

# On the exceptional series, and its descendants

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Résumé. Les articles [1], [2], [3] exhibent des ressemblances entre les propriétés des représentations adjointes des groupes de la série exceptionnelle. Nous obtenons des ressemblances analogues pour certaines représentations préférées de séries de sous-groupes. L’algèbre tensorielle de ces représentations est plus accessible (cf. [4], [5], [7]). Ceci pourrait aider à comprendre ce qui se passe.

Les sous-groupes en question forment un “triangle magique” qui prolonge le carré magique d’algèbres de Lie de Freudenthal. Nous décrivons ces sous-groupes en termes de paires duales, et leur représentations préférées en termes de leur action sur l’algèbre de Lie du groupe de la série exceptionnelle ambiant.

Abstract. Many of the striking similarities which occur for the adjoint representation of groups in the exceptional series (cf. [1], [2], [3]) also occur for certain representations of specific reductive subgroups. The tensor algebras on these representations are easier to describe (cf. [4], [5], [7]), and may offer clues to the original situation.

The subgroups which occur form a Magic Triangle, which extends Freudenthal’s Magic Square of Lie algebras. We describe these groups from the perspective of dual pairs, and their representations from the action of the dual pair on an exceptional Lie algebra.

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We consider the series of algebraic groups

$$(1) \quad e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

where an indexed capital letter stands for the corresponding simple simply connected complex group. Each group is a subgroup of the next, the inclusion morphism being unique up to conjugacy. The inclusions of the trivial group  $e$  in  $A_1$  in  $A_2$  and of  $E_6$  in  $E_7$  in  $E_8$  are given by inclusions of Dynkin diagrams. The long roots of  $G_2$  (resp.  $F_4$ ) form a root system of type  $A_2$  (resp.  $D_4$ ), providing  $A_2 \subset G_2$  and  $D_4 \subset F_4$ . Finally,  $G_2$  (resp.  $F_4$ ) is the subgroup of  $D_4$  (resp.  $E_6$ ) fixed by the group  $S_3$  (resp.  $S_2$ ) of automorphisms respecting a pinning (épinglage).

The centralizers in  $E_8$  of the subgroups (1) form (a conjugate of) the same series of subgroups of  $E_8$ , taken in the reverse order, giving rise to the dual pairs  $(e, E_8)$ ,  $(A_1, E_7)$ ,  $(A_2, E_6)$ ,  $(G_2, F_4)$  and  $(D_4, D_4)$  in  $E_8$ .

For each pair  $H \subset K$  of distinct subgroups in (1), table 1 gives the centralizer of  $H$  in  $K$ . Rows are indexed by  $H$ , columns by  $K$ . If  $H'$  and  $K'$  are the centralizers of  $H$  and  $K$  in  $E_8$ , the centralizer  $H' \cap K'$  of  $H$  in  $K$  is also the centralizer of  $K'$  in  $H'$ . This explains the symmetry of the table. If we go over to Lie algebras, the lower right  $4 \times 4$  square is Freudenthal's "magic square" of Lie algebras.

For each pair  $H \subset K$ ,  $H$  injects into the automorphism group  $\text{Aut } k$  of the Lie algebra  $k$  of  $K$ . Our *magic triangle* is the system of the centralizers  $G$  of  $H$  in  $\text{Aut } k$ . As  $H$  is connected, the centralizer of  $H$  in the adjoint group  $K^{\text{ad}}$  is the image of the centralizer of  $H$  in  $K$ : to obtain  $G$  from the corresponding entry of table 1, one divides by the center of  $K$  and takes an extension with, as quotient group, the group of outer automorphisms of  $K$ . This modifying isogeny is indicated under each column. In all but one case, the required extension is a semi-direct product, with the quotient group acting as a group of automorphisms respecting a pinning. The exception is the nonsplit extension  $(\mathbb{G}_m/\mu_3).2$  at  $A_1 \subset A_2$ .

As  $H$  injects into  $\text{Aut}(k)$ , the center  $Z(H)$  of  $H$  is a central subgroup of  $G$ . It is shown at the right of table 1. In most cases,  $G$  and  $H$  form a dual pair in  $\text{Aut}(k)$ , that is, they are each other's centralizer, hence have the same center. The exceptions are  $E_6 \subset E_7$  and  $F_4 \subset E_6$ .

