On the Hilbert polynomials and Hilbert series of homogeneous projective varieties

Benedict H. Gross and Nolan R. Wallach [1]

Among all complex projective varieties $X \hookrightarrow \mathbb{P}(V)$, the equivariant embeddings of homogeneous varieties—those admitting a transitive action of a semi-simple complex algebraic group G—are the easiest to study. These include projective spaces, Grassmannians, non-singular quadrics, Segre varieties, and Veronese varieties. In Joe Harris' book "*Algebraic Geometry: A First Course*" [H], he computes the dimension $d = \dim(X)$ and degree $\deg(X)$ of $X \hookrightarrow \mathbb{P}(V)$ for many homogeneous varieties, in a geometric fashion.

In this expository paper we redo this calculation using some representation theory of G. We determine the Hilbert polynomial $h(t)$ and Hilbert series of the homogeneous coordinate ring of $X \hookrightarrow \mathbb{P}(V)$. Since

$$h(t) = \deg(X) \cdot \frac{t^d}{d!} + \text{(lower order terms)}$$

with $d = \dim(X)$, this gives a formula for the two invariants. As a byproduct, we find that $h(t)$ is the product of linear factors over $\mathbb{Q}$.

We now state the results precisely. Fix a maximal torus T contained in a Borel subgroup B of G. The projective varieties X which admit a transitive action of G correspond to the 2^n parabolic subgroups P of G which contain B (where $n = \dim(T)$). These varieties depend only on G up to isogeny, so there is no loss of generality in assuming that G is simply-connected, and we will henceforth do so.

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The equivariant projective embeddings π_λ of $X = G/P$ into $\mathbb{P}(V)$ then correspond bijectively to the dominant weights λ for T which lie in a certain face of the closed Weyl chamber corresponding to B.

The Hilbert polynomial $h_\lambda(t)$ of the coordinate algebra of $\pi_\lambda : X \hookrightarrow P(V)$ factors as the product

$$h_\lambda(t) = \prod_\alpha (1 + c_\lambda(\alpha)t).$$

This product is taken over the set of positive roots α of G which satisfy $\langle \lambda, \alpha^\vee \rangle \neq 0$; the number d of such roots is equal to the dimension of X. In the product, $c_\lambda(\alpha)$ is the positive rational number

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where ρ is half the sum of the positive roots and $\alpha^\vee$ is the corresponding co-root (cf. [B,VI.1]). Hence

$$\deg(X) = d! \prod_\alpha \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where the product is taken over the same subset of positive roots. This simple formula for the degree was obtained by Borel and Hirzebruch [B-H, Theorem 24.10], using characteristic classes for the compact form of G.

Using the same methods we also calculate the Hilbert series of the image of the equivariant embedding corresponding to λ, yielding

$$H_\lambda(q) = h_\lambda(q\frac{\mathrm{d}}{\mathrm{d}q})\frac{1}{1-q}.$$

After sketching the proof of these results, which follow from the Borel-Weil theorem and Weyl's dimension formula, we illustrate them by calculating the degrees and Hilbert series of several equivariant embeddings. We note that the calculations for many of the Hilbert series were done with the aid of a computer. One can find the mathematica code at http://math.ucsd.edu/∼nwallach/ .

1. Equivariant embeddings

Let G be a semi-simple, simply-connected, complex algebraic group. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup. The choice of B determines a set of positive roots for G—those characters of T which occur in $\mathrm{Lie}(B)/\mathrm{Lie}(T)$—as well as a Weyl chamber of dominant weights in the character group of T. We say a weight λ is dominant if the integer $\langle\lambda, \alpha^\vee\rangle$ is $\geqslant 0$ for all positive roots α. Here $\alpha^\vee \in \mathrm{Lie}(T)$ is the coroot associated with α. If ρ is half the sum of the positive roots, then ρ is a dominant weight in the interior of the Weyl chamber: $\langle\rho, \alpha^\vee\rangle$ is strictly positive for all positive roots α.

