Arithmetic invariants of discrete Langlands parameters

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Abstract

The local Langlands correspondence can be used as a tool for making verifiable predictions about irreducible complex representations of p-adic groups and their Langlands parameters, which are homomorphisms from the local Weil-Deligne group to the L-group. In this paper we refine a conjecture of Hiraga-Ichino-Ikeda, which relates the formal degree of a discrete series representation to the value of the local gamma factor of its parameter. We attach a rational function in x with rational coefficients to each discrete parameter, which specializes to this local gamma value when $x = q$, the cardinality of the residue field. The order of this rational function at $x = 0$ is also an important invariant of the parameter - it leads to a conjectural inequality for the Swan conductor of a discrete parameter acting on the adjoint representation of the L-group. We verify this conjecture in many cases. When we impose equality, we obtain a prediction for the existence of simple wild parameters and simple supercuspidal representations, both of which are found and described in this paper.

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1 Introduction

Let G be a reductive algebraic group over the local field k. The local Langlands conjecture predicts that the irreducible complex representations π of the locally compact group $G(k)$ should correspond to objects of an arithmetic nature: homomorphisms φ from the Weil-Deligne group of k to the complex L-group of G, together with an irreducible representation ρ of the component group of the centralizer of φ. In light of this conjecture - which has been established for algebraic tori, as well as for some non-abelian groups like $GL_n(k)$ [23],[25], and $SL_n(k)$ [27] - it is reasonable to predict how representation-theoretic invariants of π relate to the arithmetic invariants of its parameters (φ_π, ρ_π).

An early example of this was the paper [19], which predicts branching laws for the restriction of irreducible representations of the group $SO_n(k)$ to the subgroup $SO_{n-1}(k)$, using the ε-factor of a distinguished symplectic representation of the L-group of $SO_n \times SO_{n-1}$. These conjectures have now been verified in several cases; see [21] and [22].

For general groups, one invariant of a discrete series representation π is its formal degree. Recently, Hiraga, Ichino, and Ikeda have formulated a conjecture [26] for the formal degree of a discrete series representation, in terms of the L-function and ε-factor of the adjoint representation of the L-group.

For the rest of this paper we suppose the local field k is non-archimedean, with residue field $\mathfrak{f}$ of finite cardinality q a power of a prime p. We also assume the center of $G(k)$ is compact.

We reformulate the conjecture of [26], using Serre's Euler-Poincare measure μ_G on $G(k)$, which is ideally constructed for the study of the discrete series. In this form, the conjecture expresses the formal degree in terms of a ratio of adjoint gamma factors $\gamma(\varphi)/\gamma(\varphi_0)$, where φ is a discrete series parameter and φ_0 is the *principal parameter* (corresponding to the Steinberg representation). If π is induced from an open compact subgroup of $G(k)$, the formal degree $\deg_{\mu_G}(\pi)$ is a rational number. Our first main result, Proposition 4.1, is that the ratio $\gamma(\varphi)/\gamma(\varphi_0)$ is also rational. In fact, for any discrete parameter φ we define a rational function $\Gamma_\varphi(x)$ with rational coefficients and show that $\gamma(\varphi)/\gamma(\varphi_0) = \Gamma_\varphi(q)$.

We then calculate the order of $\Gamma_\varphi(x)$ at the point $x = 0$. This leads to a conjectural inequality on the Swan conductor $b(\varphi)$ of the Galois representation $\operatorname{Ad}\varphi = \operatorname{Ad}\circ\varphi$, which we verify in some cases by studying the permutation representation of the Weyl group on the roots. We next focus on parameters with *minimal Swan conductor*, i.e., those for which the inequality on $b(\varphi)$ becomes an equality.

When we combine this condition on $b(\varphi)$ with the formal degree conjecture, the tension between Galois and Lie theory yields strong conditions on φ. For example, in a large family of cases we can determine the precise image $\varphi(\mathcal{W})$, along with its ramification filtration, using the theory of the Coxeter element. See Proposition 5.6. This leads to the notion of a *simple wild parameter* which will reappear later in the local Langlands correspondence. We then construct some simple wild parameters for split simply-connected groups G in two cases: when p does not divide the Coxeter number of G and when $k = \mathbf{Q}_2$.

At this point, the formal degree conjecture almost prescribes the representations π for which $\varphi = \varphi_\pi$. Putting conjectures aside, this leads to our second main result: a construction of *simple supercuspidal representations* that seem not to have been singled out before. For p larger than the Coxeter number h of G, they appear in the constructions of [1] and [62] and are the supercuspidal representations of minimal positive depth $1/h$. However, our construction is completely different and works uniformly for any non-archimedean local field; see Proposition 9.3. For simplicity, we only give the construction in this paper when G is split, simply-connected and absolutely simple. Then the formal degree of an ssc representation π with respect to Euler-Poincare measure μ_G is given by the formula

$$(-1)^{\dim T} \deg_{\mu_G}(\pi) = \frac{|T(\mathfrak{f})|}{|Z(\mathfrak{f})| \cdot \mathrm{vol}_{\mu_G}(J)},$$

where $J < G(k)$ is an Iwahori subgroup, and $Z < T < G$ are the center and a maximal torus in G over $\mathfrak{f}$ respectively. We then compare this formula with the adjoint gamma factor and show that the degree conjecture implies that the Langlands parameter of an simple supercuspidal representation is one of the simple wild parameters arising from our study of minimal Swan conductors.

Both of our main results are of a group theoretic nature, and are unconditional: they could be formulated with little or no reference to the local Langlands conjecture. But without the Langlands conjecture as our guide, we would never have found them. Likewise, the existence and number of simple supercuspidal representations leads us to conjecture some unexpected relations between complex Lie theory and local Galois groups. For example, we predict that for any complex simple adjoint group $\hat{G}$ there should exist a unique $\hat{G}$-conjugacy class of simple wild parameters

$$\varphi : \mathrm{Gal}(\bar{\mathbf{Q}}_2/\mathbf{Q}_2) \to \hat{G}.$$

The simple wild parameters mentioned above for $\mathbf{Q}_2$ provide evidence for this when $\hat{G}$ has one of the types A_{2n}, B_n, D_n, G_2 or E_8.

Simple wild parameters and simple supercuspidal representations are also of arithmetic interest in the global setting. For example, the matrix coefficients of simple supercuspidal representations provide local test functions in the trace formula which allow one to explicitly compute multiplicities in the discrete spectrum of global automorphic representations having simple supercuspidal local components [17]. On the other hand, the Kloosterman sheaves on $\mathbf{G}_m$ over finite fields studied by Deligne [13] and Katz [30] give rise to simple wild parameters when restricted to the decomposition group at infinity (cf. [14]).

This paper is organized as follows. In the first few sections we review the basic ingredients of the Langlands conjectures - the Weil-Deligne group and its representations, local L-functions and ε-factors, the dual group, and the L-group. This is foundational material, and we have attempted to fill some gaps in the literature. We then turn to the related topics of the motive of G and the principal parameter φ_0, which is used for volume computatations, and to compare our version of the degree conjecture with that of [26]. Our proof of the rationality of $\Gamma_\varphi(x)$ uses Kac's classification [29] of torsion automorphisms of simple Lie algebras (see also [43]). We find that the possible irreducible factors (in $\mathbf{Z}[x]$) of the numerator and denominator of $\Gamma_\varphi(x)$ are remarkably few in number and can be easily listed.

In chapter 5 we compute the leading term of Γ_φ and study its implications for φ, arriving at the notions of minimal Swan conductor and simple wild parameters. We show that a totally ramified parameter with minimal wild ramification must be a simple wild parameter. We then construct simple wild parameters in the two cases mentioned above. To this point, everything involves representations of the local Weil-Deligne group and is independent of the local Langlands conjecture. We then turn to the Langlands correspondence itself in chapter 7, where we reformulate the conjecture of [26] in terms of Euler-Poincare measure. Chapter 8 is additional foundational material on central characters, which allows us to predict the sign in our version of the formal degree conjecture. We conclude with chapter 9 where we give a construction of simple supercuspidal representations for a split, simply-connected group G over k, and show how the formal degree conjecture implies that their parameters must be simple wild parameters.

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Contents

2 The Weil group

Let k be a local field with finite residue field $\mathfrak{f}$ of cardinality q, a power of a prime p. We will assume that k has characteristic zero. This assumption is only used in a few places (primarily to simplify the discussion of Galois cohomology) and we believe that it can be easily removed.

Let A denote the ring of integers of k, ϖ a uniformizing element and let $P = \varpi A$ be the prime ideal of A, so that $\mathfrak{f} = A/P$. We fix an algebraic closure $\bar{k}$ of k, let K be the maximal unramified extension of k in $\bar{k}$ and let $\bar{A}$ denote the integral closure of A in K. Then $\bar{\mathfrak{f}} := \bar{A}/\varpi\bar{A}$ is an algebraic closure of $\mathfrak{f}$.

The Weil group $\mathcal{W} = \mathcal{W}(k)$ of k is the dense subgroup of $\mathrm{Gal}(\bar{k}/k)$ consisting of elements which induce on $\bar{\mathfrak{f}}$ an integer power of the automorphism $x \mapsto x^q$. Following the convention in [11], we fix a geometric Frobenius element $\mathrm{Fr} \in \mathcal{W}$, which satisfies $\mathrm{Fr}(x^q) = x$ on $\bar{\mathfrak{f}}$. Then $\mathcal{W}$ is a semidirect product: $\mathcal{W} = \langle \mathrm{Fr} \rangle \ltimes \mathcal{I}$, where $\mathcal{I}$, the inertia subgroup, is the kernel of the action of $\mathcal{W}$ on $\bar{\mathfrak{f}}$ and is also the absolute Galois group of K. The inertia subgroup $\mathcal{I}$ is a compact group with the profinite topology, so the open subgroups of $\mathcal{I}$ all have finite index, and are also closed in $\mathcal{I}$.

We normalize the isomorphism $k^\times \simeq \mathcal{W}^{ab}$ of local class field theory so that ϖ maps to the class of the geometric Frobenius Fr in $\mathcal{W}^{ab}$. The normalized valuation on $k^\times$ then gives a homomorphism

$$|\cdot| : \mathcal{W} \longrightarrow q^{\mathbf{Z}} \subset \mathbf{R}^\times_{>0}$$

which satisfies $\ker|\cdot| = \mathcal{I}$ and $|\,\mathrm{Fr}\,| = q^{-1}$.

A representation $\rho : \mathcal{W} \to GL(V)$ on a finite dimensional complex vector space V is called *admissible* if $\ker\rho$ is open and $\rho(\mathrm{Fr})$ is semisimple. The first condition implies that $\rho(\mathcal{I})$ is finite. Since $\rho(\mathrm{Fr})$ normalizes $\rho(\mathcal{I})$, it preserves the subspace $V^{\mathcal{I}}$ of invariants under $\rho(\mathcal{I})$.

The local factors of V are defined as follows. The L-function of V is given by

$$L(V, s) := \det(1 - q^{-s}\rho(\mathrm{Fr})|V^{\mathcal{I}})^{-1}.$$

The ε-factor of V is given by

$$\varepsilon_0(V, s) := \varepsilon(V \otimes |\cdot|^s, \psi, dx),$$

where dx is the Haar measure on k^+ giving A^+ volume $= 1$, ψ is an additive character of k^+ such that A^+ is the largest fractional ideal contained in the kernel of ψ , and $\varepsilon(V \otimes |\cdot|^s, \psi, dx)$ is the local ε function defined by Deligne and Tate (cf. [58]). We have

$$\varepsilon_0(V,s) = w(V)q^{a(V)(\frac{1}{2}-s)},$$

where $w(V) = \varepsilon_0(V,\frac{1}{2}) \in \mathbf{C}^\times$ is independent of s and $a(V) \in \mathbf{Z}_{\geq 0}$ is the Artin conductor of V, defined as follows. The inertial image $D_0 = \rho(\mathcal{I})$ is the Galois group of a finite extension K'/K. If A' is the ring of integers of K' with maximal ideal P', then we define the normal subgroup $D_j \leq D_0$ as the kernel of the action of D_0 on $A'/(P')^{j+1}$. This gives the lower ramification filtration

$$D_0 \geq D_1 \geq D_2 \geq \cdots \geq D_n = 1.$$

We let $d_j = |D_j|$. Now the *Artin conductor* is defined by

$$a(V) = \sum_{j\geq 0} \dim(V/V^{D_j})\frac{d_j}{d_0}.$$

Isolating the term for $j = 0$, we can write

$$a(V) = \dim(V/V^{D_0}) + b(V),$$

where the *Swan conductor* $b(V)$ measures the action of the wild inertia group D_1 on V. We sometimes write $a(\rho)$ for $a(V)$ or $b(\rho)$ for $b(V)$.

The Artin conductor is additive in V. The computation of $a(V)$ can therefore be reduced to the case where V is an irreducible representation of $\mathcal{W}$. Since each D_j is normal in $\rho(\mathcal{W})$, the subspaces V^{D_j} are $\mathcal{W}$-invariant. If V is irreducible, then either $V^{D_j} = 0$ or $V^{D_j} = V$. If $V^{D_0} = V$, we have $a(V) = 0$. If $V^{D_0} = 0$, the Artin conductor is given by the simpler formula

$$a(V) = \frac{\dim V}{d_0}\sum_{j=0}^{m} d_j, \qquad (1)$$

where $d_j = |D_j|$ and m is the largest integer such that $D_m \neq 1$.

The Artin conductor refines the valuation of the discriminant. We will use this relation to find Galois representations with a given conductor, and for completeness we review it here. Let E/k be a finite Galois extension of k with group $D = \mathrm{Gal}(E/k)$ and let R be the regular representation of D. Via the canonical map $\mathcal{W} \to \mathrm{Gal}(E/k)$, we get an admissible representation $\rho : \mathcal{W} \to GL(R)$ with $\rho(\mathcal{W}) = D$. The Artin conductor of R is given by

$$a(R) = \mathrm{val}_k(\mathfrak{d}_{E/k}),$$

where $\mathfrak{d}_{E/k}$ is the discriminant of E/k (see [48, VI.2]). On the other hand,

$$R = \sum_{V\in \mathrm{Irr}(D)} (\dim V)\cdot V,$$

so we have

$$\mathrm{val}_k(\mathfrak{d}_{E/k}) = a(R) = \sum_{V\in\mathrm{Irr}(D)} (\dim V)\cdot a(V).$$

Since R^{D_j} has dimension $[D : D_j] = d/d_j$ for all j, we have

$$a(R) = \sum_{j\geq 0} \frac{d_j}{d_0}\left(d - \frac{d}{d_j}\right) = \frac{d}{d_0}\sum_{j\geq 0}(d_j - 1) = f(e-1) + b(R),$$

where $d = |D|$ and $f = d/d_0$, $e = d_0$ are the residue degree and ramification index of E/k, respectively. In particular, we have

$$\mathrm{val}_k(\mathfrak{d}_{E/k}) \geq f(e-1),$$

with equality if and only if E/k is tamely ramified.

More generally, if L is the subfield of E fixed by the subgroup C of D and U is a representation of C, then [48, VI.2]

$$a(\mathrm{Ind}_C^D U) = \mathrm{val}_k(\mathfrak{d}_{L/k})\cdot \dim U + f_{L/k}\cdot a(U). \tag{2}$$

Taking $U = \mathbf{1}$ to be the trivial representation of C and applying Frobenius reciprocity, we find that

$$\mathrm{val}_k(\mathfrak{d}_{L/k}) = a(\mathrm{Ind}_C^D \mathbf{1}) = \sum_{V\in\mathrm{Irr}(D)} (\dim V^C)\cdot a(V).$$

2.1 The Weil-Deligne group and its representations

For the local Langlands correspondence, the classical notion of a representation of the Weil group must be refined in two ways: first by replacing $GL(V)$ by a reductive complex Lie group $\mathcal{G}$, and second by the addition of a nilpotent element N in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$; the element N plays the role of the nilpotent part in the Jordan decomposition of an element of $\mathfrak{g}$, while the role of semisimple part is played by a homomorphism $\rho : \mathcal{W} \to \mathcal{G}$.