The last row of the magic triangle, corresponding to  $H = e$ , is the exceptional series of [2]: the groups of automorphisms of the Lie algebras  $a_1, a_2, g_2, d_4, f_4, e_6, e_7, e_8$ . Under each column, we also give the value of a parameter  $\nu$ . For the exceptional series, it is  $h^\vee/6$ , for  $h^\vee$  the dual Coxeter number: the inverse of the parameter  $\mu$  used in [3].

For each pair  $H \subset K$ , giving rise to  $G$ , the representation of  $G \times H$  on  $k = \text{Lie}(K)$  has the following structure:

$$k \simeq g \otimes 1 + 1 \otimes h + \sum V_\alpha \otimes W_\alpha,$$

with  $W_\alpha$  running over the irreducible representations of  $H$  occurring in  $k/(g + h)$ . If  $H = A_1, A_2, D_4, E_6$  or  $E_7$ , the  $W_\alpha$  are the minuscule representations of  $H$ . They can be indexed by their central character, which is an arbitrary nontrivial character of the center, and are conjugate by the outer automorphisms of  $H$ . For  $H = G_2$  or  $F_4$ , there is only one  $W$ : the representation whose nonzero weights are the short roots of  $H$ .

The  $V_\alpha$  are always irreducible. In this way, for each row of the magic triangle, we obtain a system of irreducible representations of  $G$ . When  $H$  is simply laced, it is indexed by the nontrivial characters of  $Z(H)$ . When  $H = G_2$  or  $F_4$ , it is reduced to a single representation. For  $H = e$ , it is empty. We call those representations the *preferred* representations.

The preferred representations of  $G$  are conjugate by outer automorphisms and, when there are two of them, are in duality. In particular, the preferred representations  $V$  of  $G$  are all of the same dimension, given in table 2. When the restriction  $V|_{G^0}$  of  $V$  to the centralizer  $G^0$  of  $H$  in  $K^{\text{ad}}$  is reducible, the dimension is given in the form  $a.b$ ,  $a$  being the length of  $V|_{G^0}$  and  $b$  the dimension of its irreducible constituents. At  $G_2 \subset D_4$  (resp.  $F_4 \subset E_6$ ), we get the irreducible representation of  $S_3$  of dimension 2 (resp. the sign character of  $S_2$ ). In all other cases, there is a subgroup  $A$  of index  $a$  of  $\text{Out}(K)$  such that  $V$  is induced from a representation  $V_1$  of  $A \times G^0$  with  $A$  acting trivially on the dominant weight subspace of  $V_1$ . As could be inferred from the knowledge of central characters, at  $A_1 \subset F_4$  (resp.  $G_2 \subset E_7$ ), the dominant weight of the representation of  $C_3$  is  $[0, 0, 1]$  (resp.  $[0, 1, 0]$ ). For the rows 1 to 7, the group  $G$  is the subgroup of  $\prod \text{GL}(V_\alpha)$  fixing tensors whose types depend only on the row. The number of preferred representations, and those types, are given in table 2 as well; an entry such as  $S^3$  stands for  $S^3(V_\alpha)$ , all  $\alpha$ .

One of us (B.G.) first encountered the Magic Triangle (for Lie algebras) in the Harvard Ph.D. thesis of K. Rumelhart. The approach to it here was suggested by the work of G. Savin on dual pairs.

In each row, the dimension of a preferred representation is a linear function of  $\nu$ : for the row of centralizers to  $E_7, \dots, A_1$ , it is

$$(2) \quad \begin{array}{cccc} E_7: \nu - 3 & E_6: \nu - 2 & F_4: 2\nu - 3 & D_4: 2\nu - 2 \\ G_2: 6\nu - 4 & A_2: 6\nu - 3 & A_1: 12\nu - 4 & . \end{array}$$

If, in the definition of the magic triangle, one allows  $H = K$ , prefixing each row by a trivial group  $e$  and taking for its preferred representation(s) the zero representation(s), the dimension formulas (2) remain valid.