Associated to every dominant weight λ for T there is an irreducible representation $V = V_\lambda$ of G over $\mathbb{C}$ with highest weight λ for B (cf. [F-H] for an introduction to this theory). Let V^* be the dual representation, and let $\langle f\rangle$ be the unique line in $V^* = \mathrm{Hom}(V, \mathbb{C})$ fixed by B; the character of T on this line is $i(\lambda)$, where i is the opposition involution of G. Let $P \supset B$ be the parabolic subgroup of G which stabilizes the line $\langle f\rangle$ in V^*, or equivalently which stabilizes the hyperplane H annihilated by f in V.

Let $\mathbb{P}(V)$ denote the projective space of *all* hyperplanes in V. This has homogeneous coordinate ring

$$\mathbb{C}[\mathbb{P}(V)] = \mathrm{Sym}^\bullet(V) = \bigoplus_{n\geqslant 0} \mathrm{Sym}^n(V)$$

Associated to λ, we have the equivariant embedding

$$\pi_\lambda : X = G/P \hookrightarrow \mathbb{P}(V)$$

defined by mapping the coset gP to the hyperplane $g(H)$. The image of π_λ is the unique closed orbit of G on $\mathbb{P}(V)$, and is a homogeneous, nonsingular projective variety [F-H,pg. 384].

2. The Hilbert polynomial and Hilbert series

We will start with some generalities on Hilbert polynomials and Hilbert series of projective embeddings. Let X be a projective variety and let $\pi : X \hookrightarrow \mathbb{P}(V)$ be an embedding of varieties over an algebraically closed field F. Let Y denote the cone in V on πX and let $F^j[\pi X]$ be the F vector space of all regular functions on Y which are homogeneous of degree j. (This vector space is denoted $S^j(X)$ in [H]. It is the quotient of the vector space $\mathrm{Sym}^j(V)$ by the subspace of homogenous polynomials of degree j which vanish on πX.)

Then the Hilbert polynomial of the embedding is the unique polynomial $h(t)$ in $\mathbb{Q}[t]$ such that

$$\dim F^j[\pi X] = h(j), \text{for all } j \gg 1.$$

The existence of such a polynomial is established in [H, Proposition 13.2], which also shows that the degree of $h(t)$ is the dimension of X.

Let $\mathcal{L}^j$ be the pullback of the Serre twist $\mathcal{O}(j)$ of the structure sheaf of $\mathbb{P}(V)$. If X is projectively normal (cf. [Ha,pg.126]), then for all $j \geqslant 0$ we have

$$\dim H^0(X, \mathcal{L}^j) = \dim F^j[\pi X].$$

Another basic invariant of the projective embedding of X is its Hilbert series. This is the formal sum

$$H(q) = \sum_{j \geq 0} \dim F^j[\pi X] q^j.$$

Serre's vanishing theorem implies that for $j \gg 1$ one has $H^i(X, \mathcal{L}^j) = 0$ for all $i > 0$. This implies that the Hilbert series is given by

$$H(q) = \sum_{j \geq 0} h(j) q^j + p(q)$$

with $p(q) \in \mathbb{Z}[q]$.

We set $\tilde{H}(q) = \sum_{j \geq 0} h(j) q^j$. If $h(t) = \sum c_k t^k$ then we see that

$$\tilde{H}(q) = \sum c_j \sum_{m \geq 0} m^j q^m.$$

The inner series has a long history (cf. [St, 1.3]). We will make a short digression on some of its properties. Obviously it is the formal expansion of

$$f_j(q) = \left(q \frac{d}{dq}\right)^j \frac{1}{1-q} = \frac{\phi_j(q)}{(1-q)^{j+1}}$$

and since $q\frac{d}{dq}$ preserves degree, $\phi_j(q)$ is a polynomial of degree j. We write

$$\phi_j(q) = \sum a_{j,i} q^i.$$

We note that $a_{j,0} = 0$. If we arrange the $a_{j,i}$ with $i = 1, ..., j$ in a triangle with jth row $a_{j,1}, ..., a_{j,j}$ we have

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

This triangle is called Euler's triangle and it has been studied intensively. We note one property. Consider the diagonals (the second diagonal is 1,4,11,26,...) then the element in the ith diagonal and the nth row is the number of permutations with exactly i descents.

