**Rigid local systems on $\mathbb{G}_m$ with finite monodromy**

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Katz has constructed some rigid local systems of rank 7 on $\mathbb{G}_m$ in finite characteristics, whose monodromy is a finite subgroup $G$ of the complex Lie group M of type $G_2$ [K, Thm 9.1] . Among the five finite subgroups $G$ of $M$ which appear in his work, four occur in their natural characteristic:

$SL_2(8)$ in char 2

$PU_3(3)$ in char 3

$PGL_2(7)$ in char 7

$PSL_2(13)$ in char 13.

In this paper, we give another construction of these four local systems on $\mathbb{G}_m$, and find similar local systems with finite monodromy in other complex Lie groups. Some examples of finite groups which occur in exceptional complex Lie groups $M$, analogous to the ones above are

$$G = PSL_2(27) \quad \text{in char } 3 \quad M = F_4$$

$$G = PSU_3(8) \quad \text{in char } 2 \quad M = E_7$$

$$G = PGL_2(31) \quad \text{in char } 31 \quad M = E_8$$

$$G = PSL_2(61) \quad \text{in char } 61 \quad M = E_8$$

There are interesting families in the classical groups. For example, when $q = p^f > 2$, we have:

$$G = PU_3(q) \ \text{ in char } p \qquad M = Sp_{2n} \quad 2n = q(q-1).$$

Our method of construction is as follows. Let $k$ be the finite field with $q$ elements. We first consider the action of a finite group $G = G(q)$ of the type

$$G = PGL_2(q) = A_1(q)$$

$$G = PU_3(q_0) = {}^2A_2(q) \quad q = q_0^2$$

$$G = Sz(q) = {}^2B_2(q) \qquad q = 2q_0^2$$

$$G = R(q) = {}^2G_2(q) \qquad q = 3q_0^2$$

on the corresponding Deligne-Lusztig curve $Y$ over $k$. (We note that the algebraic group $PU_3(q_0)$ is defined over $q_0$, but is naturally a subgroup of $A_2(q) = PGL_3(q)$ with $q = q_0^2$.) This extends to an action of $G$ on the complete curve $X = Y \cup S$, where $S$ denotes the finite set of points in the completion. Using results of Lusztig [L], we show the quotient map $X \to X/G$ is ramified over precisely 2 points of the base, which has genus 0. Identifying $X/G$ with $\mathbb{P}^1$, and the two points of ramification with $\{0, \infty\}$, we obtain a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \to G,$$

where $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$.

We then study embeddings

$$g : G \to M$$

where $M$ is a simple, complex Lie group. Let Ad be the adjoint representation of $M$ on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \xrightarrow[f]{} G \xrightarrow[g]{} M \xrightarrow[Ad]{} GL(\mathrm{Lie}(M))$$

gives a complex respresentation of the Galois group of the global field $k(\mathbb{G}_m) = k(t)$, which is unramified outside $t = 0, \infty$. We say the homomorphism $g$ is *rigid* if the Artin $L$-function $L(\mathrm{Ad}, s)$ is equal to the constant function 1.

The embedding $g$ is rigid precisely when the $\ell$-adic sheaf on $\mathbb{P}^1$ corresponding to the middle extension of the adjoint representation has vanishing cohomology. This is equivalent to the statement that the principal $M$-bundle on $\mathbb{G}_m$ has no deformations, preserving the singularities at $t = 0$ and $t = \infty$.

Using results of Serre [S] and Greiss-Ryba [G-Ry], we construct rigid embeddings for $G = PGL_2(q)$ and $G = PU_3(q_0)$, as well as for their normal subgroups

$$G_0 = PSL_2(q) \qquad q \equiv 1 \pmod 2.$$

$$G_0 = PSU_3(q_0) \qquad q_0 \equiv 2 \pmod 3.$$

These give rise to the rigid local systems on $\mathbb{G}_m$ with finite monodromy, described above.

As a bonus, we obtain simple wild parameters (cf. [G-Re, §5.6]) for the local field $k((1/t))$, by restricting our local systems to the decomposition group at $t = \infty$. These parameters are representations of the local Galois group into $M$, which have no inertial invariants on the adjoint representation and have Swan conductor equal to the rank of $M$.