Precisely, a *representation of the Weil-Deligne group* is a triple $(\rho, \mathcal{G}, N)$ where $\mathcal{G}$ is a complex Lie group whose identity component $\mathcal{G}^\circ$ is reductive, $\rho : \mathcal{W} \to \mathcal{G}$ is a homomorphism which is continuous on $\mathcal{I}$, with $\rho(\mathrm{Fr})$ semisimple in $\mathcal{G}$, and N is a nilpotent element in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ such that $\mathrm{Ad}\,\rho(w)N = |w|\cdot N$ for all $w \in \mathcal{W}$. In particular, we have that $\mathrm{Ad}\,\rho(\mathcal{I})$ fixes N and $\mathrm{Ad}\,\rho(\mathrm{Fr})N = q^{-1}N$. Two representations $(\rho, \mathcal{G}, N)$, $(\rho', \mathcal{G}, N')$ of the Weil-Deligne group are considered equivalent if there is an element $g \in \mathcal{G}^\circ$ such that $\rho' = g\rho g^{-1}$ and $N' = \mathrm{Ad}(g)N$.

A representation $(\rho, \mathcal{G}, N)$ of the Weil-Deligne group can be put on more familiar algebraic footing by associating to it the $\mathcal{G}^\circ$-conjugacy class of a homomorphism

$$\varphi : \mathcal{W} \times SL_2(\mathbf{C}) \to \mathcal{G}.$$

Since only the case of GL_n has been discussed in the literature (cf. [44]) we shall give a proof of this correspondence.

Since $\rho(\mathcal{I})$ is finite, the centralizer $\mathfrak{h} := \mathfrak{g}^{\rho(\mathcal{I})}$ is reductive, as follows easily from [6, I.6.4 Prop.5]. Moreover, $\mathrm{Ad}\,\rho(\mathrm{Fr})$ normalizes $\mathfrak{h}$. For $\lambda \in \mathbf{C}^\times$, let $\mathfrak{h}(\lambda)$ be the λ-eigenspace of $\mathrm{Ad}(\rho(\mathrm{Fr}))$ in $\mathfrak{h}$. Finally, note that $N \in \mathfrak{h}$. We need a refinement of the Jacobson-Morozov theorem, which says that every nilpotent element is contained in an $\mathfrak{sl}_2$-triple. For background on this, see [9, chap.5].

Lemma 2.1 *There is an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{h}$ such that*

$$e = N \in \mathfrak{h}(q^{-1}), \qquad h \in \mathfrak{h}(1) = \mathfrak{g}^{\rho(\mathcal{W})}, \qquad f \in \mathfrak{h}(q).$$

Any two such triples are conjugate by an element of $\mathcal{G}^\circ$ which centralizes $\mathcal{W}$ and fixes N.

Proof: This was proved for $q = -1$ by Kostant and Rallis in [34]. We imitate their proof. Let (e, f_0, h_0) be any $\mathfrak{sl}_2$-triple containing e. Write

$$h_0 = h + h', \qquad \text{with} \qquad h \in \mathfrak{h}(1), \qquad h' \in \sum_{\lambda \neq 1} \mathfrak{h}(\lambda).$$

Comparing eigenvalues in the equation $2e = [h, e] + [h', e]$, we have

$$[h, e] = 2e. \tag{3}$$

Likewise, write

$$f_0 = f' + f'', \qquad \text{with} \qquad f' \in \mathfrak{h}(q), \qquad f'' \in \sum_{\lambda \neq q} \mathfrak{h}(\lambda).$$

Comparing eigenvalues in the equation $h + h' = [e, f'] + [e, f'']$, we have

$$h = [e, f'] \in [e, \mathfrak{h}]. \tag{4}$$

Equation (3) and the containment in (4) are the conditions in [33, Cor.3.5] guaranteeing the existence of an element $f \in \mathfrak{h}$ such that (e, f, h) is an $\mathfrak{sl}_2$-triple. Since $\mathrm{ad}(e)$ is injective on the -2-eigenspace of $\mathrm{ad}(h)$, the relations $[h, f] = -2f$, $[e, f] = h$ imply that $f \in \mathfrak{h}(q)$, as desired.

The proof of the conjugacy assertion is identical to that of [34]. We reproduce it here for completeness. Suppose that (e, f_1, h_1) is another $\mathfrak{sl}_2$-triple satisfying the conditions of the Lemma. The element $y = h_1 - h$ belongs to the Lie subalgebra $\mathfrak{m} = \mathfrak{h}^e \cap \mathfrak{h}(1)$, which is normalized by h. By the representation theory of $\mathfrak{sl}_2$, we have

$$\mathfrak{m} = \bigoplus_{i \geq 0} \mathfrak{m}_i, \qquad \text{where} \qquad \mathfrak{m}_i = \{x \in \mathfrak{m} : \ [h, x] = ix\}.$$

The subalgebra

$$\mathfrak{u} := \bigoplus_{i > 0} \mathfrak{m}_i$$

is nilpotent and the corresponding subgroup $U = \exp(\mathfrak{u})$ is unipotent. We consider the orbit $\mathrm{Ad}(U) \cdot h \subseteq h + \mathfrak{u}$. Since U is unipotent, the orbit $\mathrm{Ad}(U) \cdot h$ is closed. The tangent space to the orbit at h is $[h, \mathfrak{u}] = \mathfrak{u}$, so the orbit also open. Therefore we in fact have

$$\mathrm{Ad}(U) \cdot h = h + \mathfrak{u}. \tag{5}$$

Next, we observe that $\mathfrak{u} = \mathfrak{m} \cap [e, \mathfrak{h}]$, again by the representation theory of $\mathfrak{sl}_2$. Since $y = [e, f_1 - f]$, it follows that $y \in \mathfrak{u}$. Hence by (5), there is an element $u \in U$ such that

$$\mathrm{Ad}(u)h = h + y = h_1.$$

We clearly have $\mathrm{Ad}(u)e = e$. It remains to show that $\mathrm{Ad}(u)f = f_1$. But this follows from the same argument used to prove $f \in \mathfrak{h}(q)$ above. ■

Now, given a representation $(\rho, \mathcal{G}, N)$ of the Weil-Deligne group, let H be the identity component of the fixed-points $(\mathcal{G}^\circ)^{\mathcal{I}}$ of $\rho(\mathcal{I})$ in $\mathcal{G}^\circ$. Then H has Lie algebra $\mathfrak{h} = \mathfrak{g}^{\mathcal{I}}$. Choosing an $\mathfrak{sl}_2$-triple (e, f, h) as in Lemma 2.1 gives a homomorphism $\varphi : SL_2 \to H$ such that

$$e = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

whence a semisimple element

$$s = \varphi \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in H.$$

Note that $s = \exp(\frac{1}{2} \log(q) \cdot h)$ commutes with $\rho(\mathrm{Fr})$, since $h \in \mathfrak{h}(1)$. Moreover, we have

$$\mathrm{Ad}(s)e = qe, \qquad \text{and} \qquad \mathrm{Ad}(s)f = q^{-1}f.$$

It follows that $\mathrm{Ad}(s \cdot \rho(\mathrm{Fr}))$ centralizes $\varphi(SL_2)$. Since s commutes with $\varphi(\mathcal{I})$, we obtain the promised extension

$$\varphi : \mathcal{W} \times SL_2 \longrightarrow \mathcal{G}$$

by making $\varphi = \rho$ on $\mathcal{I}$ and setting $\varphi(\mathrm{Fr}) = s \cdot \rho(\mathrm{Fr})$. The conjugacy assertion in Lemma 2.1 implies that the $\mathcal{G}^\circ$-conjugacy class of φ is independent of the choice of triple (e, f, h). To get back from φ to the original data $(\rho, \mathcal{G}, N)$, we define

$$\rho|_{\mathcal{I}} = \varphi|_{\mathcal{I}}, \qquad \rho(\mathrm{Fr}) = \varphi(\mathrm{Fr}) \cdot \varphi \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \qquad N = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

To summarize, we have proved:

Proposition 2.2 *The correpondence $(\rho, \mathcal{G}, N) \leftrightarrow \varphi$ just described gives a bijection between the equivalence classes of representations of the Weil-Deligne group and the $\mathcal{G}^\circ$-conjugacy classes of homomorphisms $\varphi : \mathcal{W} \times SL_2 \to \mathcal{G}$ for which φ is trivial on an open subgroup of $\mathcal{I}$, $\varphi(\mathrm{Fr})$ is semisimple and the restriction of φ to SL_2 is algebraic.*

If φ is a homomorphism as in Prop. 2.2 and A is a subgroup of $\mathcal{W}$ (resp. SL_2) we often write $\varphi(A)$ instead of $\varphi(A \times 1)$ (resp. $\varphi(1 \times A)$).

The *wild inertia subgroup* of $\mathcal{W}$ is the pro-p-Sylow subgroup $\mathcal{I}_+$ of $\mathcal{I}$. If $\varphi : \mathcal{W} \times SL_2 \to \mathcal{G}$ is a homomorphism as in Prop. 2.2, then $\varphi(\mathcal{I})$ is finite, and $\varphi(\mathcal{I}_+)$ is a finite p-group contained in the centralizer $\mathcal{M}$ of $\varphi(SL_2)$. From a result of Borel and Serre [5], it follows that $\varphi(\mathcal{I}_+)$ normalizes a maximal torus $\mathcal{S}$ in $\mathcal{M}$. If p does not divide the index of $\mathcal{S}$ in its normalizer in $\mathcal{M}$, it follows that $\varphi(\mathcal{I}_+)$ is contained in $\mathcal{S}$ and is, in particular, abelian.

2.2 Local factors for Weil-Deligne representations

The local factors for representations of the Weil group generalize to representations $(\rho, \mathcal{G}, N)$ of the Weil-Deligne group as follows. Fix a finite dimensional complex representation $r : \mathcal{G} \to GL(V)$ which is algebraic on $\mathcal{G}^\circ$. Then

$$r \circ \rho : \mathcal{W} \longrightarrow \mathcal{G} \longrightarrow GL(V)$$

is a representation of $\mathcal{W}$, which we will again denote by ρ, as r is fixed. For any ρ-invariant subspace $U \subset V$, let $U^{\mathcal{I}}$ denote the vectors in U which are invariant under $\rho(\mathcal{I})$. Since $\mathcal{I}$ is normal in $\mathcal{W}$, the space $U^{\mathcal{I}}$ is preserved by the operator $\rho(\mathrm{Fr})$.

Since each operator from $\rho(\mathcal{W})$ commutes with N up to a scalar, the kernel V_N of N on V is an invariant subspace, hence the $\mathcal{I}$-invariants $V_N^{\mathcal{I}}$ are preserved by $\rho(\mathrm{Fr})$. The local L-function of (ρ, V) is defined (see [58, 4.1.6]) by

$$L(\rho, V, s) = \det\left(1 - q^{-s}\rho(\mathrm{Fr})|V_N^{\mathcal{I}}\right)^{-1}, \tag{6}$$

and the local ε-factor is defined by

$$\varepsilon(\rho, V, s) = \varepsilon_0(V, s) \cdot \det\left(-q^{-s}\rho(\mathrm{Fr})|V^{\mathcal{I}}/V_N^{\mathcal{I}}\right), \tag{7}$$

where $\varepsilon_0(V, s)$ is the ε-factor defined in section 2. We want to express these factors in terms of the corresponding representation

$$r \circ \varphi : \mathcal{W} \times SL_2 \to \mathcal{G} \to GL(V),$$

which we again denote by φ, as r is fixed. Since we will work with φ in the rest of the paper, we now write

$$L(\varphi, V, s) := L(\rho, V, s), \qquad \text{and} \qquad \varepsilon(\varphi, V, s) := \varepsilon(\rho, V, s).$$

Since V is semisimple under φ, we have

$$V = \bigoplus_{n \geq 0} V_n \otimes \mathrm{Sym}^n, \tag{8}$$

where $\mathrm{Sym}^n = \mathrm{Sym}^n(\mathbf{C}^2)$ is the irreducible SL_2-representation of dimension $n+1$, and V_n is a semisimple complex representation of $\mathcal{W}$. Hence

$$V_N^{\mathcal{I}} = \bigoplus_{n\geq 0} V_n^{\mathcal{I}} \otimes \mathrm{Sym}_N^n,$$

where Sym_N^n, the highest weight line, is the kernel of N in Sym^n. Since $\rho(\mathrm{Fr}) = q^{-n/2}\varphi(\mathrm{Fr})$ on Sym_N^n, we can rewrite (6) and (7) as

$$L(\varphi, V, s) = \prod_{n\geq 0} \det\left(1 - q^{-\frac{n}{2}-s}\varphi(\mathrm{Fr})|V_n^{\mathcal{I}}\right)^{-1}, \tag{9}$$

and

$$\varepsilon(\varphi, V, s) = \omega(\varphi, V) q^{\alpha(\varphi,V)(\frac{1}{2}-s)}, \tag{10}$$

where

$$\begin{aligned} \alpha(\varphi, V) &= \sum_{n\geq 0}(n+1)a(V_n) + \sum_{n\geq 1} n \cdot \dim V_n^{\mathcal{I}}, \qquad \text{and} \\ \omega(\varphi, V) &= \prod_{n\geq 0} w(V_n)^{n+1} \cdot \prod_{n\geq 1} \det(-\varphi(\mathrm{Fr})|V_n^{\mathcal{I}})^n, \end{aligned} \tag{11}$$

where $a(V_n)$ and $w(V_n)$ are the Artin conductor and local constants of the representations (ρ, V_n) of $\mathcal{W}$, as recalled in section 2.

Finally, the *gamma factor* of (φ, V) is defined by

$$\gamma(\varphi, V, s) = \frac{L(\varphi, \check{V}, 1-s)\varepsilon(\varphi, V, s)}{L(\varphi, V, s)},$$

where $\check{V}$ is the representation of $\mathcal{W} \times SL_2$ dual to V.

2.3 Self-dual representations

A representation

$$\varphi : \mathcal{W} \times SL_2 \longrightarrow GL(V)$$

is self-dual if it is isomorphic to the contragredient representation on the dual space $\check{V}$ or equivalently, if the image $\mathrm{im}\,\varphi = \varphi(\mathcal{W} \times SL_2)$ preserves a non-degenerate bilinear form on V. We say V is orthogonal if $\mathrm{im}\,\varphi$ preserves a non-degenerate symmetric bilinear form on V, and symplectic if $\mathrm{im}\,\varphi$ preserves a non-degenerate alternating bilinear form on V. In the symplectic case, the dimension $\dim(V)$ must be even and the determinant $\det V := \det \circ \varphi$ must be trivial on $\mathcal{W} \times SL_2$. In the orthogonal case,

$$\det V : \mathcal{W} \times SL_2 \longrightarrow \{\pm 1\}$$

is a quadratic character of the quotient group $\mathcal{W}^{ab} = k^\times$, so $\det V(-1) = \pm 1$ is well-defined.

Assume that V is orthogonal. In the decomposition

$$V = \bigoplus_{n \geq 0} V_n \otimes \mathrm{Sym}^n$$

(see (8)), each representation V_n of $\mathcal{W}$ is self-dual, and each summand $V_n \otimes \mathrm{Sym}^n$ is orthogonal. Since Sym^n is orthogonal when n is even and symplectic when n is odd, we have that V_n is also orthogonal when n is even and symplectic when n is odd. This must also hold for the invariant subspaces $V_n^{\mathcal{I}}$. Since $V \simeq \sum_{n \geq 0}(n+1)V_n$ as a $\mathcal{W}$-module, it follows that

$$\alpha(\varphi, V) = a(V) + \sum_{n \geq 1} n \cdot \dim V_n^{\mathcal{I}} \equiv a(V) \mod 2,$$

and

$$\omega(\varphi, V) = w(V) \cdot \prod_{n \geq 1} \det(-\varphi(\mathrm{Fr})|V_n^{\mathcal{I}})^n = w(V).$$

Since V is self-dual, we have (see [58, 3.6.8])

$$w(V)^2 = \det V(-1) \tag{12}$$

and by [45, Thm.1] we have

$$a(V) \equiv a(\det V) \mod 2, \tag{13}$$

so we have shown

Proposition 2.3 *If $\varphi : \mathcal{W} \times SL_2 \to GL(V)$ is an orthogonal representation, then $\varepsilon(\varphi, V, s) = \omega(\varphi, V) q^{\alpha(\varphi,V)(\frac{1}{2}-s)}$, with*

$$\alpha(\varphi, V) \equiv a(\det V) \mod 2,$$
$$\omega(\varphi, V)^2 = \det V(-1).$$

For example, if V is orthogonal and $\det V$ is an unramified quadratic character of $\mathcal{W}$, then $\alpha(\varphi, V)$ is even and $\omega(\varphi, V)^2 = 1$.