Other representations correspond to each other across the groups in a row. They give rise to uniform decomposition formulas, and their dimensions are the values at  $\nu(G)$  of a rational function, which is the quotient of products of rational linear factors. The case of the last row, the exceptional series, is considered in [1], [2], [3]; the adjoint representations correspond to each other, and they have the dimension  $2(5\nu - 1)(6\nu + 1)/(\nu + 1)$ . This formula, due to Vogel, was the beginning of the story on the exceptional series. The case of the seventh row, corresponding to  $H = A_1$ , is strikingly similar. The vertices of the Dynkin diagram  $E_7$  are labelled

$$\begin{array}{cccccc} 2 & 6 & 7 & 5 & 3 & 1 \\ & & 4 & & & \end{array},$$

and we write  $\alpha(i)$ ,  $\Omega_i$ ,  $s(i)$  for the simple roots, fundamental weights, and generators of the Weyl group so numbered. Let  $V_r$  be the ( $\mathbb{Q}$ -) linear span of the  $\Omega_i$  ( $i \leq r$ ); it is the orthogonal complement of the  $\alpha(i)$ ,  $i > r$ . For  $r = 1, 3, 5, 6$  the trace on  $V_r$  of the  $E_7$ -root

system is empty, resp. a root system  $A_1^3, A_5, D_6$ . We take as its positive roots those which are positive for  $E_7$ . For  $r = 1, 3$ , the orthogonal projection to  $V_r$  of the  $E_7$ -root system is a root system  $A_1$ , resp.  $C_3$ . We take as positive roots the nonzero projections of positive roots. For  $G$  in row seven and of rank  $r$ , this construction allows us to view the  $\Omega_i$  ( $i \leq r$ ) as dominant weights of the connected component  $G^0$  of  $G$ . To an irreducible representation  $V(\lambda)$  of  $E_7$ , whose dominant weight  $\lambda$  is linear combination of the  $\Omega_i$  ( $i \leq r$ ), will correspond the following representation  $V(G, \lambda)$  of  $G^0$ : take the irreducible representation  $V(G^0, \lambda)$  of the “same” dominant weight  $\lambda$ , and the sum of its distinct conjugates by the component group  $G/G^0$ . This representation is the restriction to  $G^0$  of an irreducible representation of  $G$ . We expect there is a best choice for this representation of  $G$ , giving rise to nicer decomposition formulas, but we have not pinned it down. Corresponding representations have the same central characters  $\chi$ , and are orthogonal (resp. symplectic) when  $\chi$  is the trivial (resp. nontrivial) character of  $\mu_2$ .

For each  $G$ ,  $V(\Omega_1)$  is the preferred representation. For  $\text{rank}(G) \geq 2$ ,  $V(\Omega_2)$  is the adjoint representation, and  $S^2V(\Omega_1) = V(2\Omega_1) + V(\Omega_2)$ . For  $\text{rank}(G) \geq 3$ ,  $\wedge^2 V(\Omega_1) = V(\Omega_3) + 1$ .

For  $\text{rank}(G) \geq 3$  and  $p \geq 1$ , one has

$$V(\Omega_1) \otimes V(p\Omega_1) = \sum V(p\Omega_1 + \mu)$$

with  $\mu$  running over the weights  $\Omega_1, -\Omega_1, \Omega_2 - \Omega_1$  and  $\Omega_3 - \Omega_1$ . Experiments using the program LiE suggest that more generally

$$V(p\Omega_1) \otimes V(q\Omega_1) = \sum V(a\Omega_1 + b\Omega_2 + c\Omega_3),$$

the sum being over the triples  $(a, b, c)$  with  $a$  of the same parity as  $p+q$  and  $|p-q| \leq a \leq p+q - 2(b+c)$ , and that for  $\text{rank}(G) \geq 4$ , a similar formula holds for  $V(p\Omega_1) \otimes V(q\Omega_2)$ .

We will exhibit rational functions  $D(\lambda; X)$  depending on a dominant weight  $\lambda$  of  $E_7$  which is a linear combination of the  $\Omega_i$  ( $i \leq 6$ ), with the property that if the coefficient of  $\Omega_i$  in  $\lambda$  vanishes whenever  $i > \text{rank}(G)$ , then

$$\dim V(G, \lambda) = D(\lambda, \nu(G))$$

for  $\nu(G)$  the parameter of  $G$ , given in table 1.