## 1. Deligne-Lusztig curves

Let $d = 2, 3, 4, 6$. For each $d$, and certain prime powers $q$, we consider the Deligne-Lusztig curve $Y = Y(w)$ over the field $k$ with $q$ elements. the restrictions on $q$ are

$$\begin{array}{rll} d = & 2 & q = \text{arbitrary} \\ & 3 & q = q_0^2 \\ & 4 & q = 2q_0^2 \\ & 6 & q = 3q_0^2 \end{array}$$

The curve $Y$ is affine and non-singular, and admits an action of the finite adjoint group $G = G(q)$ over $k$. In ATLAS notation, $G$ is given by

$$\begin{array}{rll} d = & 2 & PGL_2(q) = A_1(q) \\ & 3 & PU_3(q_0) = {}^2A_2(q) \\ & 4 & Sz(q) = {}^2B_2(q) \\ & 6 & R(q) = {}^2G_2(q) \end{array}$$

The order of $G$ is given by the formula

$$\#G = q^{d/2}(q^{d/2} + 1)(q - 1).$$

Let $\overline{k}$ be an algebraic closure of $k$ and $F : G(\overline{k}) \to G(\overline{k})$ the (geometric) Frobenius map with fixed points $G$. When $d = 2, 3$ let $\sigma = F$; when $d = 4, 6$ let $\sigma$ be a square-root of $F$. By definition, $Y$ is the variety of all Borel subgrops of $G(\overline{k})$ such that $B$ and $\sigma(B)$ are in relative position $w$, where $w$ is the Coxeter class in the Weyl group relative to $\sigma$. The integer $d$ is the order of a Coxeter element. (In [L] the affine curve $Y$ is denoted $X_f$, and the integer $d$ is denoted $h_0$.) Since $G$ acts on the function field $k(Y)$, the action on $Y$ extends to an action on the complete non-singular model $X = \overline{Y} = Y \cup S$, where $S$ denotes the finite set of points in the completion.

There are two maximal subgroups of $G$ that will play an important role in the study of the $G$ action on $X$. The first is the Borel subgroup $B = B(q)$, of order $q^{d/2}(q-1)$. This fixes a point in a 2-transitive permutation representation, which we will see occurs on the points $S$ of the completion. The $p$-Sylow subgroup $U$ of $B$ is normal, and the quotient $B/U$ is cyclic. The second maximal subgroup is the normalizer $N(T)$ of a Coxeter torus $T = T(q)$. This is the semi-direct product of the cyclic group $T$, of order

$$\begin{array}{rll} d = & 2 & q+1 \\ & 3 & q - q_0 + 1 \\ & 4 & q - 2q_0 + 1 \\ & 6 & q - 3q_0 + 1 \end{array}$$

dividing $(q^{d/2}+1)$, with a Coxeter element of order $d$ in $G$: $N(T) = T.d$.

The adjoint group $G = G(q)$ has a normal subgroup $G_0$, which is the image of $k$-rational points of the simply-connected cover. This has index $> 1$ in the following cases

$$\begin{array}{lll} d = 2 & PGL_2(q) \triangleright_2 PSL_2(q) & q \equiv 1 \pmod 2 \\ d = 3 & PU_3(q_0) \triangleright_3 PSU_3(q_0) & q_0 \equiv 2 \pmod 3 \end{array}$$

Both $B$ and $T$ have a non-trivial image in the quotient group $G/G_0$ in these cases, so the kernel defines maximal subgroups $B_0$ and $N(T_0) = T_0.d$ in $G_0$.

The action of $G$ on the affine curve $Y$ is completely described in [L]. We summarize these results in Theorem 1 below. For $m \geqslant 1$, let $k_m$ be the unique extension of $k$ of degree $m$ in $\overline{k}$.