3 Reductive groups and Langlands parameters

Recall that k is a finite extension of $\mathbb{Q}_p$ with fixed algebraic closure $\bar{k}$. Let G be a connected, reductive algebraic group over k, and let k_0 be the splitting field in $\bar{k}$ of the quasi-split inner form G_0 of G over k. Let $T \subset B$ be a maximal torus in a Borel subgroup of G_0 defined over k. Then T splits over k_0 and we let S be the maximal k-split subtorus of T. Let $r(G_0) = \dim S$ denote the k-rank of G_0 and let $r(G)$ denote the rank of G over k. We have $r(G) \leq r(G_0)$ with equality if and only if $G \simeq G_0$ over k. Later we will let ℓ_0 denote the rank of G over the maximal unramified extension K of k.

The Galois group $\mathrm{Gal}(k_0/k)$ acts as automorphisms of the based root datum $(X, \hat{X}, \Delta, \hat{\Delta})$ of G_0, where $X = \mathrm{Hom}_{\bar{k}}(T, \mathbf{G}_m)$, $\hat{X} = \mathrm{Hom}_{\bar{k}}(\mathbf{G}_m, T)$ is the dual lattice of co-weights, $\Delta \subset X$ is the set of simple roots of T in B, and $\hat{\Delta} \subset \hat{X}$ is the corresponding set of simple co-roots. Let $\Phi \subset X$ and $\hat{\Phi} \subset \hat{X}$ be the set of all roots and coroots of T in G; these generate subgroups $\mathbf{Z}\Phi < X$ and $\mathbf{Z}\hat{\Phi} < \hat{X}$.

3.1 The L-group

Let $\hat{G}$ be a connected, reductive group over $\mathbf{C}$ with the dual root datum $(\hat{X}, X, \hat{\Delta}, \Delta)$. Then $\hat{G}$ has a maximal torus in a Borel subgroup $\hat{T} \subset \hat{B}$ such that $\hat{X}$ may be identified with the group of algebraic characters $\lambda : \hat{T} \to \mathbf{C}^\times$. Then $\hat{\Phi}$ and $\hat{\Delta}$ become the sets of roots and simple roots of $\hat{T}$ in $\hat{B}$. Fix a set $\{x_{\hat{\alpha}} : \hat{\alpha} \in \hat{\Delta}\}$ of nonzero vectors in each simple root space in $\hat{\mathfrak{b}} = \mathrm{Lie}(\hat{B})$, thereby giving a pinning (*épinglage*) $\mathcal{E} := (\hat{T}, \hat{B}, \{x_{\hat{\alpha}}\})$ in $\hat{G}$. The action of $\mathrm{Gal}(k_0/k)$ on each object in the pinning gives an embedding of $\mathrm{Gal}(k_0/k)$ in the group $\mathrm{Aut}(\hat{G}, \mathcal{E})$ of automorphisms of $\hat{G}$ preserving the pinning $\mathcal{E}$. The elements of $\mathrm{Aut}(\hat{G}, \mathcal{E})$ are called *pinned automorphisms*. See, for example, [54] or [16] for more background on pinned automorphisms.

We define the L-group of G as the semidirect product

$$^L G := \mathrm{Gal}(k_0/k) \ltimes \hat{G}.$$

The center $^L Z$ of $^L G$ is the group of $\mathrm{Gal}(k_0/k)$-fixed points in the center of $\hat{G}$. Kottwitz [35] showed that the Galois cohomology $H^1(k, G)$ of G is in canonical bijection with $\mathrm{Hom}(\pi_0(^L Z), \mathbf{C}^\times)$. The group $^L Z$ is finite if and only if the maximal torus in the center of G is anisotropic over k. For the rest of this paper:

We assume that the maximal torus in the center of G is anisotropic over k.

With this assumption, we have simply

$$H^1(k, G) \simeq \mathrm{Hom}(^L Z, \mathbf{C}^\times).$$

3.2 Langlands parameters

Let $\mathsf{pr} : {}^L G \to \mathrm{Gal}(k_0/k)$ be the natural projection map with kernel $\hat{G}$. A (Langlands) *parameter* for G is a representation of the Weil-Deligne group $(\rho, {}^L G, N)$ such that the composite mapping

$$\mathcal{W} \xrightarrow{\rho} {}^L G \xrightarrow{\mathsf{pr}} \mathrm{Gal}(k_0/k)$$

is the canonical surjection $\mathcal{W} \to \mathrm{Gal}(k_0/k)$. Two parameters are considered equivalent if they are conjugate by $\hat{G}$.

By the results in section 2.1 (with $\mathcal{G} = {}^L G$ and $\mathcal{G}^\circ = \hat{G}$), we may reinterpret parameters as homomorphisms

$$\varphi : \mathcal{W} \times SL_2 \longrightarrow {}^L G$$

such that

1) $\varphi : SL_2 \to \hat{G}$ is a homomorphism of algebraic groups over **C**;
2) φ is continuous on $\mathcal{I}$ and $\varphi(\mathrm{Fr})$ is semisimple;
3) the composite $\mathcal{W} \xrightarrow{\rho} {}^L G \xrightarrow{\mathsf{pr}} \mathrm{Gal}(k_0/k)$ is the canonical surjection $\mathcal{W} \to \mathrm{Gal}(k_0/k)$.

As before, two parameters are considered equivalent if they are conjugate by $\hat{G}$.

Given a parameter φ, let A_φ denote the centralizer in $\hat{G}$ of the image $\varphi(\mathcal{W} \times SL_2)$. We say that φ is *discrete* if A_φ is finite. Since we are assuming that the maximal torus in the center of G is k-anisotropic, φ is discrete if and only if $\varphi(\mathcal{W} \times SL_2)$ is not contained in a proper parabolic subgroup of ${}^L G$ [4]. In other words, φ is discrete when its image is "irreducible" in ${}^L G$, in the sense of Serre [51]. We always have ${}^L Z \subset A_\varphi$.

Lemma 3.1 *If $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ is a discrete parameter, then the element $\varphi(\mathrm{Fr}) \in {}^L G$ has finite order and the image $\varphi(\mathcal{W})$ is finite.*

Proof: Since $\varphi(\mathcal{I})$ is finite and normalized by $\varphi(\mathrm{Fr})$, some power $\varphi(\mathrm{Fr})^n$ must centralize $\varphi(\mathcal{I})$. Since $\varphi(\mathrm{Fr})$ must also centralize $\varphi(SL_2)$, it follows that $\varphi(\mathrm{Fr})^n$ lies in the finite group A_φ. Since $\varphi(\mathrm{Fr})$ and $\varphi(\mathcal{I})$ generate $\varphi(\mathcal{W})$, it follows that $\varphi(\mathcal{W})$ is finite. ■

For any parameter φ, the representation

$$\mathrm{Ad}\,\varphi : \mathcal{W} \times SL_2 \xrightarrow{\varphi} {}^L G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\hat{\mathfrak{g}})$$

is a self-dual orthogonal representation. Indeed, ${}^L G$ preserves the Killing form on $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ and the action of ${}^L G$ on the center $\hat{\mathfrak{z}}$ of $\hat{\mathfrak{g}}$ factors through the finite group $\mathrm{Gal}(k_0/k)$ and preserves the co-weight lattice of the connected center of $\hat{G}$, so that $\hat{\mathfrak{z}}$ is orthogonal for ${}^L G$. Thus the representation $\mathrm{Ad}\,\varphi$ on $\hat{\mathfrak{g}}$ is the direct sum of two orthogonal representations.

We have the adjoint L-function $L(\varphi, \hat{\mathfrak{g}}, s)$, the epsilon factor $\varepsilon(\varphi, \hat{\mathfrak{g}}, s)$ and the gamma factor

$$\gamma(\varphi, \hat{\mathfrak{g}}, s) = \frac{L(\varphi, \hat{\mathfrak{g}}, 1-s)\varepsilon(\varphi, \hat{\mathfrak{g}}, s)}{L(\varphi, \hat{\mathfrak{g}}, s)}$$

attached, as in section 2.2, to the representation $\mathrm{Ad}\,\varphi$.

Proposition 3.2 *A parameter $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ is discrete if and only if the adjoint L-function $L(\varphi, \hat{\mathfrak{g}}, s)$ is regular at $s = 0$.*

Proof: The centralizer A_φ is finite if and only if there are no nonzero invariants of $\varphi(\mathcal{W} \times SL_2)$ in $\hat{\mathfrak{g}}$. Writing

$$\hat{\mathfrak{g}} = \bigoplus_{n \geq 0} \hat{\mathfrak{g}}_n \otimes \mathrm{Sym}^n$$

as in (8), we see that A_φ is finite if and only if the $\varphi(\mathcal{W})$-invariants in $\hat{\mathfrak{g}}_0$ are zero. Consider the adjoint L-function

$$L(\varphi, \hat{\mathfrak{g}}, s) = \prod_{n \geq 0} \det \left(1 - q^{-s-n/2} \varphi(\mathrm{Fr}) | \hat{\mathfrak{g}}_n^{\mathcal{I}}\right)^{-1}.$$

If φ is a discrete parameter then by Lemma 3.1 the eigenvalues of $\varphi(\mathrm{Fr})$ are roots of unity , so the only possible pole of $L(\varphi, \hat{\mathfrak{g}}, s)$ at $s = 0$ can come from the factor for $n = 0$, which is

$$\det \left(1 - q^{-s} \varphi(\mathrm{Fr}) | \hat{\mathfrak{g}}_0^{\mathcal{I}}\right)^{-1}.$$

But this factor is regular at $s = 0$, since $\hat{\mathfrak{g}}_0^{\varphi(\mathcal{W})} = 0$. Conversely, if $L(\varphi, \hat{\mathfrak{g}}, s)$ is regular at $s = 0$ then so is its factor for $n = 0$, so we have $\hat{\mathfrak{g}}_0^{\varphi(\mathcal{W})} = 0$ and φ is discrete. ■

Note that $L(\varphi, \hat{\mathfrak{g}}, 1)$ is finite for any discrete parameter, by Lemma 3.1. Hence the adjoint gamma value:

$$\gamma(\varphi, \hat{\mathfrak{g}}, 0) = \frac{L(\varphi, \hat{\mathfrak{g}}, 1) \cdot \varepsilon(\varphi, \hat{\mathfrak{g}}, 0)}{L(\varphi, \hat{\mathfrak{g}}, 0)} \tag{14}$$

is nonzero when φ is discrete.

For the rest of this paper we will only concern ourselves with the values at $s = 0$ of local gamma and epsilon factors attached to $\mathrm{Ad}\,\varphi$, where $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ is a discrete parameter. To simplify the notation, we henceforth abbreviate:

$$\gamma(\varphi) := \gamma(\varphi, \hat{\mathfrak{g}}, 0), \quad \varepsilon(\varphi) := \varepsilon(\varphi, \hat{\mathfrak{g}}, 0), \quad \omega(\varphi) := \omega(\varphi, \hat{\mathfrak{g}}), \quad \alpha(\varphi) := \alpha(\varphi, \hat{\mathfrak{g}}), \quad b(\varphi) := b(\varphi, \hat{\mathfrak{g}}).$$

From our results in section 2.3, we have that

$$\varepsilon(\varphi) = \omega(\varphi) q^{\alpha(\varphi)/2},$$

with

$$\alpha(\varphi) = a(\hat{\mathfrak{g}}) + \sum_{n \geq 1} n \cdot \dim \hat{\mathfrak{g}}_n^{\mathcal{I}} \equiv a(\det \hat{\mathfrak{g}}) \mod 2 \tag{15}$$

and

$$\omega(\varphi)^2 = \det \hat{\mathfrak{g}}(-1), \tag{16}$$

where

$$\det \hat{\mathfrak{g}} = \det X : \mathrm{Gal}(k_0/k) \to \{\pm 1\},$$

by results in [18], where we recall that $X = X_*(\hat{T})$. In particular, if k_0 is an unramified extension of k then $\alpha(\varphi)$ is even and $\omega(\varphi) = \pm 1$.

3.3 The principal parameter

Recall that the torus T splits over k_0, so its character group X is a $\mathbf{Z}$-representation of the finite group

$$^LW := \mathrm{Gal}(k_0/k) \ltimes W,$$

where $W = N_{G_0}(T)/T$ is the absolute Weyl group of T in G_0.

The choice of pinning $\mathcal{E}$ in $\hat{G}$ determines a principal nilpotent element

$$N_0 = \sum_{\hat{\alpha}\in\hat{\Pi}} x_{\hat{\alpha}} \in \hat{\mathfrak{g}} := \mathrm{Lie}(\hat{G}),$$

which is fixed by $\mathrm{Gal}(k_0/k)$. Applying the Jacobson-Morozov theorem to the group of $\mathrm{Gal}(k_0/k)$-invariants in $\hat{G}$, there exists a homomorphism $SL_2 \to \hat{G}$ which is pointwise fixed by $\mathrm{Gal}(k_0/k)$, whose differential maps a nonzero nilpotent element in $\mathfrak{sl}_2$ to N_0. Thus, the choice of pinning gives a homomorphism

$$\varphi_0 : \mathrm{Gal}(k_0/k) \times SL_2 \to {}^LG. \tag{17}$$

The pullback of φ_0 to $\mathcal{W} \times SL_2$ via the natural surjection $\mathcal{W} \to \mathrm{Gal}(k_0/k)$, also denoted by φ_0, is called the *principal parameter*. The centralizer $A_{\varphi_0} = C_{\hat{G}}(\varphi_0)$ of the principal parameter is just the center ${}^LZ$ of ${}^LG$, which is finite by our hypothesis that the maximal torus in the center of G is anisotropic (see section 3.2). Hence the principal parameter is discrete.

