} \arrow[uu, shift left] \arrow[dl, shift left] \\
 & \left\{\begin{matrix}\text{quadric del Pezzo surfaces}\\ S \text{ with regular birational}\\ \text{map } \pi : S \to \mathbb{P}^2\end{matrix}\right\} \arrow[ur, shift left] \arrow[uul, shift left] &
\end{tikzcd}$$

Lastly, if we specify only the curve C and an even theta characteristic Θ on C, we get a typical net of quadrics in $\mathbb{P}3$, but no distinguished base point; and we cannot complete the cycle in any way.

There are a number of questions, we can ask about this correspondence. For one, all the various moduli spaces referred to in the diagrams above have known compactifications— some, several. To what extent do these associations extend to regular isomorphisms between these moduli spaces? To mention one particularly interesting case, we could enlarge the set of smooth nonhyperelliptic curves of genus 3 to all smooth curves of genus three, and try to extend the definition of the maps α, β and γ to this locus. What happens to the net of quadrics when the curve becomes hyperelliptic? What happens to its eight base points?

Similarly, if we consider some singular curves of genus 3—for example, to take the simplest cases, nodal plane quartics—what happens to the nets, to their base points, and to the del Pezzo surfaces?

Finally, it is known that (some) plane curves with theta characteristics, as well a nets of quadrics are related to the theory of vector bundles on $\mathbb{P}^2$. In the particular case relevant to the paper one takes (stable) rank 2 bundles with $c_1 = 0$ and $c_2 = 4$. Then the quartic curves which arise in this way can be characterized as *Luroth* quartics; see again [B].

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