**Theorem 1 (Lusztig)**

a) *The set $Y(k_m)$ is empty, for $1 \leqslant m < d$.*

b) *$G$ acts transitively on $Y(k_d)$ with stabilizer $T$.*

c) *$G$ acts freely on $Y(\overline{k}) - Y(k_d)$, and acts simply-transivitely on $Y(k_{d+1})$.*

d) *The linear action of $G$ on the compactly supported $\ell$-adic cohomology groups $H^i_c(Y/\overline{k}, \mathbb{Q}_\ell)$ commutes with the action of $F$, and the distinct eigenspaces for $F$ give distinct unipotent representations $W$ of $G$. A unipotent representation occurs in $H^i_c(Y)$ if and only if $\mathrm{Tr}(t|W) = (-1)^i$, for all regular semi-simple elements $t$ in $T$. If such a representation occurs, its multiplicity in $H^i_c(Y)$ is equal to one.*

One can make part d) more explicit by enumerating the eigenvalues $\alpha$ of $F$ on $H^i_c(Y)$, and the unipotent representation $W_\alpha$ that appears in the $\alpha$-eigenspace. They are

| | | |
|---|---|---|
| $\alpha = q$ | on $H^2_c(Y)$ | $W =$ trivial representation |
| $\alpha = 1$ | on $H^1_c(Y)$ | $W =$ Steinberg representation |
| | | $\dim W = q^{d/2}$ |

There are $(d-2)$ further eigenvalues on $H_c^1(Y)$, which correspond to unipotent cuspidal representation

$d = 3$

$\alpha = -q_0$ $\qquad$ $\dim W = q_0(q_0 - 1)$

$d = 4$

$\alpha_\pm = -q_0(1 \pm i)$ $\qquad$ $\dim W_\pm = q_0(q - 1)$

$d = 6$

$\alpha_\pm = -q_0\left(\frac{3\pm\sqrt{-3}}{2}\right)$ $\qquad$ $\dim W_\pm = q_0(q^2 - 1)$

$\beta_\pm = -q_0(\pm\sqrt{-3})$ $\qquad$ $\dim W_\pm = q_0(q-1)(q+3q_0+1)/2$

The two unipotent cuspidal representations $W$ of the Ree groups with

$$\dim W = q_0(q-1)(q-3q_0+1)/2$$

do **not** appear in the cohomology of $Y$, as they restrict to the regular representation of $T$.

From Lusztig's fundamental results, we deduce the following facts on the complete curve $X = \overline{Y} = Y \cup S$.

**Corollary**

a) *$G$ acts transitively on $X(k) = S(k) = S(\overline{k})$, with stabilizer $B$.*

b) *The covering $X \to X/G = Z$ is ramified over two rational points of the base, which has genus 0.*

c) *We may fix an isomorphism $Z \longrightarrow \mathbb{P}^1$ over $k$ so that the resulting cover $\pi : X \to \mathbb{P}^1$ satisfies*

$$\pi(X(k)) = \infty$$

$$\pi(X(k_d)) - X(k)) = 0$$

$$\pi(X(k_{d+1}) - X(k)) = 1$$

d) *The decomposition group $D$, inertia group $I$, and wild inertia group $P$ at the points $\{\infty, 0, 1\}$ are*

$D_\infty = I_\infty = B \rhd P_\infty = U \quad I_\infty/P_\infty = (q-1)$

$D_0 = N(T) \rhd I_0 = T \quad P_0 = 1 \quad D_0/I_0 = (d)$

$D_1 = (d+1) \quad I_1 = 1.$

**Proof.** Let $W_0$ be the representation of $G$ on the functions on $S$ with values in $\mathbb{Q}_\ell$ and with sum equal to 0. Then $W_0$ appears in $H^1_c(Y)$, and the eigenvalues of $F$ on $W_0$ are roots of unity. The only such eigenvalue is $\alpha = 1$, so $S(k) = S(\overline{k})$. This eigenspace affords the Steinberg representation of $G$, which is the functions on $G/B$ with sum equal to 0. Hence $G$ acts transitively on $S(k) = X(k)$ with stabilizer $B$.

Since $G$ fixes no non-zero vector in $H^1(X) = \oplus_{i=1}^{d-2} W_i$ the quotient curve $Z = X/G$ has genus 0. The decomposition groups at $\{\infty, 0, 1\}$ and their inertia subgroups follows from the structure of the $G$-orbits on $X(\overline{k})$, and their fields of rationality. We note that the element of order $(d+1)$ generating $D_1$ is a torsion class of Kac type (2; 1, 1) in $G(q)$, just as the element of $d$ normalizing $T$ is of Coxeter type (1; 1, 1).