Let $E = \oplus_{d\geq 1} E_d$ be the graded $\mathbf{Q}$-vector space given by the homogeneous generators of the W-invariants in the symmetric algebra on $\mathbf{Q} \otimes X$. The subspace E_d, spanned by the invariant generators in degree d, is a $\mathbf{Q}$-representation of $\mathrm{Gal}(k_0/k)$. Refining results of Kostant, it was proved in [16] that the action of $\mathrm{Gal}(k_0/k) \times SL_2$ on $\hat{\mathfrak{g}}$ via $\mathrm{Ad}\,\varphi_0$ decomposes as

$$\hat{\mathfrak{g}} \simeq \bigoplus_{d\geq 1} E_d \otimes \mathrm{Sym}^{2d-2}. \tag{18}$$

In particular, we see that $\mathrm{Gal}(k_0/k) \hookrightarrow \mathrm{Aut}(\hat{G}, \mathcal{E})$ acts on $\hat{\mathfrak{g}}$ by the representation:

$$\bigoplus_{d\geq 1} (2d-1)E_d.$$

Hence the gamma factor (see section 2.2)

$$\gamma(\varphi_0) = \omega(\varphi_0) q^{\alpha(\varphi_0)/2} \frac{L(\varphi_0, \hat{\mathfrak{g}}, 1)}{L(\varphi_0, \hat{\mathfrak{g}}, 0)} \tag{19}$$

can be written explicitly as follows. The L-function is given by

$$L(\varphi_0, \hat{\mathfrak{g}}, s) = \prod_d \det(1 - q^{1-d-s} F_0 | E_d^{\mathcal{I}})^{-1},$$

where $F_0 = \varphi_0(\mathrm{Fr}) \in \mathrm{Aut}(\hat{G}, \mathcal{E})$. The Artin conductor is given by

$$\begin{aligned}
\alpha(\varphi_0) &= a(\varphi_0) + \sum_d (2d-2)\dim E_d^{\mathcal{I}} \\
&= \sum_d (2d-1)a(E_d) + \sum_d (2d-2)\dim E_d^{\mathcal{I}} \\
&= \sum_d (2d-1)[\dim(E_d/E_d^{\mathcal{I}}) + b(E_d)] + \sum_d (2d-2)\dim E_d^{\mathcal{I}} \\
&= \sum_d (2d-1)[\dim E_d + b(E_d)] - \dim E^{\mathcal{I}} \\
&= \dim \hat{\mathfrak{g}} - \dim E^{\mathcal{I}} + \sum_d (2d-1)b(E_d),
\end{aligned} \tag{20}$$

where we recall that $b(E_d)$ denotes the Swan conductor of the representation of $\mathcal{W}$ on E_d. Since F_0 is orthogonal and rational on each $E_d^{\mathcal{I}}$, it follows that F_0 and F_0^{-1} have the same set of eigenvalues and

$$\prod_d \det(-F_0|E_d^{\mathcal{I}})^{-1} = \det(-F_0|E^{\mathcal{I}})^{-1} = (-1)^{r(G_0)},$$

where $r(G_0)$ is the k-rank of G_0. From equation (11) we find that the root number of the principal parameter is given by

$$\omega(\varphi_0) = \prod_d w(E_d)^{2d-1} \cdot \prod_d \det(-F_0|E_d^{\mathcal{I}})^{2d-2} = \prod_d w(E_d)^{2d-1}. \tag{21}$$

If k_0/k is unramified then $b(E_d) = 0$ and $w(E_d) = +1$ for all d, so we have

$$\alpha(\varphi_0) = \dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{t}} \quad \text{and} \quad \omega(\varphi_0) = +1.$$

3.4 The motive of G

The adjoint L-function of the principal parameter φ_0 is closely related to the L-function of the Artin-Tate motive $M = M_G$ of G [15], defined by

$$\begin{aligned}
M = M_G &= \bigoplus_{d \geq 1} E_d(1-d), \\
\check{M}(1) &= \bigoplus_{d \geq 1} E_d(d).
\end{aligned} \tag{22}$$

Here the twisting $E(n)$ of a $\mathrm{Gal}(\bar{k}/k)$-module E over $\mathbf{Q}$ is defined by:

$$E(n) = E \otimes \mathbf{Q}(1)^{\otimes n},$$

and for any prime ℓ (recall that k has characteristic zero) the ℓ-adic realization of the Tate motive $\mathbf{Q}(1)$ is given by

$$\mathbf{Q}_\ell(1) = \varprojlim_n \mu_{\ell^n}$$

and μ_{ℓ^n} is the ℓ^n-torsion subgroup of $\bar{k}^\times$. A better definition of $M \otimes \mathbf{Q}_\ell$ in terms of the ℓ-adic cohomology of G has been given by Yu [63].

Since k is p-adic, the Tate motive $\mathbf{Q}_\ell(n)$ affords (for any prime $\ell \neq p$) the unramified character of $\mathrm{Gal}(\bar{k}/k)$ sending $\mathrm{Fr} \mapsto q^{-n}$. It follows that

$$\begin{aligned} L(M,s) &= \prod_{d\geq 1} \det\left(1 - q^{d-1-s} F_0 | E_d^{\mathcal{I}}\right)^{-1}, \\ L(\check{M}(1),s) &= \prod_{d\geq 1} \det\left(1 - q^{-d-s} F_0 | E_d^{\mathcal{I}}\right)^{-1}. \end{aligned} \tag{23}$$

In particular, we have

$$L(\check{M},s) = L(\varphi_0, \hat{\mathfrak{g}}, s), \tag{24}$$

where φ_0 is the principal parameter. We are interested in the values

$$L(M) := L(M,0), \qquad \text{and} \qquad L(\check{M}(1)) := L(\check{M}(1),0).$$

Since Fr acts on E via a finite order automorphism defined over $\mathbf{Q}$, $L(\check{M}(1),s)$ is always regular at $s=0$ and $L(\check{M}(1))$ is a positive rational number.

The function $L(M,s)$ is regular at $s=0$ if and only if $\mathrm{Gal}(k_0/k)$ has no non-trivial invariants in E_1. Since E_1 is the $\mathbf{Q}$-vector space spanned by the character group of the maximal torus C in the center of G, we see that $L(M)$ is finite precisely when C is anisotropic over k, which we have assumed. Thus, $L(M)$ is a nonzero rational number with sign $(-1)^{r(G_0)}$.

The conductor and root number of the motive M are defined in terms of their classical counterparts by

$$a(M) = \sum_{d\geq 1}(2d-1)a(E_d), \qquad w(M) = \prod_{d\geq 1} w(E_d).$$

Since each E_d is an orthogonal representation of $\mathrm{Gal}(k_0/k)$ and $\det(\oplus E_d) = \det(X) = \det(\hat{\mathfrak{g}})$ as a character of $k^\times$, we find that

$$a(M) \equiv a(\oplus E_d) \equiv a(\det(\hat{\mathfrak{g}})) \mod 2, \qquad \text{and} \qquad w(M)^2 = \det(\hat{\mathfrak{g}})(-1).$$

The epsilon factor of the motive is then defined by

$$\varepsilon(M,s) = w(M) \cdot q^{a(M)(\frac{1}{2}-s)}.$$

We set $\varepsilon(M) = \varepsilon(M, 0) = w(M) \cdot q^{a(M)/2}$. Finally, the gamma factor of M is defined by

$$\gamma(M) = \frac{L(\check{M}(1)) \cdot \varepsilon(M)}{L(M)}. \tag{25}$$

We now show that the nonzero complex numbers $\gamma(M)$ and $\gamma(\varphi_0)$ agree up to a sign. From equation (24), we have

$$L(\varphi_0, \hat{\mathfrak{g}}, 1) = L(\check{M}(1)). \tag{26}$$

Since

$$L(\varphi_0, \hat{\mathfrak{g}}, 0) = \prod_d \det(-q^{1-d} F_0 | E_d^{\mathcal{I}})^{-1} \cdot \det(1 - q^{d-1} F_0^{-1} | E_d^{\mathcal{I}})^{-1},$$

we have

$$L(\varphi_0, \hat{\mathfrak{g}}, 0) = (-1)^{r(G_0)} q^{\nu} L(M), \tag{27}$$

where

$$\nu = \sum_d (d-1) \dim E_d^{\mathcal{I}}.$$

It follows that

$$\frac{\gamma(\varphi_0)}{\gamma(M)} = (-1)^{r(G_0)} \frac{\omega(\varphi_0)}{w(M)} = \pm 1. \tag{28}$$

This sign can be computed from the action of $\mathrm{Gal}(k_0/k)$ on E. Indeed, from equation (12), we have

$$\frac{\omega(\varphi_0)}{w(M)} = \prod_d w(E_d)^{2d-2} = \prod_d \det(E_d)(-1)^{d-1} \tag{29}$$

If G is absolutely simple, one can check in each case that the ratio (29) is $+1$ unless G has type D_{2n}, and the splitting field k_0 of G is a ramified quadratic extension of k where -1 is not a norm.

4 Rationality of adjoint gamma factors

We have seen that our assumption that the connected center C of G is anisotropic is equivalent to assuming that the principal adjoint gamma value

$$\gamma(\varphi_0) = \frac{L(\varphi_0, \hat{\mathfrak{g}}, 1) \cdot \varepsilon(\varphi_0, \hat{\mathfrak{g}}, 0)}{L(\varphi_0, \hat{\mathfrak{g}}, 0)}$$

is nonzero.

We have also seen that a Langlands parameter $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ is discrete precisely when the adjoint gamma value

$$\gamma(\varphi) = \frac{L(\varphi, \hat{\mathfrak{g}}, 1) \cdot \varepsilon(\varphi, \hat{\mathfrak{g}}, 0)}{L(\varphi, \hat{\mathfrak{g}}, 0)}$$

is nonzero. Our goal in this chapter is to define a rational function $\Gamma_\varphi(x)$ with rational coefficients, which is an invariant of the discrete parameter φ. We will show that $\gamma(\varphi) = \Gamma_\varphi(q)$, and in particular that $\gamma(\varphi)$ is a nonzero rational number.

To state this result precisely, we first recall from Lemma 3.1 that if φ is a discrete parameter the image $D = \varphi(\mathcal{W})$ is finite. Let d be the cardinality of D and let $m \geq 1$ be an integer with $m \equiv 1 \mod 4d$. (This congruence condition will be explained in the proof of Prop. 4.1 below.) Let $\mathcal{W}_m = \mathcal{I} \ltimes \langle \mathrm{Fr}^m \rangle$ be the Weil group of the unramified extension k_m of k of degree m. The restriction φ^m of φ to $\mathcal{W}_m \times SL_2$ is a discrete parameter for $\mathcal{W}_m$, as

$$\varphi^m(\mathrm{Fr}^m) = \varphi(\mathrm{Fr}^m) = \varphi(\mathrm{Fr})^m = \varphi(\mathrm{Fr}).$$

Hence we may consider the ratios $\gamma(\varphi^m)/\gamma(\varphi_0^m) \in \mathbf{C}^\times$ for all $m \equiv 1 \mod 4d$. We can now state our rationality result as follows.

Proposition 4.1 *Let $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ be a discrete parameter and let $d = |\varphi(\mathcal{W})|$. Then there is a unique rational function $\Gamma_\varphi(x) \in \mathbf{Q}(x)$ such that*

$$\gamma(\varphi^m)/\gamma(\varphi_0^m) = \Gamma_\varphi(q^m)$$

for all $m \geq 1$ with $m \equiv 1 \mod 4d$. In particular, the ratio $\gamma(\varphi)/\gamma(\varphi_0) = \Gamma_\varphi(q)$ is a nonzero rational number.

Individually, the values

$$\varepsilon(\varphi) = \omega(\varphi) q^{\alpha(\varphi)/2} \quad \text{and} \quad \varepsilon(\varphi_0) = \omega(\varphi_0) q^{\alpha(\varphi_0)/2}$$

are both powers of i (a complex square root of -1) times an integral power of $q^{1/2}$, hence need not be rational. Likewise, the values $L(\varphi, \hat{\mathfrak{g}}, 0)$ and $L(\varphi, \hat{\mathfrak{g}}, 1)$ are products of terms of the form $(1-q^{k/2}\lambda)^{-1}$, where k is an integer, possibly odd, and λ is a root of unity. To prove Proposition 4.1 we first consider the ratio $\varepsilon(\varphi)/\varepsilon(\varphi_0)$, and then the values $L(\varphi, \hat{\mathfrak{g}}, 0)$ and $L(\varphi, \hat{\mathfrak{g}}, 1)$. The proof will give more information about these gamma ratios than is stated in Proposition 4.1. For example, we will see that there is a polynomial $\Delta_G \in \mathbf{Z}[x]$, depending only on G and k, such that $\Delta_G \Gamma_\varphi \in \mathbb{Z}[x]$.

4.1 The epsilon ratio

Since φ and φ^m have the same image, it is clear that their Artin conductors coincide:

$$\alpha(\varphi) = \alpha(\varphi^m). \tag{30}$$

Though it is less obvious, the root numbers also coincide:

$$\omega(\varphi) = \omega(\varphi^m). \tag{31}$$

To see this, recall that

$$\omega(\varphi) = \prod_{n\geq 0} w(\varphi, \hat{\mathfrak{g}}_n)^{n+1} \cdot \prod_{n\geq 1} \det(-\varphi(\mathrm{Fr})|\hat{\mathfrak{g}}_n^{\varphi(\mathcal{I})}),$$

and similarly for φ^m. We have $\varphi^m(\mathrm{Fr}) = \varphi(\mathrm{Fr})$ since $m \equiv 1 \mod d$. The congruence $m \equiv 1 \mod 4$ implies that

$$w(\varphi, \hat{\mathfrak{g}}_n) = w(\varphi^m, \hat{\mathfrak{g}}_n)$$

for all n. Indeed, take any integer $m \geq 1$, any self-dual representation $\rho : \mathcal{W} \to GL(V)$ and let ρ^m be the restriction of ρ to the Weil group $\mathcal{W}_m$ of the unramified extension k_m of degree m over k. Then since k_m/k is unramified, we have [58, 3.4]

$$w(\rho^m, V) = w(\rho, \mathrm{Ind}_{\mathcal{W}_m}^{\mathcal{W}} V) = w(\rho, V \otimes \mathrm{Ind}_{\mathcal{W}_m}^{\mathcal{W}} \mathbf{C}) = w(\rho, V)^m \cdot (-1)^{(m-1)a(V)}.$$

Hence if $m \equiv 1 \mod 4$, we have

$$w(\rho^m, V) = w(\rho, V).$$

Thus equation (31) is verified.

Proposition 4.2 *For every integer $m \equiv 1 \mod 4d$, we have*

$$\frac{\varepsilon(\varphi^m)}{\varepsilon(\varphi_0^m)} = \pm q^{m\cdot k(\varphi)},$$

where $k(\varphi)$ is an integer and the sign $\pm$ is independent of m and given by $\omega(\varphi)/\omega(\varphi_0) = \pm 1$.

Proof: By equations (30) and (31), it suffices to prove this for $m = 1$. We have

$$\frac{\varepsilon(\varphi)}{\varepsilon(\varphi_0)} = \frac{\omega(\varphi)}{\omega(\varphi_0)} \cdot q^{(\alpha(\varphi)-\alpha(\varphi_0))/2}.$$

By Prop. 2.3 we have

$$\alpha(\varphi_0) \equiv a(\det(\hat{\mathfrak{g}})) \mod 2.$$

But also

$$\alpha(\varphi) \equiv a(\det(\hat{\mathfrak{g}})) \mod 2,$$

by (15). Hence $\alpha(\varphi) \equiv \alpha(\varphi_0) \mod 2$. Likewise, we have

$$\omega(\varphi_0)^2 = \det(X)(-1) = \det(\hat{\mathfrak{g}})(-1) = \omega(\varphi)^2,$$

this last by (16). Thus we have shown that $\varepsilon(\varphi)/\varepsilon(\varphi_0)$ has the form $\pm q^{k(\varphi)}$, with $k(\varphi) = (\alpha(\varphi) - \alpha(\varphi_0))/2$ an integer, and sign $\omega(\varphi)/\omega(\varphi_0) = \pm 1$ as claimed. ■

4.2 Rationality of L-values

Let $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ be a discrete parameter and set $F := \varphi(\mathrm{Fr})$. In this section we complete the proof of Proposition 4.1 by analyzing the adjoint L-function

$$L(\varphi, \hat{\mathfrak{g}}, s) = \prod_{n \geq 0} \det\left(1 - q^{-s-n/2} F | \hat{\mathfrak{g}}_n^{\mathcal{I}}\right)^{-1},$$

where we recall that

$$\hat{\mathfrak{g}} \simeq \bigoplus_n \hat{\mathfrak{g}}_n \otimes \mathrm{Sym}^n$$

as a representation of $\mathcal{W} \times SL_2$ under $\mathrm{Ad}(\varphi)$.

Proposition 4.3 *For each $n \geq 0$ the polynomial $P_n(x) = \det(1 - xF|\hat{\mathfrak{g}}_n^{\mathcal{I}})$ has integral coefficients and is an even function of x when n is odd. More precisely, for $n \geq 1$ the factorization of $P_n(x)$ has the form*

$$P_n(x) = \prod R_i(x^{e_i}),$$

where each e_i is a positive integer such that ne_i is even, and each R_i is one of the six cyclotomic polynomials:

$$1 - x, \quad 1 + x, \quad 1 + x + x^2, \quad 1 + x^2, \quad 1 + x + x^2 + x^3 + x^4, \quad 1 - x + x^2,$$

satisfied by the m^{th} roots of unity for $1 \leq m \leq 6$. The polynomial $P_0(x)$ can have arbitrary cyclotomic factors, with the single restriction that $P_0(1) \neq 0$.

Proof: We first prove our assertions about $P_0(x)$. Since $\mathcal{I}$ is finite, the Lie algebra $\hat{\mathfrak{g}}^{\mathcal{I}}$ is reductive, as is the SL_2-invariant subalgebra $\hat{\mathfrak{g}}_0^{\mathcal{I}}$ (use [6, I.6.4 Prop.5], as in section 2.1).