The upshot is that we have

$$H(q) = p(q) + \sum c_j \frac{\phi_j(q)}{(1-q)^{j+1}} = \frac{g(q)}{(1-q)^{d+1}}$$

with $d = \dim X$ and

$$g(q) = \sum_{j=0}^{d} c_j \phi_j(q)(1-q)^{d-j} + (1-q)^{d+1} p(q).$$

We now return to the main point of this paper. We will henceforth assume that $F = \mathbb{C}$. Fix an equivariant embedding

$$\pi_\lambda : X \hookrightarrow \mathbb{P}(V_\lambda).$$

The line bundle $\mathcal{L} = \pi^* \mathcal{O}(1)$ on X is equivariant and has sections

$$H^0(X, \mathcal{L}) = V_\lambda$$

Then $\mathcal{L}^n = \pi^* \mathcal{O}(n)$ is also equivariant, and by the theorem of the highest weight

$$H^0(X, \mathcal{L}^n) = V_{n\lambda}$$

for all $n \geqslant 0$ (cf. [B-H,p.393]).

Since the restriction homomorphism ($n \geqslant 0$)

$$\begin{tikzcd} H^0(\mathbb{P}(V_\lambda), \mathcal{O}(n)) \arrow[r] \arrow[d, equal] & H^0(X, \mathcal{L}^n) \arrow[d, equal] \\ \mathrm{Sym}^n(V_\lambda) \arrow[r] & V_{n\lambda} \end{tikzcd}$$

is G-equivariant and non-zero, and $V_{n\lambda}$ is irreducible, it must be surjective for all $n \geqslant 0$.

Hence the embedding of X is projectively normal and the homogeneous coordinate ring of the embedding is

$$\mathbb{C}[\pi_\lambda X] = \bigoplus_{n \geqslant 0} V_{n\lambda}.$$

In particular, the Hilbert polynomial $h_\lambda(t)$ of $\pi_\lambda : X \hookrightarrow \mathbb{P}(V)$ satisfies

$$h_\lambda(n) = \dim V_{n\lambda}$$

for $n \gg 0$.

But the Weyl dimension formula [S] states that

$$\dim V_{n\lambda} = \prod_{\alpha > 0} \frac{\langle n\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where the product is taken over all positive roots α. Hence

$$\dim V_{n\lambda} = \prod_{\alpha > 0} (1 + n \cdot c_\lambda(\alpha))$$

with

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}.$$

Therefore the polynomial

$$h_\lambda(t) = \prod_{\alpha > 0} (1 + t \cdot c_\lambda(\alpha))$$

satisfies $h_\lambda(n) = \dim V_{n\lambda}$ for all $n \geqslant 0$. This completes the determination of the Hilbert polynomial of $X \hookrightarrow \mathbb{P}(V)$ using representation theory. We note that Manivel

[M] has established some interesting results on the roots of this polynomial, for maximal parabolic subgroups P.

We now turn to the study of the Hilbert series. The fact that $h(n)$ gives the dimension of $F^n[\pi X]$ for all $n \geqslant 0$ implies that the polynomial $p(q)$ in the general Hilbert series is zero in this case. This yields the following formula for the Hilbert series

$$H(q) = \sum_{n\geq 0} \left(\prod_{\langle \lambda, \alpha^\vee \rangle > 0} \frac{\langle n\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \right) q^n = \sum_{n\geq 0} \left(\prod_{\langle \lambda, \alpha^\vee \rangle > 0} (nc_\lambda(\alpha) + 1) \right) q^n.$$

Let $\beta_1, \ldots, \beta_d$ be an enumeration of the set of roots α such that $\langle \lambda, \alpha^\vee \rangle > 0$. Let e_j be the jth elementary symmetric function in d variables. Then we have (after a bit of manipulation and in the notation of the beginning of this section)

$$H(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) f_j(q).$$

By the above, this implies that $H(q) = \frac{g(q)}{(1-q)^{d+1}}$ with

$$g(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) \phi_j(q)(1-q)^{d-j}.$$

In particular, since $g(1) = \deg \pi_\lambda$, we have the formula

$$\deg \pi_\lambda = e_d(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) \phi_d(1).$$

This agrees with the formula for the degree in the introduction since $\phi_d(1) = d!$.