To describe  $D(\lambda; X)$ , we will use the language of multisets (sets with integer multiplicities, positive or negative). For  $\mathcal{E}$  a set, a multiset  $E$  in  $\mathcal{E}$  is simply a function  $m_E: \mathcal{E} \rightarrow \mathbb{Z}$ , its multiplicity function. We will only use multisets whose multiplicity function has finite support. For  $e$  in  $\mathcal{E}$ , the characteristic function of  $\{e\}$  is noted  $\delta[e]$ . Multisets can be added, and  $E = \sum m_E(e)\delta[e]$ .

Let  $(, )$  be the Weyl group invariant symmetric bilinear form on the  $E_7$  root system for which roots have square length 2. It induces a bilinear form  $(, )$  on  $V_r$ , and hence on the root systems  $A_1, A_1^3, C_3, A_5, D_6$ . For those of rank  $\geq r$ , define

$$\begin{aligned} W(G^0, \lambda) &= (\text{multiset of the } (\lambda + \rho, \alpha), \text{ for } \alpha \text{ a positive root}) \\ &\quad - (\text{multiset of the } (\rho, \alpha), \text{ for } \alpha \text{ a positive root}) \end{aligned}$$

In this definition,  $\rho$  is the sum of fundamental weights of  $G^0$ , and the multiplicity  $m(x)$  of a rational number  $x$  in  $W(G^0, \lambda)$  is the number of positive roots such that  $(\lambda + \rho, \alpha) = x$ , minus the number of positive roots such that  $(\rho, \alpha) = x$ . Weyl's dimension formula says

$$\dim V(G^0, \lambda) = \prod x \quad (x \text{ in } W(G^0, \lambda)) := \prod x^{m(x)}.$$

It is not only the integers  $\dim V(G, \lambda)$  which we will interpolate from group to group, but also the multisets  $W(G^0, \lambda)$ . Define  $W'(G^0, \lambda)$  by its multiplicity function

$$(3) \quad m'(x) := m(x) - m(x+1).$$

One has  $m(x) = \sum m'(x+n)$  (sum over  $n \geq 0$ ). We will exhibit a multiset  $\mathcal{W}'$  of affine linear forms in the coordinates  $p_i$  ( $i \leq 6$ ) of  $\lambda = \sum p_i \Omega_i$ , and an indeterminate  $X$ , such that each  $W'(G^0, \lambda)$  is deduced from  $\mathcal{W}'$  by specialization at  $\lambda$  and  $\nu(G)$ . Further, for a given dominant weight  $\lambda = \sum p_i \Omega_i$ , if we specialize  $\mathcal{W}'$  at  $\lambda$  to obtain a multiset  $\mathcal{W}'_\lambda$  of linear forms in  $X$ ,  $\mathcal{W}'_\lambda$  is derived by (3) from a finite multiset  $\mathcal{W}_\lambda$ , and

$$D(\lambda, X) = \prod w \quad (w \text{ in } \mathcal{W}_\lambda).$$

As in [3], the discrepancy between  $\dim V(G^0, \lambda)$  and  $\dim V(G, \lambda)$  is accounted for by true values of factors  $0/0$  (see example below).

Let  $\mathcal{W}'_{\text{mv}}$  be the sum of the following contributions ("mv" is for "moving": linear forms actually depending on  $\lambda$ ). A positive root  $\alpha = \sum a_i \alpha(i)$  of  $E_7$  fixed by  $s(7)$  contributes

$$\delta \left[ a_7 X + n + \sum a_i p_i \right] - \delta \left[ a_7 X + n - 1 + \sum a_i p_i \right],$$

with  $5a_7 + n = \sum_{i \leq 7} a_i$ . A pair of positive roots exchanged by  $s(7)$ ,  $\alpha = \sum a_i \alpha(i)$  and  $\alpha' = \alpha - \alpha(7) = \sum a'_i \alpha(i)$ , contributes

$$\delta \left[ a_7 X + n + \sum a_i p_i \right] - \delta \left[ a'_7 X + n' - 1 + \sum a'_i p_i \right]$$

with  $5a_7 + n = \sum_{i \leq 7} a_i$  and  $5a'_7 + n' = \sum_{i \leq 7} a'_i$ . One has  $a'_7 = a_7 - 1$ ,  $a'_i = a_i$  for  $i \leq 6$  and  $n' - 1 = n + 3$ .