From [56, Thm. 7.5], it follows (see, for example, [43]) that any semisimple automorphism τ of a complex semisimple Lie algebra $\mathfrak{h}$ may be written as a commuting product $\tau = \sigma s$, where s is inner and σ preserves a pinning in $\mathfrak{h}$. The fixed-point subalgebras $\mathfrak{h}^\tau$ and $\mathfrak{h}^\sigma$ both have rank equal to the number of σ-orbits on simple roots of $\mathfrak{h}$. In particular $\mathfrak{h}^\tau$ is nonzero.

Hence the discreteness of φ forces the derived subalgebra of $\hat{\mathfrak{g}}_0^{\mathcal{I}}$ to be zero. Hence $\hat{\mathfrak{g}}_0^{\mathcal{I}}$ is the Lie algebra of a torus $\hat{S} \subset \hat{G}$, and is spanned by the co-weight lattice of $\hat{S}$. Since $\hat{\mathfrak{g}}_0^{\mathcal{I}}$ is preserved by F, it follows that $\hat{S}$ and its co-weight lattice are preserved by F. Hence $P_0(x)$ has integral coefficients. We have already noted that $P_0(1) \neq 0$. Finally, given an integral polynomial $Q(x)$ of degree d whose zeros are roots of unity such that $Q(0) = 1$ and $Q(1) \neq 0$, there exists a $d \times d$ integer matrix A such that $\det(1 - xA) = Q(x)$. Taking G to be an unramified anisotropic torus with character group $\mathbf{Z}^d$ on which Fr acts via A, we have $P_0(x) = Q(x)$.

For $n \geq 1$, the factor $P_n(x)$ arises in the derived subalgebra $[\hat{\mathfrak{g}}^{\mathcal{I}}, \hat{\mathfrak{g}}^{\mathcal{I}}]$, since the center of $\hat{\mathfrak{g}}^{\mathcal{I}}$ is contained in $\hat{\mathfrak{g}}_0^{\mathcal{I}}$. Thus we are reduced to a question about semisimple Lie algebras. We change notation and replace $[\hat{\mathfrak{g}}^{\mathcal{I}}, \hat{\mathfrak{g}}^{\mathcal{I}}]$ by an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g})$ be its automorphism group and let $G = \mathrm{Aut}(\mathfrak{g})^\circ$. Fix a pinning $\mathcal{E} = (T, B, \{x_\alpha\})$ in G. Then $\mathrm{Aut}(G, \mathcal{E})$ may be viewed as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ and we have

$$\mathrm{Aut}(\mathfrak{g}) = G \rtimes \mathrm{Aut}(G, \mathcal{E}).$$

We have an element $F \in \mathrm{Aut}(\mathfrak{g})$ of finite order, and an algebraic homomorphism $\varphi : SL_2 \to G^F$ whose centralizer A_φ in G^F is finite. This implies the center of G^F is finite, hence $\mathfrak{g}^F$ is semisimple. We have

$$\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}_n \otimes \mathrm{Sym}^n$$

under $\langle F \rangle \times SL_2$, and we must show that each $P_n(x) = \det(1 - xF|\mathfrak{g}_n)$ has the asserted form. Since $P_n(x)$ is the product of factors arising from the orbits of F on the simple factors of $\mathfrak{g}$, we may assume that $\mathfrak{g} = \mathfrak{h}^e$ is a direct sum of e copies of a simple Lie algebra $\mathfrak{h}$ and that F acts transitively on these summands. Then the representation of $\langle F \rangle \times SL_2$ on $\mathfrak{g}$ is induced from the representation of $\langle F^e \rangle \times SL_2$ on $\mathfrak{h}$ and we have

$$\det(1 - xF|\mathfrak{g}_n) = \det(1 - x^e F^e|\mathfrak{h}_n).$$

We may therefore assume that $\mathfrak{g}$ is simple.

We invoke Kac's classification [29] of torsion automorphisms of $\mathfrak{g}$, considering only those $F \in \mathrm{Aut}(\mathfrak{g})$ for which $\mathfrak{g}^F$ is semisimple (see also [43]). The projection of F in $\mathrm{Aut}(G, \mathcal{E})$ is a pinned automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ of order $f \in \{1, 2, 3\}$. Each σ determines a Dynkin diagram of type ${}^f X$, where X is the type of $\mathfrak{g}$, whose nodes correspond to simple roots of the connected group G^σ, which is also of adjoint type. Each node i in the diagram is labelled with an integer $a_i \geq 1$, giving the multiplicity of that root in the highest root of G (when $f = 1$), the highest short root in G^σ (when $f = 2$ or 3 except in type ${}^2A_{2n}$), or twice the highest short root of G^σ in type ${}^2A_{2n}$.

For each node i there is an inner automorphism $s_i \in G^F$ of order $m_i = fa_i$, and F is G-conjugate to σs_i for some i. Examining the various cases, we find that $a_i \leq 6$ always, that $a_i \leq 3$ if $f = 2$ and $a_i \leq 2$ if $f = 3$. It follows that the order m of F satisfies $m \leq 6$ in all cases.

We can now prove the key result for the rationality of L-values:

Lemma 4.4 *Assume $\mathfrak{g}$ is a simple complex Lie algebra with $G = \mathrm{Aut}(\mathfrak{g})^\circ$. Let $F \in \mathrm{Aut}(\mathfrak{g})$ have finite order and suppose $\varphi : SL_2 \to G^F$ is an algebraic homomorphism whose centralizer A_φ in G^F is finite. Decompose*

$$\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}_n \otimes \mathrm{Sym}^n$$

as a representation of $\langle F \rangle \times SL_2$. Then for each $n \geq 0$, the characteristic polynomial $\det(1 - xF|\mathfrak{g}_n)$ *has integer coefficients and its irreducible factors over* $\mathbf{Q}$ *are among the six cyclotomic polynomials listed in Proposition 4.3.*

Proof: As $\mathfrak{g}$ is an orthogonal representation of $\langle F\rangle$, each representation $\mathfrak{g}_n$ is self-dual. It follows that $\det(1-xF|\mathfrak{g}_n)$ has real coefficients. We have seen that the eigenvalues of F are m^{th} roots of unity with $m \leq 6$. Hence the coefficients in $\det(1-xF|\mathfrak{g}_n)$ are integers, except possibly when $m=5$.

The case $m=5$ occurs only once, when $G=E_8$. In this case, $F \in G$ and $G^F = C_G(F)$ is isogenous to $SL_5 \times SL_5$. The finiteness of A_φ forces $\varphi : SL_2 \to G^F$ to be principal. But then φ has Bala-Carter type $E_8(a_7)$ [9] and the centralizer $C_G(\varphi)$ is the symmetric group S_5. Each $\mathfrak{g}_n$ is actually an S_5-module, and every S_5-module is rational, so $\det(1-xF|\mathfrak{g}_n)$ has integral coefficients, completing the proof of Lemma 4.4.

Proposition 4.3 is now proved in the case that $\mathfrak{g}_n$ is nonzero only for even n. This occurs precisely when the map $\varphi : SL_2 \to G^F \subset \mathrm{Aut}(\mathfrak{g})$ factors through PGL_2. So we must consider the cases where $\varphi(-I) \neq 1$, where $-I$ is the central element of SL_2, and we must show that $\det(1-xF|\mathfrak{g}_n)$ is an even polynomial when n is odd.

Lemma 4.5 *If $\varphi(-I) \neq 1$, then F has order four and $\varphi(-I) = F^2$.*

Proof: Recall that $F = \sigma s$, where s, of order $m = fa$, belongs to the connected adjont group G^σ, and σ has order f. Hence $F^f = s^f$ has order a and lies in the center C of $(G^F)^\circ$. In fact, C is generated by F^f [43]. Since A_φ is finite, the mapping $\varphi : SL_2 \to (G^F)^\circ$ is distinguished and therefore even [9]. It follows that $\varphi(-I) \in C$, so that $\varphi(-I) = F^{kf}$ for some integer k, taken modulo a, with $2k \equiv 0 \mod a$. Since $\varphi(-I) \neq 1$, we must have a even, so $m \in \{2,4,6\}$ by Kac's classification. It suffices to show that $m=4$.

If $m=2$, so that $F^2=1$, then $\varphi(-I)=F$, since $\varphi(-I)\neq 1$. but then A_φ is the centralizer of $\varphi(SL_2)$ in G. Since A_φ is finite, this means φ is distinguished (see [9]), hence even, so $\varphi(-I)=1$, a contradiction. Therefore $m \in \{4,6\}$.

If $m=6$ there are only two possibilities:

$$\begin{aligned} G &= E_8 \quad &\text{and}\quad G^F &= (SL_2 \times SL_3 \times SL_6)/\Delta\mu_6, \quad \text{or}\\ G &= PSO_8 \quad &\text{and}\quad G^F &= (SL_2 \times SL_2)/\Delta\mu_2, \end{aligned}$$

where Δ is a diagonal mapping of μ_d onto the center of each factor. In both cases, the only distinguished $\varphi : SL_2 \to G^F$ is principal and we again have $\varphi(-I)=1$, so $m \neq 6$. The lemma is proved. ■

We pause to remark that a case where $\varphi(-I) \neq 1$ and F has order four occurs in E_7. Here $\sigma=1$ and i corresponds to the branch node. The group $(G^F)^\circ$ is isogenous to $SL_4 \times SL_2 \times SL_4$. Under the principal homomorphism $\varphi : SL_2 \to (G^F)^\circ$, the F-eigenspace $\mathfrak{g}(i)$ decomposes as

$$\mathfrak{g}(i) = \mathrm{Sym}^7 + 2\,\mathrm{Sym}^5 + 2\,\mathrm{Sym}^3 + 2\,\mathrm{Sym}^1$$

and $\varphi(-I) = -1$ on each summand.

Returning to our task, we complete the proof of Proposition 4.3 as follows. By Lemma 4.5, we may assume that F has order four, with eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}(1) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(i) \oplus \mathfrak{g}(-i).$$

Since $\varphi(-I) = F^2$ acts by -1 on Sym^n , we have

$$\mathfrak{g}_n \otimes \mathrm{Sym}^n \subset \mathfrak{g}(i) + \mathfrak{g}(-i),$$

and the eigenspaces $\mathfrak{g}(i)$ and $\mathfrak{g}(-i)$ are dual to each other, since $\mathfrak{g}$ is orthogonal for F. Hence the only eigenvalues of F on $\mathfrak{g}_n$ are $\pm i$ and these come in pairs. It follows that

$$\det(1 - xF|\mathfrak{g}_n) = (1 + x^2)^k$$

where the symplectic representation $\mathfrak{g}_n$ of $\langle F \rangle$ has dimension $2k$. Thus, Proposition 4.3 is proved.

For integer $k \geq 0$, the polynomial $P_{2k+1}(x)$ in Proposition 4.3 is even. Let $Q_{2k+1}(x)$ be the polynomial with integer coefficients such that $P_{2k+1}(x) = Q_{2k+1}(x^2)$. Let $P_n^0(x)$ be the analogous polynomials for the principal parameter φ_0; these are zero for odd n because $-I \in \ker \varphi_0$.

We have shown that

$$\frac{\gamma(\varphi^m)}{\gamma({\varphi_0}^m)} = \Gamma_\varphi(q^m),$$

where $\Gamma_\varphi(x) \in \mathbf{Q}(x)$ is the rational function

$$\Gamma_\varphi(x) = \pm x^{m(\varphi)} \cdot \prod_{k \geq 0} \frac{P_{2k}(x^{-k})}{P_{2k}(x^{-1-k})} \cdot \frac{Q_{2k+1}(x^{-2k-1})}{Q_{2k+1}(x^{-2k-3})} \cdot \frac{P_{2k}^0(x^{-1-k})}{P_{2k}^0(x^{-k})} \tag{32}$$

Here the integer $m(\varphi) = (\alpha(\varphi) - \alpha(\varphi_0))/2$ and the sign $\pm 1 = \omega(\varphi)/\omega(\varphi_0)$ are as in Proposition 4.2. This completes the proof of Proposition 4.1.

5 An inequality for the Swan conductor

We have seen that the adjoint gamma ratio $\gamma(\varphi)/\gamma(\varphi_0)$ is the value at q of a rational function $\Gamma_\varphi(x) \in \mathbf{Q}(x)$. In this chapter, we find the order of $\Gamma_\varphi(x)$ at $x = 0$. This leads to a conjectural inequality for the Swan conductor of a discrete parameter, which we verify in some interesting cases. When equality is achieved, we obtain conditions on φ and we are led to the notion of a *simple wild parameter*. Later in the paper, simple wild parameters will arise in the local Langlands correspondence.

5.1 The leading term in the gamma ratio

We will compute the order of $\Gamma_\varphi(x)$ at $x = 0$ in terms of the ramification of φ and the variety of Borel subalgebras of $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}$. This generalizes a result for unramified parameters in [41] which was in turn inspired by a conjecture of Lusztig [39] on formal degrees for affine Hecke algebras.

Our result is as follows. Let $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ be a discrete parameter, and let N be the image under the differential $d\varphi : \mathfrak{sl}_2 \to \hat{\mathfrak{g}}$ of a nonzero nilpotent element in $\mathfrak{sl}_2$. Let $\hat{\mathfrak{g}}_N^{\varphi(\mathcal{I})}$ denote the kernel of $\mathrm{ad}(N)$ on the space $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}$ of inertial invariants, and set

$$\ell_\varphi := \dim \hat{\mathfrak{g}}_N^{\varphi(\mathcal{I})}.$$

If N is a principal nilpotent element in $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}$, then ℓ_φ is the rank of $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}$. For the principal parameter φ_0 we write

$$\ell_0 := \ell_{\varphi_0}$$

Then $\ell_0 = \dim M^{\mathcal{I}}$ and is the rank of G over the maximal unramified extension K of k.

Proposition 5.1 *At $x = 0$, the rational function $\Gamma_\varphi(x)$ has order equal to $\frac{1}{2}[\ell_\varphi - \ell_0 + b(\varphi) - b(\varphi_0)]$.*

Proof: Given two rational functions $f(x), g(x)$ in the variable x, we write

$$f(x) \sim g(x) \qquad \text{if} \qquad \lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

As before, we set

$$\hat{\mathfrak{h}} = \hat{\mathfrak{g}}^{\varphi(\mathcal{I})}, \qquad \hat{\mathfrak{h}}_n = \hat{\mathfrak{g}}_n^{\varphi(\mathcal{I})}, \qquad F = \varphi(\mathrm{Fr}),$$

so that

$$L(\varphi, \hat{\mathfrak{g}}, s) = L_\varphi(q, s),$$

where

$$L_\varphi(x, s) = \prod_{n \geq 0} \det(1 - x^{-s-n/2} F | \hat{\mathfrak{h}}_n)^{-1} \tag{33}$$

is a rational function in x with complex coefficients. Since $\det F = \pm 1$ on $\hat{\mathfrak{h}}_n$, we have

$$\begin{aligned} L_\varphi(x, 0) &= \prod_{n \geq 0} \det(1 - x^{-n/2} F | \hat{\mathfrak{h}}_n)^{-1} \\ &= \prod_{n \geq 0} \det(-x^{-n/2} F | \hat{\mathfrak{h}}_n)^{-1} \cdot \det(1 - x^{n/2} F | \hat{\mathfrak{h}}_n)^{-1} \\ &\sim x^{d(\varphi)/2} \cdot \det(-F | \hat{\mathfrak{h}})^{-1} \cdot \det(1 - F | \hat{\mathfrak{h}}_0)^{-1}, \end{aligned}$$

where

$$d(\varphi) = \sum_{n \geq 0} n \dim \hat{\mathfrak{h}}_n.$$

Now $\hat{\mathfrak{h}}$ is the Lie algebra of the (possibly disconnected) subgroup

$$\hat{H} = C_{\hat{G}}(\varphi(\mathcal{I})) \subset \hat{G},$$

and $\hat{\mathfrak{h}}_0$ is the Lie algebra of the subgroup

$$M_\varphi = C_{\hat{H}}(\varphi(SL_2)) = C_{\hat{G}}(\varphi(\mathcal{I} \times SL_2))$$

whose identity component

$$C_\varphi := M_\varphi^\circ$$

is a torus with finite fixed point set under F, since φ is discrete. Indeed, we have

$$|C_\varphi^F| = \det(1 - F|\hat{\mathfrak{h}}_0).$$

We note that $A_\varphi = M_\varphi^F$ contains C_φ^F as a normal subgroup and since C_φ^F is finite, we have

$$A_\varphi / C_\varphi^F = \pi_0(M_\varphi)^F.$$

Returning to our calculation, we have

$$L_\varphi(x, 0) \sim \frac{x^{d(\varphi)/2}}{|C_\varphi^F| \cdot \det(-F|\hat{\mathfrak{h}})}. \tag{34}$$

A similar calculation gives

$$L_\varphi(x, 1) \sim \frac{x^{\ell_\varphi + d(\varphi)/2}}{\det(-F|\hat{\mathfrak{h}})}. \tag{35}$$

Thus, we find that

$$\frac{L_\varphi(x, 1)}{L_\varphi(x, 0)} \sim |C_\varphi^F| \cdot x^{\ell_\varphi}. \tag{36}$$

Applying this result to the principal parameter φ_0, we get

$$\frac{L_{\varphi_0}(x, 1)}{L_{\varphi_0}(x, 0)} \sim |C_{\varphi_0}^{F_0}| \cdot x^{\ell_0}, \tag{37}$$

where $F_0 = \varphi_0(\mathrm{Fr})$. We note that C_{φ_0} is the identity component of the subgroup of $\varphi_0(\mathcal{I})$-invariants in the center of $\hat{G}$. It follows that $C_{\varphi_0} \subset C_\varphi$ for any discrete parameter φ, and $C_{\varphi_0}^{F_0} = C_{\varphi_0}^F$, where $F = \varphi(\mathrm{Fr})$.