There is another more geometric way of writing the above formula for $H(q)$. We note that if we consider the case of the standard Segre embedding of $(\mathbb{P}^1)^j = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (j copies) into $\mathbb{P}(\otimes^j \mathbb{C}^2)$ then the Hilbert series is

$$\sum_{n\geq 0} (n+1)^j q^n = \frac{\phi_j(q)/q}{(1-q)^{j+1}}.$$

So the degree of this embedding is $d!$. This also says that the formula above for $H(q)$ expresses the Hilbert series of $\pi_\lambda(G/P)$ in terms of the Hilbert series of $(\mathbb{P}^1)^j$ for $j = 1, \ldots, d$. The simplest example is the case of G/B with $\lambda = \rho$. Then the formula becomes

$$H_{G/B}(q) = H_{(\mathbb{P}^1)^d}(q)$$

with d equal to the number of positive roots.

We also record the following result.

Theorem. *The Hilbert series of the embedding* π_λ *of* G/P *is*

$$\prod_{\langle \lambda, \check{a} \rangle > 0} \left(\frac{\langle \lambda, \check{a} \rangle}{\langle \rho, \check{a} \rangle} q \frac{d}{dq} + 1 \right) \frac{1}{1-q}.$$

3. Veronese varieties

For the first examples, we observe that the variety X remains unchanged as we scale λ by an integer $m \geqslant 1$. If $\dim(X) = d$, then

$$\deg(\pi_{m\lambda}) = m^d \cdot \deg(\pi_\lambda)$$

as every factor $c(\alpha)$ in the product for the degree is scaled by m.

We apply this to $G = SL(V)$ and $V = V_\lambda$ the standard representation. Then $X = \mathbb{P}(V) = \mathbb{P}^n$, where $\dim(V) = n + 1$, and $\deg(\pi_\lambda) = 1$. Hence the Veronese embedding

$$\pi_{m\lambda} : \mathbb{P}^n \longrightarrow \mathbb{P}(\mathrm{Sym}^m V) = \mathbb{P}^{\binom{m+n}{n} - 1}$$

has degree $= m^n$.

For $n = 1$, this is the rational normal curve, of degree m in $\mathbb{P}^m$. For $n = 2$ and $m = 2$ this gives the degree $(= 4)$ of the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

4. The flag variety

Another simple case is the embedding of the full flag variety $X = G/B$ using the representation V_ρ. (The dominant weight ρ is the simplest weight in the interior of the Weyl chamber; the stabilizer of its highest weight vector $\langle v_\rho \rangle$ is equal to B.)

In this case, $\dim(V_\rho) = 2^d$ by the Weyl dimension formula, where $d = \dim(X)$ is the number of positive roots. Moreover, for every positive root α we have

$$c_\rho(\alpha) = \frac{\langle \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = 1.$$

Hence $h_\rho(t) = (t+1)^d$ (the corresponding Hilbert series has been studied in section 2) and

$$\pi_\rho : X = G/B \hookrightarrow \mathbb{P}^{2^d - 1}$$

has degree $= d!$.

It is interesting to compare these results with the linear system $|2\Theta|$ on a principally polarized abelian variety A of dimension d, which maps $A \to \mathbb{P}^{2^d-1}$ with degree $2^d \cdot d!$.

5. Segre varieties

We next consider the representation of $G = SL(W) \times SL(U)$ on $V = \mathrm{Hom}(W, U) = V_\lambda$. The closed orbit X of G on $\mathbb{P}(V)$ consists of the linear maps of rank 1; this gives the Segre embedding

$$\pi_\lambda = \pi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$$

where $n + 1 = \dim(W)$ and $m + 1 = \dim(U)$.

Let $\{e_1, \ldots, e_{n+1}\}$ be the weights for $SL(W)$ on W and $\{f_1, \ldots, f_{m+1}\}$ be the weights for $SL(U)$ on U. The highest weight of $V_\lambda = W^* \otimes U = \bigwedge^n W \otimes U$ is

$$\lambda = (e_1 + e_2 + \cdots + e_n) + f_1.$$

There are $(n+m) = d$ positive roots α with $c_\lambda(\alpha) \neq 0$:

$$\begin{aligned} \alpha &= e_i - e_{n+1} \qquad & i &= 1, 2, \ldots, n \\ \alpha &= f_1 - f_j \qquad & j &= 2, 3, \ldots, m+1. \end{aligned}$$