Let $K = k(\mathbb{P}^1) = k(t)$, and let $K' = k(X)$ be the Galois extension of $K$ corresponding to the covering $X \to \mathbb{P}^1$. Let $V$ be a complex representation of $G = \mathrm{Gal}(K'/K)$, and for each place $v$ of $K$, let $F_v$ be an geometric Frobenius element in the quotient $G_v/I_v$. We define the Artin $L$-function of $V$ by an Euler product over the places $v$ of $K$:

$$L(V,x) = \prod_v \det(1 - F_v x^{degv} | V^{I_v})^{-1}.$$

This is a rational function of $x$, and is a polynomial in $x$ when $V^G = 0$. We define the Artin conductor $f_v(V)$ and the Swan conductor $sw_v(V)$ as in [S2, Ch VI]. Then

$$f_0(V) = \dim(V/V^T)$$

$$f_\infty(V) = \dim(V/V^B) + sw_\infty(V).$$

Since the degree of $L(V,x)$ is given by [W, Appendix V]

$$\deg L(V,x) = f_0(V) + f_\infty(V) - 2\,\dim V$$

we find the formula

$$sw_\infty(V) = \deg L(V,x) + \dim V^B + \dim V^T.$$

Lusztig's calculation of the cohomology of $Y$ gives us a determination of the Artin $L$-functions of the irreducible representations $W$ of $G$, as we have the factorization

$$\prod_{\substack{\text{irred} \\ W}} L(W,x)^{\dim W} = \zeta_{K'}(x) = \frac{\det(1 - Fx|H^1(X))}{(1-x)(1-qx)}.$$

**Corollary.** *If $W$ is an irreducible representation of $G$, then $L(W,x)$ has degree = -2, 0, or 1. More precisely*

a) *if* $W$ *is the trivial representation*

$$L(W,x) = \frac{1}{(1-x)(1-qx)}$$

b) *if* $W$ *is a unipotent cuspidal representation, which occurs as the* $\alpha$-*eigenspace for* $F$ *in* $H^1(X)$, *then*

$$L(W,x) = (1-\alpha x)$$

c) *for all other irreducible representations* $W$, $L(W,x) = 1$.

**Corollary.** a) *If* $W$ *is the trivial representation of* $G$, *or the quadratic character of* $PGL_2(q)$ *when* $q \equiv 1 \pmod 2$, *or one of the cubic characters of* $PU_3(q_0)$ *when* $q_0 \equiv 2$ (mod) *then* $sw_\infty(W) = 0$.

b) *If* $W$ *is an irreducible representation of* $PGL_2(q)$ *and* $\dim W > 1$, *then* $sw_\infty(W) = 1$.

c) *If* $W$ *is the unipotent cuspidal representation of* $PU_3(q_0)$, *of dimension* $q_0^2 - q_0$, *then* $sw_\infty(W) = 1$.

**Proof.** a) These are precisely the irreducible representations which are tamely ramified. For all other irreducible representations, $sw_\infty(W) \geqslant 1$.

b) For these representations, $L(W,x) = 1$ has degree 0, so $sw_\infty(W) = \dim W^B + \dim W^T$. For the Steinberg representation $W = St$ of dimension $q$, we have $\dim W^B = 1$ and $\dim W^T = 0$. For all other irreducibles (of dimensions $q-1$, $q$, $q+1$) we have $\dim W^B = 0$ and $\dim W^T = 1$.

c) In this case, $L(W,x) = (1+q_0 x)$ has degree 1. We find $\dim W^B =$

$\dim W^T = 0$, as $W$ is cuspidal and the restriction of $W$ to $T$ is the regular representation minus the trivial character. Hence $sw_\infty(W) = 1$.