Turning to the epsilon ratio, we express the integer $k(\varphi) = \frac{1}{2}(\alpha(\varphi) - \alpha(\varphi_0))$ in terms of Swan conductors. We have

$$\alpha(\varphi) = a(\varphi, \hat{\mathfrak{g}}) + \sum_{n \geq 1} n \cdot \dim \hat{\mathfrak{h}}_n.$$

Since

$$\dim \hat{\mathfrak{h}} = \sum_{n \geq 0} (n+1) \dim \hat{\mathfrak{h}}_n \quad \text{and} \quad \dim \hat{\mathfrak{h}}_N = \sum_{n \geq 0} \dim \hat{\mathfrak{h}}_n = \ell_\varphi,$$

it follows that

$$\begin{aligned} \alpha(\varphi) &= \dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{h}} + b(\varphi) + \dim \hat{\mathfrak{h}} - \dim \hat{\mathfrak{h}}_N \\ &= \dim \hat{\mathfrak{g}} - \ell_\varphi + b(\varphi). \end{aligned} \tag{38}$$

Applying this result to φ_0, we get

$$\alpha(\varphi_0) = \dim \hat{\mathfrak{g}} - \ell_0 + b(\varphi_0), \tag{39}$$

so that

$$k(\varphi) = \tfrac{1}{2}[\alpha(\varphi) - \alpha(\varphi_0)] = \tfrac{1}{2}[\ell_0 - \ell_\varphi + b(\varphi) - b(\varphi_0)]. \tag{40}$$

Combining equations (40), (36) and (37), we get the following formula for the leading term of $\Gamma_\varphi(x)$:

$$\Gamma_\varphi(x) \sim \pm |C^F_\varphi / C^F_{\varphi_0}| \cdot x^{[\ell_\varphi - \ell_0 + b(\varphi) - b(\varphi_0)]/2}, \tag{41}$$

which gives the order of vanishing

$$\operatorname*{ord}_{x=0}(\Gamma_\varphi) = \tfrac{1}{2}[\ell_\varphi - \ell_0 + b(\varphi) - b(\varphi_0)]$$

asserted in Prop. 5.1. ∎

A variant: By a result of Spaltenstein [52, 2.8] we have

$$\ell_\varphi = \operatorname{rank}(\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}) + 2 \dim \mathcal{B}_N, \tag{42}$$

where $\mathcal{B}_N$ is the variety of Borel subalgebras of $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}$ containing N. Hence we may also express the order of vanishing as

$$\operatorname*{ord}_{x=0}(\Gamma_\varphi) = \dim \mathcal{B}_N + \tfrac{1}{2}[\operatorname{rank}(\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}) - \ell_0 + b(\varphi) - b(\varphi_0)]. \tag{43}$$

Example 1: Suppose φ is tamely ramified. Then the splitting field k_0 of G is also tamely ramified over k and we have $b(\varphi) = b(\varphi_0) = 0$. The image $\varphi_0(\mathcal{I}) \subset \mathrm{Gal}(k_0/k)$ is cyclic; let ϑ be a generator. Then

$$\ell_0 = \dim \hat{\mathfrak{t}}^\vartheta.$$

The image $\varphi(\mathcal{I}) = D_0$ is also cyclic, generated by an element $\sigma = \vartheta s$ for some $s \in \hat{G}$. Conjugating by $\hat{G}$, we may assume that $s \in \hat{T}^\vartheta$ (see [43]). Then $\hat{\mathfrak{t}}^\vartheta$ is a Cartan subalgebra of $\mathfrak{g}^{\varphi(\mathcal{I})} = \mathfrak{g}^\sigma$, so that

$$\operatorname{rank}(\hat{\mathfrak{g}}^{\varphi(\mathcal{I})}) = \dim \hat{\mathfrak{t}}^\vartheta = \ell_0.$$

It follows that

$$\operatorname*{ord}_{x=0}(\Gamma_\varphi) = \dim \mathcal{B}_N \tag{44}$$

(cf. [41, (7.2)] for some cases where φ is unramified).

Example 2: Suppose $\varphi(\mathcal{I}) \subset \hat{T}$ (so k_0/k is unramified). Then a similar argument shows that

$$\operatorname*{ord}_{x=0}(\Gamma_\varphi) = \dim \mathcal{B}_N + \tfrac{1}{2} b(\varphi). \tag{45}$$

(cf. [42, Prop. 4.1] for some cases where $N = 0$).

5.2 An inequality

We conjecture that the rational function $\Gamma_\varphi(x)$ is regular at $x = 0$, for any discrete parameter φ. By Prop. 5.1, this is equivalent to the following inequality.

Conjecture 5.2 *Let $\varphi : \mathcal{W} \times SL_2 \to {}^LG$ be a discrete parameter, let $\ell_\varphi = \dim \hat{\mathfrak{g}}_N^{\varphi(\mathcal{I})}$ and let $b(\varphi)$ be the Swan conductor of the representation $\mathrm{Ad}(\varphi)$ on $\hat{\mathfrak{g}}$. Then*

$$\ell_\varphi + b(\varphi) \geq \ell_0 + b(\varphi_0). \tag{46}$$

One can also be led to Conjecture 5.2 by considering formal degrees. See section 7.3 below.

Conjecture 5.2 holds if φ is tamely ramified or if $\varphi(\mathcal{I})$ is contained in a torus in $\hat{G}$, by Examples 1 and 2 above. In section 5.3 below we will verify Conjecture 5.2 when $\varphi(\mathcal{I}_+) \subset \hat{T}$ and $\hat{\mathfrak{g}}^{\varphi(\mathcal{I}_+)} = \hat{\mathfrak{t}}$.

If k_0/k is tamely ramified and $N = 0$ then

$$\ell_\varphi \leq \ell_0, \tag{47}$$

as we will show, so Conjecture 5.2 amounts to the nontrivial lower bound

$$b(\varphi) \geq \ell_0 - \ell_\varphi. \tag{48}$$

To see (47), let $\vartheta \in \mathrm{Gal}(k_0/k)$ be a generator of the inertia subgroup of $\mathrm{Gal}(k_0/k)$ and let $D_0 = \varphi(\mathcal{I})$. Then $D_0 = D_1 \ltimes \langle \sigma \rangle$, where $\sigma \in \vartheta\hat{G}$ is an element of ${}^LG$ of order prime to p. Since φ is discrete, the fixed point subalgebra $\hat{\mathfrak{g}}^{D_0} = (\hat{\mathfrak{g}}^{D_1})^\sigma$ is contained in a Cartan subalgebra of $\hat{\mathfrak{g}}^\sigma$. The Lie algebra $\hat{\mathfrak{g}}^\sigma$ has rank equal to ℓ_0 (see, e.g., [43]). It follows that $\ell_\varphi = \dim \hat{\mathfrak{g}}^{D_0} \leq \ell_0$, as claimed in (47).

5.3 Minimal Swan conductors

If equality holds in (46), that is, if

$$\operatorname*{ord}_{x=0}(\Gamma_\varphi) = 0,$$

we say that φ has *minimal Swan conductor*. The principal parameter φ_0 clearly has minimal Swan conductor. If φ is tamely ramified, then Example 1 above shows that φ has minimal Swan conductor precisely when N is a principal nilpotent element in $\mathfrak{g}^{\varphi(\mathcal{I})}$. For example, if $\mathfrak{g}^{\varphi(\mathcal{I})} = \hat{\mathfrak{t}}$ we have $N = 0$, which is principal in $\hat{\mathfrak{t}}$. Such parameters were studied in [10]. If $\varphi(\mathcal{I}) \subset \hat{T}$, as in Example 2, then φ can have minimal Swan conductor only if φ is tamely ramified, putting us in the previous case.

In this section, we will consider some wildly ramified parameters when G is split, simply-connected, and simple over k. We will verify the Swan inequality (46) for these parameters, and describe their image when equality holds. The dual group $\hat{G} = {}^L G$ is a simple complex Lie group of adjoint type. Let $\varphi : \mathcal{W} \times SL_2 \to \hat{G}$ be a discrete parameter for G.

Lemma 5.3 *We have $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$ if and only if $L(\varphi, \hat{\mathfrak{g}}, s) = 1$.*

Proof: If $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$ then it follows directly from the definition (6) that $L(\varphi, \hat{\mathfrak{g}}, s) = 1$. Conversely, suppose $L(\varphi, \hat{\mathfrak{g}}, s) = 1$. If $\varphi(SL_2)$ is nontrivial, then the representation $1 \otimes \mathrm{Sym}^2$ of $\mathcal{W} \times SL_2$ appears as a summand in $\hat{\mathfrak{g}}$, so that $L(\varphi, \hat{\mathfrak{g}}, s)$ is divisible by $(1 - q^{-1-s})^{-1}$. Hence $\varphi(SL_2) = 1$, and $L(\varphi, \hat{\mathfrak{g}}, s) = \det(1 - q^{-s}\varphi(\mathrm{Fr})|\hat{\mathfrak{g}}^{\varphi(\mathcal{I})})^{-1} = 1$, so we must have $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$. ∎

We say that a discrete parameter $\varphi : \mathcal{W} \to \hat{G}$ is *inertially discrete* if $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$. In this case, we have

$$\gamma(\varphi) = \varepsilon(\varphi, \hat{\mathfrak{g}}, 0) = \pm q^{a(\varphi)/2} \qquad \text{and} \qquad a(\varphi) = b(\varphi) + \dim(\hat{\mathfrak{g}}) \equiv 0 \mod 2,$$

as $\det \hat{\mathfrak{g}} = 1$. Since $\dim \hat{\mathfrak{g}} = \ell + 2 \cdot |\Phi^+|$, where ℓ is the rank of $\hat{G}$ and $\Phi^+ \subset \Phi$ is the set of positive roots, we have the congruence

$$b(\varphi) \equiv \ell \mod 2.$$

In this situation, we have $\ell_\varphi = b(\varphi_0) = 0$, so the Swan inequality conjecture (5.2) is simply

$$b(\varphi) \geq \ell. \tag{49}$$

We will first verify the inequality (49) in the case where the residual characteristic p of k does not divide the order of the Weyl group W of $\hat{G}$ (or equivalently, the Weyl group of G). A key role is played by the elements $w \in W$ for which $\hat{\mathfrak{t}}^w = 0$. We call such elements of W *elliptic*.

Proposition 5.4 *Assume that φ is inertially discrete and that p does not divide $|W|$. Let $D_0 = \varphi(\mathcal{I})$ and $D_1 = \varphi(\mathcal{I}_+)$ be the image of the inertia and wild inertia subgroups of $\mathcal{W}$ in $\hat{G}$. Then*

1. *D_1 lies in a unique maximal torus $\hat{T}$ of $\hat{G}$;*

2. *D_0 lies in the normalizer $N(\hat{T})$ of $\hat{T}$ in $\hat{G}$, and the image under the projection*

$$\mathcal{I} \xrightarrow{\varphi} N(\hat{T}) \longrightarrow N(\hat{T})/\hat{T} = W$$

of the tame quotient D_0/D_1 is a cyclic subgroup $\langle w \rangle$ of W, generated by an elliptic element w.

3. *We have the inequalities*

$$b(\varphi) \geq |\Phi/w| \geq \ell,$$

where Φ/w is a set of representatives for the orbits of $\langle w \rangle$ on the set Φ of roots of $\hat{T}$ in $\hat{\mathfrak{g}}$.

Proof: The image D_1 of wild inertia is a finite p-group inside $\hat{G}$. It is therefore nilpotent, and by a theorem of Borel and Serre [5], is contained in the normalizer $N(\hat{T})$ of a maximal torus $\hat{T}$. Since we have assumed that p does not divide $|W|$, the projection of D_1 to W is trivial, so D_1 is contained in $\hat{T}$.

To show that $\hat{T}$ is the *unique* maximal torus containing D_1, we need to show that $\hat{\mathfrak{g}}^{D_1} = \hat{\mathfrak{t}}$. If not, the derived subalgebra of $\hat{\mathfrak{g}}^{D_1}$ is a nontrivial semisimple Lie algebra $\mathfrak{s}$ with the property that $\mathfrak{s}^{D_0/D_1} = 0$. This is impossible, as D_0/D_1 is finite cyclic and any torsion automorphism of a semisimple Lie algebra has nontrivial invariants (see [43], for example). This proves 1.

Since $\mathcal{I}$ normalizes $\mathcal{I}_+$, it follows that D_0 normalizes $\hat{T}$, the connected centralizer of D_1 in $\hat{G}$. Let $\langle w \rangle$ be the image of the cyclic quotient D_0/D_1 in $W = N(\hat{T})/\hat{T}$. Since

$$0 = \hat{\mathfrak{g}}^{D_0} = \left(\hat{\mathfrak{g}}^{D_1}\right)^{D_0/D_1} = \hat{\mathfrak{t}}^w,$$

we see that w is elliptic. This proves 2.

As a representation of $D_1 \subset \hat{T}$, we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{t}} \oplus \sum_{\alpha \in \Phi} \hat{\mathfrak{g}}_\alpha,$$

according to the root-space decomposition of $\hat{\mathfrak{g}}$. Each root space $\hat{\mathfrak{g}}_\alpha$ is one-dimensional. The action of D_1 on $\hat{\mathfrak{g}}$ is trivial on $\hat{\mathfrak{t}}$ and non-trivial on each root space $\hat{\mathfrak{g}}_\alpha$. As a representation of $D_0 \subset N(\hat{T})$, we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{t}} \oplus \sum_{\alpha \in \Phi/w} \mathrm{Ind}_{D_\alpha}^{D_0} \hat{\mathfrak{g}}_\alpha,$$

where $D_1 \subset D_\alpha \subset D_0$ and D_α is the stabilizer of α in D_0. The action of D_0 on $\hat{\mathfrak{t}}$ is tame, generated by the elliptic element w, so $b(\hat{\mathfrak{t}}) = 0$. Thus, we have

$$b(\varphi) = \sum_{\alpha \in \Phi/w} b\left(\mathrm{Ind}_{D_\alpha}^{D_0} \hat{\mathfrak{g}}_\alpha\right).$$

By the inductivity of the Swan conductor, we have

$$b\left(\mathrm{Ind}_{D_\alpha}^{D_0}\hat{\mathfrak{g}}_\alpha\right) = b(\hat{\mathfrak{g}}_\alpha),$$

where the right-hand side is the Swan conductor of the one-dimensional representation $\hat{\mathfrak{g}}_\alpha$ of the local Galois group D_α. Since D_1 acts nontrivially on $\hat{\mathfrak{g}}_\alpha$, we have $b(\hat{\mathfrak{g}}_\alpha) \geq 1$ for all $\alpha \in \Phi/w$. Hence we have shown the first inequality in 3, namely $b(\varphi) \geq |\Phi/w|$.