Since $\rho = ne_1 + (n-1)e_2 + \cdots + e_n + mf_1 + (m-1)f_2 + \cdots + f_m$ we find

$$\begin{aligned} c_\lambda(\alpha) &= \frac{1}{(n+1-i)} \qquad && \text{in the first case} \\ &= \frac{1}{(j-1)} \qquad && \text{in the second case.} \end{aligned}$$

Hence

$$\begin{aligned} \deg(\pi_{n,m}) &= d! \prod c_\lambda(\alpha) \\ &= (m+n)! \cdot \frac{1}{n!} \cdot \frac{1}{m!} \\ &= \binom{m+n}{n}. \end{aligned}$$

For example, the degree of the Segre 3-fold

$$\pi_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

is equal to $\binom{3}{1} = 3$.

Using the same data we can compute the Hilbert series of this embedding of $\mathbb{P}^n \times \mathbb{P}^m$ for $n \leq m$ yielding

$$\frac{\sum_{1 \leq j \leq n} \binom{n}{j} \binom{m}{j} q^j}{(1-q)^{n+m+1}}.$$

6. Grassmannians

We now consider the Plucker embedding of the Grassmannian $G(k,n)$ of $(n-k)-$ planes (i.e., subspaces of codimension k) in $\mathbb{C}^n$. In this case $G = SL_n$ and $V = V_\lambda = \overset{k}{\bigwedge}\mathbb{C}^n$.

The highest weight λ of V is

$$\lambda = e_1 + e_2 + \cdots + e_k$$

(here and below we will be using e_i to mean the Bourbaki ϵ_i restricted to the trace 0 diagonal matrices) and there are $d = k(n-k)$ positive roots α with $c_\lambda(\alpha) = \langle \lambda, \alpha^\vee\rangle / \langle \rho, \alpha^\vee\rangle$ non-zero. We recall that

$$\rho = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}.$$

The relevant roots are those of the form $\alpha = e_i - e_j$ with $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$. All of these roots have $\langle \lambda, \alpha^\vee\rangle = 1$, and we find that $c_\lambda(\alpha) = 1/(j-i)$. Hence

$$\begin{aligned} \deg(G(k,n)) = & \qquad d! \prod_{\substack{1\leqslant i\leqslant k \\ k+1\leqslant j\leqslant n}} \frac{1}{(j-i)} \\ = & \quad (k(n-k))! \prod_{1\leqslant i\leqslant k} \frac{(k-i)!}{(n-i)!}. \end{aligned}$$

For example, the degree of $X = Gr(2, n+2)$ in $\mathbb{P}(\overset{2}{\bigwedge}\mathbb{C}^{n+2}) = \mathbb{P}^{(n^2+3n)/2}$ is equal to

$$(2n)!\frac{1}{(n+1)!}\frac{1}{n!} = \frac{1}{n+1}\binom{2n}{n},$$

the Catalan number c_n.

The corresponding Hilbert series for X is

$$\frac{\sum_{1\leq j\leq n} \frac{1}{n}\binom{n}{j}\binom{n}{j-1} q^{j-1}}{(1-q)^{2n+1}}.$$

The polynomial in the numerator has coefficients the Narayana numbers. If these numbers are laid out in a triangle they yield the so called Catalan triangle. For details on Narayana and Catalan polynomials see [St].

A similar case is the Lagrangian Grassmannian X of maximal isotropic subspaces (of dimension n) in a symplectic space of dimension $2n$. Here $G = Sp_{2n}$ and $V = V_\lambda = \bigwedge^{n} \mathbb{C}^{2n} - \bigwedge^{n-2} \mathbb{C}^{2n}$ has dimension $\frac{1}{(n+2)(n+1)}(4n+2)\binom{2n}{n}$. The highest weight is $\lambda = e_1 + e_2 + \cdots + e_n$ and there are $d = n(n+1)/2$ positive roots with $c_\lambda(\alpha)$ non-zero. These roots have the form $\alpha = e_i + e_j$ with $1 \leqslant i \leqslant j \leqslant n$. We have $c_\lambda(\alpha) = 2/(2n+2-i-j)$, so

$$deg(X) = 2^d d! \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{(2n+2-i-j)}.$$