The Swan conductors of all other irreducible representations $W$ of $G$ grow with the size of $q$. For example, the Steinberg representation $St$ of $G(q)$ has

$$\begin{aligned} sw_\infty(St) &= \dim St^B + \dim St^T \\ &= 1 + \dim St^T \\ &= (q^{d/2}+1)/\#T \\ &= \begin{cases} 1 & d=2 \\ q_0+1 & d=3 \\ q+2q_0+1 & d=4 \\ q^2+3qq_0+2q+3q_0+1 & d=6 \end{cases} \end{aligned}$$

The two unipotent cuspidal representations $W$ of the Ree group ${}^2G_2(q)$ which do **not** appear in the cohomology of $X$ have

$$sw_\infty(W) = \dim W^T = q_0\left(\frac{q-1}{2}\right).$$

We now make some observations about the subfields of $K' = k(X)$ fixed by certain subgroups of the Galois group $G$. In the cases where we have a normal subgroup $G \triangleright G_0$:

$$PGL_2(q) \underset{2}{\triangleright} PSL_2(q) \qquad q \equiv 1 \pmod 2.$$

$$PU_3(q_0) \underset{3}{\triangleright} PSU_3(q_0) \qquad q_0 \equiv 2 \pmod 3.$$

The fixed field $K_0$ is tamely ramified, of degree $d = 2$ or 3, over $K = k(t)$. Since it is also split at the place $t = 1$, we find

$$K_0 = k(\sqrt[d]{t}).$$

Since the fixed field $K_0$ of $G_0$ also has genus 0, we obtain $d$ homomorphisms up to conjugacy

$$f_0 : \pi_1(\mathbb{G}_m/k) \to G_0.$$

These are all conjugate under the other action of $G$. The different classes of $f_0$ correspond to fixing a $d^{th}$ root of $t$ in $K_0$, or equivalently, to fixing a place above the place $t = 1$ in $K$.

Now consider the subfield $M$ of $K'$ fixed by the subgroup $B$ of $G$:

$$\begin{array}{ccc} & & K' = k(X) \\ & B \diagup & \\ M & & \\ & \diagdown & \\ & & K = k(t) \end{array}$$

This has degree $(q^{d/2} + 1)$ over $K$. Since $H^1(X)^B = 0$, it also has genus 0. The zeta function of $M$ factors

$$\zeta_M(x) = \zeta_K(x) L(St, x)$$

and the discriminant divisor $D(M/K)$ of $M/K$ is given by the conductor of the Steinberg representation:

$$D(M/K) = f_\infty(St).(\infty) + f_0(St)(0).$$

We have

$$\begin{aligned} f_\infty(St) &= \dim(St/St^B) + sw_\infty(St) \\ &= (q^{d/2}-1) + sw_\infty(St) \\ &= \begin{cases} q & d=2 \\ q_0^3+q_0 & d=3 \\ q^2+q+2q_0 & d=4 \\ q^3+q^2+3qq_0+2q+3q_0 & d=6 \end{cases} \end{aligned}$$

and

$$\begin{aligned} f_0(St) &= \dim(St/St^T) \\ &= \begin{cases} q & d=2 \\ q_0^3-q_0 & d=3 \\ q^2-q-2q_0 & d=4 \\ q^3-q^2-3qq_0-2q-3q_0 & d=6 \end{cases} \end{aligned}$$

The prime $P_\infty$ factors in $M$ as

$$P_\infty = R \cdot (R')^{q^{d/2}} \qquad \text{with} \quad \deg R = \deg R' = 1$$

The prime $P_0$ factors in $M$ as

$$P_0 = \prod_{i=1}^{n} (R_i)^{\#T} \quad \deg R_i = 1$$

where $n = (q^{d/2}+1)/\#T$. All other primes are unramified.

In the case where $G = PGL_2(q)$, the curve $X$ is the projective line over $k = \mathbb{F}_q$ and the action is by fractional linear transformations. Here, specific polynomials of degree $(q+1)$ over $K = k(t)$ are known, with root field $M$ and splitting field $K' = k(X)$. Can one find such equations, of degree $(q^{d/2}+1)$ over $K$, in general?

## 2. Rigid local systems

In the previous section, we defined a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \to G$$

where $k$ is a finite field with $q$ elements and $G$ is the finite group acting on a Deligne-Lusztig curve $X$ over $k$ (or a normal subgroup $G_0$ of $G$).