To complete the proof, we will show that $|\Phi/w| \geq \ell$ for any elliptic element $w \in W$. Let n be a lift of w in $N(\hat{T})$. We compute

$$\ell \leq \dim \hat{\mathfrak{g}}^n = \dim \hat{\mathfrak{t}}^w + \sum_{\alpha \in \Phi/w} \dim\left(\mathrm{Ind}_{D_\alpha}^{D_0}\hat{\mathfrak{g}}_\alpha\right)^n.$$

But $\hat{\mathfrak{t}}^w = 0$ and n permutes the root spaces in each $\mathrm{Ind}_{D_\alpha}^{D_0}\hat{\mathfrak{g}}_\alpha$ transitively. Hence

$$\dim\left(\mathrm{Ind}_{D_\alpha}^{D_0}\hat{\mathfrak{g}}_\alpha\right)^n \leq 1$$

for all $\alpha \in \Phi/w$, hence $\ell \leq |\Phi/w|$, as claimed. ■

We now consider inertially discrete parameters φ where equality holds in the Swan inequality (49):

$$\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0 \qquad \text{and} \qquad b(\varphi) = \ell.$$

We will determine the structure of $D_0 = \varphi(\mathcal{I})$ when p does not divide the order of W. From Prop. 5.4, the equality $b(\varphi) = \ell$ implies that the elliptic element w has exactly ℓ orbits on the set Φ of roots.

Lemma 5.5 *Suppose $w \in W$ is an elliptic element with exactly ℓ orbits in Φ. Let $\Phi^\circ \subset \Phi$ be union of the free orbits of w in Φ. Then Φ° is nonempty and the intersection $\bigcap_{\alpha \in \Phi^\circ} \ker \alpha|_{\hat{T}}$ is contained in the 2-torsion subgroup $\hat{T}[2] \subset \hat{T}$. If $\Phi^\circ = \Phi$ then w is a Coxeter element.*

Proof: Suppose w belongs to a subgroup of W generated by reflections about roots from a proper root-subsystem Φ' of Φ. Since w elliptic, the rank of Φ' equals ℓ, so w has at least ℓ orbits in Φ', hence has more than ℓ orbits in Φ, a contradiction. It now follows from [53, 8.2] that $\det(1 - w|\hat{\mathfrak{t}})$ is as small as possible, namely equal to the order $|Z(\hat{G}_{sc})|$ of the simply-connected cover $\hat{G}_{sc}$ of $\hat{G}$.

Let $n \in N(\hat{T})$ be a lift of w. Since w has exactly ℓ orbits in Φ, the inequality $\dim \hat{\mathfrak{g}}^n \geq \ell$ forces each w-orbit in Φ to contribute a line to $\hat{g}^n$. This implies that n has the same order as w.

It was shown by Kostant [33, Cor. 6.8] that $|\Phi| = h \cdot \ell$, where h is the Coxeter number of $\hat{G}$. Since w has exactly ℓ orbits in Φ, the order of w is at least h. All orbits are free exactly when w has order h, in which case $\dim \hat{\mathfrak{g}}^n = \ell$ and n belongs to the unique $\hat{G}$-conjugacy class of regular elements of minimal order h, which means [33, Cor. 8.8] that w is a Coxeter element.

Assume that the order of w is greater than h and that $\det(1-w|\hat{\mathfrak{t}}) = |Z(\hat{G}_{sc})|$. If $\hat{G}$ has type A_n, G_2, F_4, E_6 or E_8, then W has no elliptic elements of order $> h$. In type B_n, C_n, the only elliptic class with $\det(1-w|\hat{\mathfrak{t}}) = 2$ is the Coxeter class.

In type D_n, the only elliptic classes with $\det(1-w|\hat{\mathfrak{t}}) = 4$ are the products of Coxeter classes for B_r and B_s with $r+s=n$. (If $r=1$ or $s=1$, this is the Coxeter class in D_n.) The number of orbits on roots is $(r-1)+(s-1)+2(r,s)$, which is equal to $\ell = r+s$ precisely when $(r,s)=1$. In this case, the two "mixed" orbits, containing $\pm e_i \pm e_j$ with $1 \leq i \leq r$ and $r+1 \leq j \leq s$, are both free. The remaining orbits have nontrivial stabilizers and we have $\bigcap_{\alpha\in\Phi^\circ} \ker \alpha|_{\hat{T}} = 1$.

Finally, in type E_7 with $h = 18$, there is a unique elliptic class of elements of order $> h$. An element w in this class has order 30; it is the commuting product of -1 with Coxeter classes w_2 of A_2 and w_4 of A_4. We may view $w_2w_4 = (123)(45678)$ as a product of cycles in the symmetric group S_8, generated by A_7 subdiagram of E_7. The simple root in this subdiagram outside A_2A_4 is then $e_3 - e_4$, whose orbit $\mathcal{O}$ under $\langle -w_2w_4\rangle$ is easily checked to be free. Viewing $\hat{T}$ in (a finite quotient of) SL_8, one also checks that $\bigcap_{\alpha\in\Phi^\circ} \ker\alpha|_{\hat{T}} \subset \hat{T}[2]$, as claimed. This completes the proof. ∎

We now show that the equality $b(\varphi) = \ell$ implies that the elliptic element w appearing in an inertially discrete parameter φ (see Prop. 5.4) is always a Coxeter element.

Proposition 5.6 *Assume that $\varphi : \mathcal{W} \to \hat{G}$ is discrete parameter for G satisfying the two conditions*

$$\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0 \qquad \textit{and} \qquad b(\varphi) = \ell$$

and that p does not divide the order of the Weyl group W of $\hat{G}$. Let $D = \varphi(\mathcal{W})$ and let $D > D_0 > D_1 > \cdots > D_m = 0$ be the lower ramification filtration filtration of this finite Galois group, so that $D_0 = \varphi(\mathcal{I})$ and $D_1 = \varphi(\mathcal{I}_+)$. Then

1. *D_1 lies in a unique maximal torus $\hat{T}$ of $\hat{G}$ and D lies in the normalizer $N(\hat{T})$ of $\hat{T}$ in $\hat{G}$.*
2. *The image of D_0/D_1 in $N(\hat{T})/\hat{T}$ is generated by a Coxeter element of order h.*
3. *$D_2 = 1$ and D_1 is a simple $\mathbf{F}_p[w]$-submodule of $\hat{T}[p]$, whose eigenvalues for w are primitive h^{th}-roots of unity in $\bar{\mathbf{F}}_p$. Hence D_1 is an elementary abelian p-group of order p^a, where a is the order of p in the group $(\mathbf{Z}/h\mathbf{Z})^\times$.*
4. *The Galois group D has upper breaks at the points 0 and $1/h$.*

Proof: Part 1 follows from Prop. 5.4 and its proof, which also shows that w is an elliptic element having exactly ℓ-orbits on Φ and that $b(\hat{\mathfrak{g}}_\alpha) = 1$ for all $\alpha \in \Phi$.

In Lemma 5.5, we saw that the union Φ° of the free orbits is nonempty. For $\alpha \in \Phi^\circ$ we have

$$1 = b(\hat{\mathfrak{g}}_\alpha) = 1 + \sum_{j\geq 2} \frac{\dim(\hat{\mathfrak{g}}_\alpha/\hat{\mathfrak{g}}_\alpha^{D_j})}{[D_1 : D_j]}.$$

This implies that D_2 acts trivially on $\hat{\mathfrak{g}}_\alpha$ for all $\alpha \in \Phi^\circ$. By the first assertion of Lemma 5.5, we have that $D_2 \subset \hat{T}[2]$. Since $|W|$ is even, we are assuming $p > 2$. Hence $D_2 = 1$. We then find, for *all* $\alpha \in \Phi$, that

$$1 = b(\hat{\mathfrak{g}}_\alpha) = \frac{1}{[D_\alpha : D_1]}.$$

This shows that w acts freely on Φ, hence w is a Coxeter element, by the last assertion of Lemma 5.5.

Since $D_2 = 1$, the image D_1 of wild inertia is an elementary abelian p-group and lies in $\hat{T}[p]$. Write $D_0 = D_1 \rtimes \langle n \rangle$, where n is the image of a generator of tame inertia. By [48, IV.2 Prop.9], we may identify D_1 with the additive group $\mathfrak{F}^+$ of the residue field of a finite unramified extension of k, such that the action of n on D_1 is given by

$$ndn^{-1} = \zeta \cdot d,$$

where $\zeta \in \mathfrak{F}^\times$ is a primitive h^{th}-root of unity. Hence the eigenvalues of w on $D_1 \subset \hat{T}[p]$ are the conjugates of ζ over $\mathbf{F}_p$, all of which have order h. Since each primitive h^{th}-root of unity occurs as an eigenvalue of w with multiplicity one [7, V.6.2 Cor.2], it follows that D_1 is a simple $\mathbf{F}_p[w]$-submodule of $\hat{T}[p]$ and has order p^a, where a is the number of conjugates of ζ over $\mathbf{F}_p$. This proves part 3.

Finally, the upper breaks in D follow from a calculation of the Herbrand transition function [48, IV.3]. ■

Examples: Suppose p and n are odd primes, and p is a generator of the cyclic group $(\mathbf{Z}/n)^\times$. Then for $\hat{G} = PGL_n$ we have $D_1 = \hat{T}[p]$, and for $\hat{G} = SO_{2n+1}$, the wild inertia group D_1 is the w-stable complement to the line in $\hat{T}[p]$ on which w acts by inversion. We can choose coordinates for $\hat{T} = (\mathbf{C}^\times)^n$ so that

$$D_1 = \{(t_1, t_2, \dots, t_n) \in \hat{T}[p] : \ t_1 t_3 \cdots t_n = t_2 t_4 \cdots t_{n-1}\}.$$

At the other extreme, suppose $\hat{G}$ is a simple adjoint group whose Coxeter number h is one less than a prime p. Let k have residue cardinality $p = h + 1$. Then there is a unique W-orbit $\mathcal{O} \subset \hat{T}[p]$ whose elements are regular in $\hat{G}$. The wild inertia group $D_1 \simeq \mathbf{F}_p$ is generated by some $t \in \mathcal{O}$, the normalizer in W of $\langle t \rangle$ is generated by a Coxeter element with lift $n \in N(\hat{T})$, and the inertia group $D_0 = \langle t, n \rangle$, of order $p(p-1)$, is a Borel subgroup in a $PGL_2(p)$ embedded in $\hat{G}$ (see [49]).

6 Construction of simple wild parameters

Assume that G is split over k. We say that a discrete parameter $\varphi : \mathcal{W} \to \hat{G}$ is a *simple wild parameter* if it satisfies the two conditions

$$\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0 \quad \text{and} \quad b(\varphi) = \ell.$$

In Prop. 5.6 we analyzed such parameters when p does not divide the order of the Weyl group of G. It remains to show that simple wild parameters actually exist.

In section 6.2 we will construct simple wild parameters when p does not divide the Coxeter number h of G. Simple wild parameters also exist when p does divide h; in section 6.3 we give examples of them for $k = \mathbf{Q}_2$. In both cases, we rely on the following construction of certain wild extensions of an arbitrary non-archimedean local field k.

6.1 Some wild extensions

Let q be a power of a prime p and fix an integer $m \geq 2$ not divisible by p. Choose a positive integer f such that $q^f \equiv 1 \mod m$ and choose an integer r modulo m belonging to the kernel of $q-1$ on $\mathbf{Z}/m$. To the data $\mathsf{d} := (q, m, f, r)$ we associate the group Γ_{d} with two generators s and F, and relations

$$s^m = 1, \qquad F^f = s^r, \qquad FsF^{-1} = s^q.$$

Thus, Γ_{d} is an extension

$$1 \longrightarrow \mathbf{Z}/m \longrightarrow \Gamma_{\mathsf{d}} \longrightarrow \mathbf{Z}/f \longrightarrow 1, \tag{50}$$

where a generator of $\mathbf{Z}/f$ acts as multiplication by q on $\mathbf{Z}/m$. Replacing F by the new generator $F' = s^j F$ changes r to $r' = r + Nj$, where $N = 1 + q + q^2 + \cdots + q^{f-1} \in \mathrm{End}(\mathbf{Z}/m)$, and the class of the extension (50) is given by the class of r in

$$\ker(q-1)/\operatorname{im} N = H^2(\mathbf{Z}/f, \mathbf{Z}/m).$$

For example, $r \in \operatorname{im} N$ precisely when we can choose F' so that Γ_{d} is a semidirect product

$$\Gamma_{\mathsf{d}} = \langle s \rangle \rtimes \langle F' \rangle.$$

We are interested in representations of Γ_{d} on finite dimensional vector spaces over the prime field $\mathbf{F}_p$, or what is the same, finite modules over the group algebra $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$.

Lemma 6.1 *Let V be a finite $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-module with $V^s = 0$. Then any extension of Γ_{d} by V splits uniquely.*

Proof: The hypotheses imply that $H^n(\langle s \rangle, V) = 0$ for all $n \geq 0$. The Hochschild-Serre spectral sequence then shows that $H^n(\Gamma_{\mathsf{d}}, V) = 0$ for all $n \geq 0$. Taking $n = 1$ and $n = 2$ yields the result.

Splitting can also be proved directly, as follows. Suppose $\tilde{\Gamma}$ is an extension

$$1 \longrightarrow V \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma_{\mathsf{d}} \longrightarrow 1$$

of Γ_{d} by V. Lift s to an element of $\tilde{\Gamma}$ of the same order m, and denote it again by s. Let F_0 denote an arbitrary lift of F to $\tilde{\Gamma}$. Then we have $F_0 s F_0^{-1} = x s^q$, for some $x \in V$. Since $V^s = 0$ and V is finite, we have $V = (1-s)V$. It follows that there is $y \in V$ with $ysy^{-1}s^{-1} = F_0^{-1}x^{-1}F_0$. If we set $F = F_0 y$, then $FsF^{-1} = s^q$.

Let H be the subgroup of $\tilde{\Gamma}$ generated by s and F. Since F normalizes $\langle s \rangle$, every element of H is of the form $F^a s^b$ for some integers a and b. Suppose that $F^a s^b = z \in V$. Then z normalizes $\langle s \rangle$. Hence the commutater $z^{-1}szs^{-1}$ belongs to both $\langle s \rangle$ and V. Since $p \nmid m$, this commutator is trivial, so that we have $szs^{-1} = z$. But then $z = 1$ by the assumption that s has trivial invariants in V. Hence we have $H \cap V = 1$ and the subgroup H is a complement to V in $\tilde{\Gamma}$. ∎

We are interested in $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-modules V having a particular restriction to the group algebra $\mathbf{F}_p[s]$ of $\langle s \rangle$ over $\mathbf{F}_p$. For such V we will construct a Galois extension $\mathrm{Gal}(L/k)$ with wild inertia subgroup V and tame quotient Γ_{d}. By Lemma 6.1, we will then have $\mathrm{Gal}(L/k) = V \rtimes \Gamma_{\mathsf{d}}$.

Before describing V we first recall that $\mathbf{F}_p[s]$ is a semisimple algebra, since $p \nmid m$. In the field $\mathfrak{F} = \mathbf{F}_{q^f}$, we fix an element $\zeta \in \mathfrak{F}^\times$ of order m, which is possible since $m \mid q^f - 1$, and view the finite field $\mathbf{F}_p(\zeta)$ generated by ζ as an $\mathbf{F}_p[s]$-module, where s acts as multiplication by ζ. Up to isomorphism, $\mathbf{F}_p(\zeta)$ is the unique simple $\mathbf{F}_p[s]$-module containing ζ as an eigenvalue of s when scalars are extended to $\bar{\mathbf{F}}_p$. We have $\dim_{\mathbf{F}_p} \mathbf{F}_p(\zeta) = a$, where a is the order of p in $(\mathbf{Z}/m\mathbf{Z})^\times$, and $\mathbf{F}_p(\zeta)$ has the $\mathbf{F}_p$-basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{a-1}\}$.