7. Two exceptional homogeneous varieties

We now consider an exceptional variety $X \hookrightarrow \mathbb{P}^{26}$ of dimension $d = 16$. Here $G = E_6$ and $V = V_\lambda$ is a minuscule representation of dimension 27. In the notation of [B] the positive roots α with $\langle \lambda, \alpha^\vee \rangle \neq 0$ have the form

$$\alpha = \frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i \right)$$

with $\sum_{i=1}^{5} \nu(i)$ even. They all satisfy $\langle \lambda, \alpha^\vee \rangle = 1$ so it suffices to compute their inner products with

$$\rho = 4(e_8 - e_7 - e_6 + e_5) + 3e_4 + 2e_3 + e_2$$

We find

$$\langle \rho, \alpha^\vee \rangle = 6 + 2(-1)^{\nu(5)} + \frac{3}{2} \cdot (-1)^{\nu(4)} + (-1)^{\nu(3)} + \frac{1}{2}(-1)^{\nu(2)}$$

$\nu(5)$	$\nu(4)$	$\nu(3)$	$\nu(2)$	$\langle \rho, \alpha^\vee \rangle$
0	0	0	0	11
0	0	0	1	10
0	0	1	0	9
0	0	1	1	8
0	1	0	0	8
0	1	0	1	7
0	1	1	0	6
0	1	1	1	5
1	0	0	0	7
1	0	0	1	6
1	0	1	0	5
1	0	1	1	4
1	1	0	0	4
1	1	0	1	3
1	1	1	0	2
1	1	1	1	1

A table of these inner products appears below, and we find

$$\begin{aligned}\deg(X) &= 16!/11!(8.7.6.5.4) \\ &= 16.15.14.13.12./8.7.6.5.4 \\ &= 78\end{aligned}$$

Is there any reason that this degree is equal to the dimension of the algebraic group E_6 which acts on X? This question also appears in [I-L].

We note that if we use the formula for the Hilbert series of this embedding and the above table we find the formula

$$\frac{1 + 10q + 28q^2 + 28q^3 + 10q^4 + q^5}{(1-q)^{17}.}$$

Similarly, for the minuscule representation V_λ of dimension 56 for the exceptional group E_7, we find that X has dimension $d = 27$ and degree $= 13110 = 2.3.5.19.23$ in $\mathbb{P}^{55}$. The Hilbert series of this embedding is given by the formula

$$\frac{(1 + 28q + 273q^2 + 1248q^3 + 3003q^4 + 4004q^5 + 3003q^6 + 1248q^7 + 273q^8 + 28q^9 + q^{10})}{(1-q)^{28}.}$$

Finally, the degree of the homogeneous variety X of dimension $d = 57$ corresponding to the adjoint representation V_λ (of dimension 248) of the exceptional group E_8 is equal to $126937516885200 = 2^4.3^2.5^2.7.31.37.41.43.47.53$.

8. Bibliography

[A-M] Atiyah, M.F. and I.G.MacDonald. Introduction to commutative algebra. Westview Press, 1969.

[B-H] Borel, A. and F. Hirzebruch. Characteristic classes and homogeneous spaces II. Amer. J. Math. 81 (1959).

[B] Bourbaki, N. Lie groups and Lie algebras. Springer-Verlag, 2002.

[F-H] Fulton, W. and J. Harris. Representation Theory. Springer GTM 129, 1991.

[H] Harris, J. Algebraic Geometry. Springer GTM 133, 1992.

[Ha] Hartshorne, R. Algebraic Geometry. Springer GTM 52, 1077.

[I-M Iliev, A. and Manivel, L. On the Chow ring of the Cayley plane. Compositio 141 (2005), 146–160.

[M] Manivel, L. The canonical strip hypothesis for complete intersections in homogeneous spaces. Preprint. ArXiv .0904.2470

[S] Serre, J.-P. Complex semi-simple Lie algebras. Springer-Verlag, 1987.

[St] Stanley, R. Enumerative combinatorics, Volume 1, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 1997.

Benedict H. Gross, Department of Mathematics, Harvard University, Cambridge, MA 02138 gross@math.harvard.edu

Nolan R. Wallach, Department of Mathematics, UCSD, La Jolla, CA, 92093 nwallach@math.ucsd.edu (This research was partially supported by NSF grant DMS-0200305.)