In this section, we will consider the local systems on $\mathbb{G}_m/k$ that we obtain by composing $f$ with a homomorphism

$$g : G \to M,$$

where $M$ is a connected, simple complex Lie group. Consider the adjoint representation of $M$ on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \underset{f}{\longrightarrow} G \underset{g}{\longrightarrow} M \longrightarrow GL(\mathrm{Lie}(M))$$

defines a complex orthogonal representation Ad of the Galois group of the global field $K = k(t)$, which is unramified outside $t = 0$ and $t = \infty$. We say the homomorphism $g$ is *rigid* if the Artin-$L$-function $L(\mathrm{Ad}, s)$ is identically equal to 1.

If we identify the complex numbers with an algebraic closure of $\mathbb{Q}_\ell$, the representation Ad of $\pi_1(\mathbb{G}_m/k)$ gives a lisse $\lambda$-adic sheaf $\mathcal{F}$ of rank equal to $\dim(M)$ on $\mathbb{G}_m$. Let $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ be the inclusion. Then the middle extension $j_*\mathcal{F}$, whose first cohomology gives the image of cohomology with compact supports in the cohomology of $\mathcal{F}$ over $\mathbb{G}_m$, is pure of weight 0 on $\mathbb{P}^1$. By Grothendieck's cohomological formula for $L(\mathrm{Ad}, s) = L(j_*\mathcal{F}, s)$, the adjoint $L$-function is constant if and only if the $\ell$-adic cohomology groups of $j_*\mathcal{F}$ all vanish: $H^i(\mathbb{P}^1, j_*\mathcal{F}) = 0$ for $i = 0, 1, 2$. This implies that the $M$-local system on $\mathbb{G}_m$ has no deformations preserving the singularities at $t = 0, \infty$. Therefore rigid maps $g$ give rise to rigid $M$-local systems on $\mathbb{G}_m$.

First, from the determination of $L(V, s)$ for irreducible representations $V$ of $G$ in §1, we obtain

**Proposition.** *The homomorphism $g : G \to M$ is rigid if and only if*

1) $\mathrm{Ad}^G = 0$

2) *the restriction of Ad to $G$ does not contain any of the $(d-2)$ unipotent cuspidal representations which appear in $H^1(X)$.*

When $G = PGL_2(q)$ or $PSL_2(q)$, we have $d = 2$ and there are no unipotent cuspidal representations. Hence the only condition for rigidity is that $\mathrm{Ad}^G = 0$, or equivalently that the centralizer of the image of $G$ in $M$ is finite. We now describe some rigid embeddings in this case, by appealing to the work of Serre [S] and Greiss-Ryba[G-Ry] on finite subgroups of Lie groups.

**Proposition.** 1) *Assume that $G = PGL_2(q)$ and that $M$ is of adjoint type, with Coxeter number $h = q - 1$. Then there are rigid homomorphism*

$$g : G \to M.$$

2) *Assume $q \equiv 1 \pmod 2$, that $G = PSL_2(q)$, and that $M$ is of adjoint*

*type, with Coxeter number* $h = \frac{q-1}{2}$*. Then there are rigid homomorphisms*

$$g : G \to M.$$

**Proof.** If $q$ is a prime number $p$, Serre [S] has constructed are principal embeddings:

$$g : PGL_2(p) \to M \qquad p = h + 1,$$

$$g : PSL_2(p) \to M \qquad p = 2h + 1.$$

These satisfy $\mathrm{Ad}^G = 0$. The only case where $q \neq p$ in the exceptional series is for $M$ of type $F_4$ and $E_6$, where $h = 12$ and $2h+1 = 25 = 5^2$. In this case, a rigid embedding

$$g : PSL_2(25) \to M$$

has been constructed by [G-Ry].

Hence, we may assume that $M$ is an adjoint group of classical type $A, B, C, D$. We tabulate $M$, its Coxeter number $h$, and a central covering $\widetilde{M} \to M$ which is a connected classical group: $\widetilde{M} \subset GL(V)$.

| $M$ | $h$ | $\widetilde{M}$ |
|---|---|---|
| $PGL_n$ | $n$ | $GL_n$ |
| $PSp_{2n}$ | $2n$ | $Sp_{2n}$ |
| $SO_{2n+1}$ | $2n$ | $SO_{2n+1}$ |
| $PSO_{2n+2}$ | $2n$ | $SO_{2n+2}$ |

We will construct a rigid homomorphism $g : G \to M$ by constructing a complex representation $V$ of $G$ (or a cover $2.G$) with the appropriate dimension and self-duality.