Finally one obtains  $\mathcal{W}'$  by adding to  $\mathcal{W}'_{\text{mv}}$  a sum of  $\delta[BX + n]$ , such that  $\mathcal{W}'$  specialized at  $\lambda = 0$  is the empty multiset of linear forms  $BX + n$ . It follows that

$$D(\lambda, X) = \prod \left[ (BX + n + 1) \dots (BX + n + \sum a_i p_i) \right],$$

the product being over  $\mathcal{W}'_{\text{mv}}$ . Example: for  $\lambda = p\Omega_1$ ,

$$\dim V(G, p\Omega_1) = \frac{(4X - 3 + 1) \dots (4X - 3 + p) \cdot (3X - 2 + 1) \dots (3X - 2 + p) \cdot (2X + p - 1)}{(X - 1 + 1) \dots (X - 1 + p) \cdot (1) \dots (p) \cdot (2X - 1)}$$

evaluated at  $X = \nu(G)$ . For  $p = 1$ , we obtain the dimension of the preferred representation:

$$\dim V = \frac{(4X - 2)(3X - 1)(2X)}{X \cdot 1 \cdot (2X - 1)}$$

evaluated at  $X = \nu(G)$ . For  $G = (\mathbb{G}_m/\mu_3).2$ , one has  $\nu := \nu(G) = 1/2$ ; the numbers  $4\nu - 2$  and  $2\nu - 1$ ,  $3\nu - 1$  and  $\nu$ , as well as  $2\nu$  and  $1$  cancel to give  $W(\mathbb{G}_m, \Omega_1) = \emptyset$ , and  $4X - 2/2X - 1$  ( $= 0/0$  at  $X = 1/2$ ) contributes the factor 2 in the dimension 2.1 of  $V$ .

We expect a similar story for row 6, if the vertices of  $E_6$  are labelled

$$\begin{array}{cccccc} 1 & 4 & 6 & 5 & 2 & \\ & & & 3 & & \end{array} .$$

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Note pour l'imprimeur: les tables 1 et 2 sont à imprimer en haut des pages 2 et 3 de la note, pour qu'elles se fassent face. Les 7 lignes de la table 2 devraient être alignées avec les 7 premières lignes de la table 1. Cette disposition permet de ne pas répéter des informations d'une table à l'autre.

**Table 1**

							$A_1 \leftarrow E_7 \supset \mu_2$		
							$\mathbf{G}_m A_2 \leftarrow E_6 \supset \mu_3$		
							$\mu_3 A_1 G_2 \leftarrow F_4 \supset 1$		
							$\mu_2^2 \mathbf{G}_m^2 A_1^3 D_4 \leftarrow D_4 \supset \mu_2^2$		
							$\mu_2^2 A_1/\mu_2 A_2 C_3 F_4 \leftarrow G_2 \supset 1$		
							$\mu_3 \mathbf{G}_m^2 A_2 A_2^2 A_5 E_6 \leftarrow A_2 \supset \mu_3$		
							$\mathbf{G}_m A_1 A_1^3 C_3 A_5 D_6 E_7 \leftarrow A_1 \supset \mu_2$		
$A_1$	$A_2$	$G_2$	$D_4$	$F_4$	$E_6$	$E_7$	$E_8$	$\leftarrow$	$e \supset 1$
$\uparrow$	$\uparrow$		$\uparrow$		$\uparrow$	$\uparrow$			
$/\mu_2$	$/\mu_3).2$		$/\mu_2^2).S_3$		$/\mu_3).2$	$/\mu_2$			
$\nu :$	1/3	1/2	2/3	1	3/2	2	3	5	

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**Table 2**

					2	(1)	$\overset{2}{\wedge}$	
				1	3	(2)	$\overset{3}{\wedge}, V' \otimes V''$	
			1	3	7	(1)	$\overset{3}{\wedge}, S^2$	
		1	2.1	4	8	(3)	$S^2, V' \otimes V'' \otimes V'''$	
	2.1	5	8	14	26	(1)	$S^2, S^3$	
	1	3.1	6	9	15	27	(2)	$S^3, V' \otimes V''$
2.1	4	8	14	20	32	56	(1)	$\overset{2}{\wedge}, S^4$