Assume that $q = h + 1$. If $M = PGL_n$, we let $V$ be an irreducible representation of $PGL_2(q)$ in the discrete series. These representations are all of dimension $q - 1$, so give

$$G \to GL_{q-1} \to PGL_{q-1} = PGL_n = M$$

. Since $Ad = V \otimes V^* - 1$, we have $Ad^G = 0$.

In the other cases, $h = 2n$ is even and $q = h + 1 = 2n + 1$ is odd. For $M = SO_{2n+1}$, we let $S$ be the Steinberg representation of $G = PGL_2(q)$, which is orthogonal of dimension $q = 2n + 1$. The determinant $\epsilon = \det(S)$ is the nontrivial quadratic character of $G$, so the representation $V = S \otimes \epsilon$ gives a homomorphism

$$G \to SO_q = SO_{2n+1} = M.$$

In this case, $Ad = \wedge^2\ V$, so the irreducibility forces $Ad^G = 0$. We could also take $V = W \oplus \epsilon$, where $W$ is an (orthogonal) discrete series representation of dimension $q - 1 = 2n$ and determinant $\epsilon$. Then $\mathbb{A}d = \wedge^2\ W \oplus (W \otimes \epsilon)$ also has no $G$-invariants.

For $M = PSO_{2n+2}$ we can take either $V = St \oplus \epsilon$ or $V = W \oplus \epsilon \oplus 1$. Both are orthogonal representations of dimension $2n + 2$ and determinant 1, so give

$$g : G \to SO_{2n+2} \to PSO_{2n+2} = M$$

with $Ad^G = (\wedge^2 V)^G = 0$.

Finally, for $M = PSp_{2n}$ we take $V$ an irreducible representation of dimension $2n = q - 1$ of a double cover $\widetilde{G} = 2.G = 2.PSL_2(q).2$ which is isoclinic to the group $GL_2(q)/\left\{\begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}\right\}$. The representation $V$ has a real character, but is obstructed. It gives a symplectic representation:

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & Sp_{2n} \\ \downarrow & & \downarrow \\ G & \longrightarrow & PSp_{2n} \end{array}$$

The case when $q = 2h + 1$ is similar. Here we use the half-discrete series of dimension $\frac{q-1}{2}$ for $G = PSL_2(q)$ in the case where $\widetilde{M} = Sp$, and the half-principal series of dimension $\frac{q+1}{2}$ for $G = PSL_2(q)$ in the cases where $\widetilde{M} = SO$. We leave the details to the reader.

**Note.** In contrast to the case for $G = PGL_2(q)$ when $q = h + 1$, where there are many rigid embeddings, when $G = PSL_2(q)$ and $q = 2h+1$ we can only construct *two* rigid embeddings, into $M$, which are equivalent under the outer action of $PGL_2(q)$.

**Corollary.** *For the maps $g : G \to M$ described above, when $q = h + 1$ and $q = 2h + 1$, the restriction of $g \circ f$ to the decomposition group $D_\infty$ in* $\mathrm{Gal}(K^s/K)$ *is a simple wild parameter for the split, simply-connected group over the local field $K_\infty = k((1/t))$ which is dual to $M$.*

**Proof.** We must check the two conditions [G-Re,§5.6]:

$$\mathrm{Ad}^{D_\infty} = 0$$

$$sw_\infty(\mathrm{Ad}) = \ell = \mathrm{rank}(M).$$

The first follows from the fact that the Steinberg representation $St$ does not appear in Ad. In the exceptional cases, this follows from the work of Serre [S] and Greiss-Ryba [G-Ry], and in the classical cases from a determination of the decomposition of $S^2V$ and $\wedge^2V$. We then have the formula

$$sw_\infty(\mathrm{Ad}) = \dim \mathrm{Ad}^T.$$

But $T$ acts regularly on the discrete series representations, and this allows us to calculate $\dim \mathrm{Ad}^T = \ell$.

We now consider rigid embeddings of the groups $G = PU_3(q_0)$ and $G = PSU_3(q_0)$. The group $G = PU_3(q_0)$ has a unique unipotent cuspidal representation $V$ of dimension $2n = q_0(q_0-1)$. This representation has rational character, but is obstructed by the rational quaternion algebra ramified at $\infty$ and $p$ (cf. [G]). It therefore gives a homomorphism

$$g : G \to M = Sp(V) \hookrightarrow SL(V).$$

**Proposition.** 1) *The restriction of $g \circ f$ to the decomposition group $D_\infty$ in* $\mathrm{Gal}(K^s/K)$ *is a simple wild parameter for both groups $SO_{2n+1}$ and $PGL_{2n}$ over $K_\infty = \mathbb{F}_q((1/t))$.*

2) *The homomorphism $g$ is rigid.*

**Proof.** The restriction of $V$ to the subgroup $D_\infty = B = q_0^{1+2}(q_0^2-1)$ is irreducible, so there are no $B$-invariants in either $S^2V$ or in $(\wedge^2V)_0$. We have also seen that $sw_\infty(V) = 1$ and that $T$ acts as the regular representation on $V \oplus 1$.

We now show that

$$sw_\infty(S^2V) = \dim(S^2V)^T = n$$

$$sw_\infty(\wedge^2 V_0) = \dim(\wedge^2 V_0)^T = n - 1$$

which together imply that

$$L(S^2V, t) = L(\wedge^2 V)_0, t) = 1.$$

The $T$-invariants in $S^2V$ and in $(\wedge^2 V)_0$ follow immediately from the determination of the characters of $T$ on $V$. To calculate the Swan conductor, we observe that the group $B = q_0^{1+2}(q_0^2 - 1)$ has irreducible representations of dimensions $q_0^2 - q_0$, $q_0^2 - 1$, and 1, with Swan conductors 1, 1, and 0, respectively. The representations of dimension $q_0^2 - q_0$ have nontrivial central characters on $q_0^{1+2}$, and the representation of dimension $q_0^2 - 1$ has trivial central character, but is nontrivial on $q_0^{1+2}$. From this, the Swan conductors of $S^2V$ and $(\wedge^2 V)_0$ are easily determined.

Since the adjoint representations are

$$\mathrm{Ad} = S^2V \qquad \text{for } M = Sp(V)$$

$$\mathrm{Ad} = S^2V + (\wedge^2 V)_0 \qquad \text{for } M = SL(V)$$

We see that $g \circ f$ is a simple wild parameter for the groups $SO_{2n+1}$ and $PGL_2$ over $K_\infty$, and that the homomorphism $g$ is rigid. This completes the proof.

**Note.** When $q_0 = 8$, we have $2n = q_0^2 - q_0 = 56$. In this case, the subgroup $G_0 = PSU_3(8)$ of index 3 in $G$ stabilizes a symmetric quartic form on $V$, and we obtain an embedding [G-Ry]:

$$g : PSU_3(8) \to E_7 \subset Sp(V)$$

which is also rigid. The restriction of

$$\pi_1(\mathbb{G}_m/\mathbb{F}_{64}) \xrightarrow[f]{} PSU_3(8) \xrightarrow[g]{} E_7$$

to $D_\infty = 8^{1+2}.21 = 2^{3+6}.21$ is a simple wild parameter for $E_7$ over $\mathbb{F}_{64}((1/t))$. We have:

$$sw_\infty(V) = 3 \qquad \deg\ L(V,t) = 1$$

$$sw_\infty(\text{Ad}) = 7 \qquad \deg\ L(\text{Ad},t) = 0.$$

We have now constructed all of Katz's local systems for $G_2$, with the exception of the group $SL_2(8)$ in char 2. The example of $SL_2(8)$ fits into a series of embeddings [S, pg. 422]

| | | |
|---|---|---|
| $SL_2(4)$ | in | $SO_3 = A_1^{ad}$ |
| $SL_2(8)$ | in | $G_2$ |
| $SL_2(16)$ | in | $D_8^{ad}$ |
| $SL_2(32)$ | in | $E_8$ |
| $PSL_2(9)$ | in | $A_2^{ad}$ |
| $PSL_2(27)$ | in | $F_4$ |

where the image of the Sylow $p$-subgroup is not contained in a maximal torus. We leave it to the reader to check that these embeddings are all rigid (the first four in char 2 and the last two in char 3). In each case, the restriction to $D_\infty$ is a simple wild parameter.

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