Suppose that V is an $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-module whose restriction to $\mathbf{F}_p[s]$ is isomorphic to $\mathbf{F}_p(\zeta)$. Then V is determined by the action of F on $\mathbf{F}_p(\zeta)$. Let

$$v := F \cdot 1 \in \mathbf{F}_p(\zeta).$$

For $1 \le i \le a-1$ we have

$$F \cdot \zeta^i = Fs^i \cdot 1 = s^{qi}F \cdot 1 = \zeta^{qi}v = v\zeta^{iq},$$

where the last two terms are products in $\mathbf{F}_p(\zeta)$. It follows that

$$F \cdot x = vx^q, \qquad \forall\, x \in \mathbf{F}_p(\zeta).$$

The relation $F^f = s^r$ forces $\mathsf{N}(v) = \zeta^r$, where $\mathsf{N} : \mathfrak{F}^\times \to \mathbf{F}_q^\times$ is the norm mapping. Likewise, on the $\mathbf{F}_p$-vector space underlying $\mathfrak{F}$, the operators

$$s \cdot x = \zeta x, \qquad F \cdot x = vx^q, \qquad \text{for} \quad x \in \mathfrak{F}$$

define a $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-module, which we denote by $V_{\mathsf{d}}(\zeta, v)$. Since ζ and v belong to $\mathbf{F}_p(\zeta)$, the trace mapping

$$\mathrm{tr} : \mathfrak{F} \longrightarrow \mathbf{F}_p(\zeta)$$

gives an epimorphism of $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-modules

$$\mathrm{tr} : V_{\mathsf{d}}(\zeta, v) \longrightarrow V.$$

We now construct the extension L/k in stages. First let K be the unramified extension of k of degree f, with residue field $\mathfrak{F}$. Identify $\mathfrak{F}^\times$ with the roots of unity of $K^\times$ of order prime to p. Since

$\mathsf{N}(v^m) = \mathsf{N}(v)^m = \zeta^{rm} = 1$, Hilbert's Theorem 90 gives an element $u \in \mathfrak{F}^\times$ such that $u^{q-1} = v^m$. Let $E = K(\pi)$ be the extension of K generated by a root π of the Eisenstein polynomial equation $x^m = u\varpi$, where ϖ is a uniformizer in k. Then E/k is a tamely ramified Galois extension of degree $[E : k] = mf$. The inertia subgroup $\mathrm{Gal}(E/K)$ is generated by the element s such that

$$s(\pi) = \zeta\pi.$$

Let $F \in \mathrm{Gal}(E/k)$ be any Frobenius element. Since

$$F(\pi)^m = F(\pi^m) = F(u\varpi) = u^q\varpi = u^{q-1}\pi^m = (v\pi)^m,$$

we have $F(\pi) = \zeta^j v\pi$ for some integer j. Replacing F by $s^{-j}F$, we may choose F so that

$$F(\pi) = v\pi.$$

Then

$$F^f(\pi) = \mathsf{N}(v)\pi = \zeta^r\pi = s^r(\pi).$$

Finally, we have

$$s^qF(\pi) = \zeta^q v\pi = F(\zeta\pi) = Fs(\pi).$$

Thus, $\mathrm{Gal}(E/k)$ is generated two elements s and F satisfying the relations of Γ_{d}. Since

$$|\,\mathrm{Gal}(E/k)| = [E : k] = m \cdot f = |\Gamma_{\mathsf{d}}|,$$

It follows that

$$\mathrm{Gal}(E/k) \simeq \Gamma_{\mathsf{d}},$$

with inertia subgroup

$$\mathrm{Gal}(E/K) \simeq \langle s\rangle \simeq \mathbf{Z}/m,$$

and unramified quotient

$$\mathrm{Gal}(K/k) \simeq \mathbf{Z}/f.$$

For the final stage, we note that multiplicative group $E^\times$ is a direct product

$$E^\times = \langle\pi\rangle \times \mathfrak{F}^\times \times (1 + \pi A_E). \tag{51}$$

Let

$$\theta : E^\times \longrightarrow (1 + \pi A_E)/(1 + \pi^2 A_E)$$

be the map induced by projecting $E^\times$ onto $1 + \pi A_E$ according to the direct product (51). The mapping $1 + \pi A_E \to A_E$ given by $1 + \pi x \mapsto x$ induces an isomorphism of $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-modules

$$(1 + \pi A_E)/(1 + \pi^2 A_E) \xrightarrow{\sim} V_{\mathsf{d}}(\zeta, v).$$

Via the trace, we get an epimorphism

$$E^\times \xrightarrow{\theta} (1+\pi A_E)/(1+\pi^2 A_E) \xrightarrow{\sim} V_{\mathsf{d}}(\zeta, v) \xrightarrow{\mathrm{tr}} V$$

which corresponds, by local class field theory, to an abelian extension L/E which is Galois over k with wild inertia group $\mathrm{Gal}(L/E) \simeq V$ as modules over $\mathrm{Gal}(E/k) = \Gamma_{\mathsf{d}}$. We have

$$\mathrm{Gal}(L/k) \simeq V \rtimes \Gamma_{\mathsf{d}},$$

with inertia subgroup

$$\mathrm{Gal}(L/k)_0 = \mathrm{Gal}(L/K) \simeq V \rtimes \langle s \rangle \simeq \mathbf{F}_p(\zeta)^+ \rtimes \langle s \rangle,$$

and wild inertia subgroup

$$\mathrm{Gal}(L/k)_1 = \mathrm{Gal}(L/E) \simeq V \simeq \mathbf{F}_p(\zeta)^+.$$

Finally, the higher ramification group

$$\mathrm{Gal}(L/k)_2 = 1,$$

since the map $E^\times \to V$ giving rise to L is trivial on $1 + \pi^2 A_E$. This extension L/k depends on the initial data $\mathsf{d} = (q, m, f, r)$ used to define Γ_{d}, the choice of $\zeta \in \mathfrak{F}^\times$, and the $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-module V whose restriction to $\mathbf{F}_p[s]$ is isomorphic to $\mathbf{F}_p(\zeta)$.

6.2 Simple wild parameters when p does not divide the Coxeter number

Assume that G is absolutely simple, simply connected and split over k, and that the Coxeter number h of G is not divisible by p. The dual group ${}^L G = \hat{G}$ is a simple adjoint group. Let $\hat{T}$ be a maximal torus in $\hat{G}$ with normalizer $N(\hat{T})$ and Weyl group $W = N(\hat{T})/\hat{T}$. Let $\sigma \in N(\hat{T})$ be a lift of a Coxeter element in W; all such lifts are conjugate to one another by $\hat{T}$. Since $\hat{G}$ is adjoint, σ has order h. Let a denote the order of p in $(\mathbf{Z}/h\mathbf{Z})^\times$ and let $\zeta \in \mathbf{F}_{p^a}^\times$ have order h.

We have seen that the semisimple $\mathbf{F}_p[\sigma]$-module $\hat{T}[p] = \{t \in \hat{T} :\ t^p = 1\}$ contains a unique simple submodule $\hat{T}(\zeta)$ which is isomorphic to the $\mathbf{F}_p[\sigma]$-module $\mathbf{F}_p(\zeta)$, on which σ acts as multiplication by ζ. The σ-eigenvalues in $\mathbf{F}_{p^a} \otimes \hat{T}(\zeta)$ are $\zeta, \zeta^p, \ldots, \zeta^{p^{a-1}}$, each with multiplicity one. Thus, we have a subgroup

$$\hat{T}(\zeta) \rtimes \langle \sigma \rangle \subset N(\hat{T}).$$

From the theory of the Coxeter element, no root vanishes on $\hat{T}(\zeta)$. Hence the centralizer of $\hat{T}(\zeta)$ in $\hat{\mathfrak{g}}$ is just $\hat{\mathfrak{t}}$. Since σ has no nonzero invariant vectors in $\mathfrak{t}$, it follows that the group $\hat{T}(\zeta) \rtimes \langle \sigma \rangle$ has no nonzero invariants in $\mathfrak{g}$. We will construct simple wild parameters whose inertial image is $D_0 = \hat{T}(\zeta) \rtimes \langle \sigma \rangle$.

Since all irreducible characters of W are defined over $\mathbf{Q}$ [55, 1.15], all generators of a given cyclic subgroup of W are conjugate. Since $p \nmid h$, the group $\langle\sigma\rangle$ is also generated by σ^q. Hence there is an element $F \in N(\hat{T})$ such that $F\sigma F^{-1} = t_1\sigma^q$ for some $t_1 \in \hat{T}$. In fact, since $\hat{T} = (1-\sigma)\hat{T}$, we may choose F in its coset modulo $\hat{T}$ so that $t_1 = 1$ and $F\sigma F^{-1} = \sigma^q$.

Lemma 6.2 *The centralizers $N(\hat{T})^\sigma$ and $\hat{T}^\sigma$ of σ in $N(\hat{T})$ and $\hat{T}$ are related by: $N(\hat{T})^\sigma = \hat{T}^\sigma \times \langle\sigma\rangle$. Moreover, $\hat{T}^\sigma$ is isomorphic to the fundamental group $\pi_1(\hat{G})$ and the exponent of $\hat{T}^\sigma$ divides h.*

Proof: The group $N(\hat{T})^\sigma$ projects to the centralizer of the Coxeter element $\sigma\hat{T} \in W$. Since Coxeter elements generate their own centralizers, any element $g \in N(\hat{T})^\sigma$ is of the form $g = t\sigma^n$, where $t \in \hat{T}$ and n is an integer. Since both g and σ centralize σ, we have $t \in \hat{T}^\sigma$. Hence, $N(\hat{T})^\sigma = \hat{T}^\sigma \cdot \langle\sigma\rangle$. The product is direct, since $\langle\sigma\rangle$ projects injectively into W. Let $\hat{G}_{sc}$ be the simply-connected cover of $\hat{G}$, and let $\hat{T}_{sc}$ denote the preimage of $\hat{T}$ in $\hat{G}_{sc}$ and let $\tilde{\sigma}$ denote a lift of σ in $\hat{G}_{sc}$. The group generated by $\tilde{\sigma}$ acts on each term in the exact sequence

$$1 \longrightarrow \pi_1(\hat{G}) \longrightarrow \hat{T}_{sc} \longrightarrow \hat{T} \longrightarrow 1$$

The coboundary from the resulting long exact sequence gives an isomorphism $\hat{T}^\sigma \simeq \pi_1(\hat{G})$. The last assertion is a well-known property of the Coxeter number, and follows from the fact that $|\pi_1(\hat{G})| = \det(1-\sigma|\hat{\mathfrak{t}})$. ■

Since $F^a\sigma F^{-a} = \sigma^{q^a} = \sigma$, Lemma 6.2 implies that $F^a = t\sigma^b$ for some integer b and $t \in \hat{T}^\sigma$. It follows that some power of F lies in $\langle\sigma\rangle$. Let f be the order of F modulo σ, and let $F^f = \sigma^r$. The group $\langle F, \sigma\rangle$ generated by F and σ is isomorphic to Γ_{d}, where $\mathsf{d} = (q, m, f, r)$.

Since $F\sigma F^{-1} = \sigma^q$, it follows that F permutes the set $\{\zeta,\ \zeta^q,\ \zeta^{q^2},\ \dots,\ \zeta^{q^{a-1}}\}$ of σ-eigenvalues in $\bar{\mathbf{F}}_p \otimes T(\zeta)$. Since $\hat{T}(\zeta)$ has multiplicity one in $\hat{T}[p]$, it is preserved by F. Thus, $\hat{T}(\zeta)$ is an $\mathbf{F}_p[\Gamma_{\mathsf{d}}]$-module whose restriction to $\mathbf{F}_p[\sigma]$ is isomorphic to $\mathbf{F}_p[\zeta]$. The second part of the proof of Lemma 6.1 shows that $\langle F, \sigma\rangle \cap \hat{T}(\zeta) = 1$. Thus, we have a subgroup

$$D = \hat{T}(\zeta) \rtimes \langle F, \sigma\rangle \subset \hat{N}.$$

The construction of section 6.1 gives a Galois extension L/k with

$$\mathrm{Gal}(L/k) \simeq D = \hat{T}(\zeta) \rtimes \Gamma_{\mathsf{d}},$$

inertia subgroup

$$\mathrm{Gal}(L/K) \simeq D_0 = \hat{T}(\zeta) \rtimes \langle\sigma\rangle$$

and wild inertia subgroup

$$\mathrm{Gal}(L/E) \simeq D_1 = \hat{T}(\zeta)$$

Thus, we have a discrete parameter

$$\varphi : \Gamma \xrightarrow{\sim} \mathrm{Gal}(L/k) \simeq D.$$

Since $\hat{T}(\zeta)$ contains a regular element in $\hat{G}$, we have $\hat{\mathfrak{g}}^{D_1} = \hat{\mathfrak{t}}$. Since the Coxeter element w is elliptic, we have $\hat{\mathfrak{g}}^{D_0} = \hat{\mathfrak{t}}^w = 0$. Finally, since $D_2 = 1$, we have

$$b(\varphi) = \frac{\dim(\hat{\mathfrak{g}}/\hat{\mathfrak{g}}^{D_1})}{[D_0 : D_1]} = \frac{|\Phi|}{h} = \ell,$$

so φ is indeed a simple wild parameter.

6.3 Some simple wild parameters for $\mathbb{Q}_2$

We can also apply the construction of L/k in section 6.1 to the field $k = \mathbf{Q}_2$. Let $m = 2n+1$ be an odd integer ≥ 3. Let f be the order of 2 in $(\mathbf{Z}/m)^\times$, let $\mathfrak{F} = \mathbf{F}_{2^f}$ and let $r = 0$, so that we have the data $\mathsf{d} = (2, m, f, 0)$. The group Γ_d may be realized concretely as a group

$$\Gamma_\mathsf{d} = A \rtimes C = \{x \mapsto a x^{2^c} :\ a \in A,\ c \in C\},$$

of linear transformations of the additive group $\mathfrak{F}^+$, where $A \subset \mathfrak{F}^\times$ is the unique subgroup of order m and $C = \mathbf{Z}/f$ is generated by the Frobenius automorphism $\gamma(x) = x^2$. Choose $\zeta \in A$ of order m and let $v = 1$. The $\mathbf{F}_2[\Gamma_\mathsf{d}]$-modules V and $V_\mathsf{d}(\zeta, v) = \mathbf{F}_2(\zeta)^+ = \mathfrak{F}^+$ coincide; we denote this group by B.

The tame extension is $E = K(\sqrt[m]{2})$, where K is the unramified extension of k of degree f in $\bar{k}$, with Galois group $\mathrm{Gal}(E/k) = AC$. We get a Galois extension L/k whose Galois group $D = \mathrm{Gal}(L/k)$ can be realized concretely as

$$D = BAC = \{x \mapsto a x^{2^c} + b :\ a \in A,\ b \in B,\ c \in C\},$$

with structure $2^f \cdot m \cdot f$, consisting of Galois-twisted affine transformations of $B = \mathfrak{F}^+$ with dilation factor a constrained to lie in A.

There is a unique nontrivial character $\chi_0 : B \to \{\pm 1\}$ which is invariant under C, given by

$$\chi_0(b) = (-1)^{\mathrm{Tr}(b)},$$

where $\mathrm{Tr} : \mathfrak{F} \to \mathbf{F}_2$ is the absolute trace. Since C is cyclic, χ_0 extends to a character BC; the various extensions are determined by their values on γ. Each extension of χ_0 to BC induces to an m-dimensional irreducible representation of D and any such representation of D is obtained in this way. There is a unique extension $\chi : BC \to \{\pm 1\}$ for which the induced representation

$$M := \mathrm{Ind}_{BC}^{D}\, \chi$$

has $\det(M) = 1$. Using [48, p.122] we find that this character χ is given by the Jacobi symbol:

$$\chi(\gamma) = \left(\frac{2}{m}\right) = \begin{cases} +1 & \text{if} \quad m \equiv 1, 7 \mod 8 \\ -1 & \text{if} \quad m \equiv 3, 5 \mod 8. \end{cases}$$

Thus, M is the unique irreducible representation of D of dimension m which is nontrivial on B and has trivial determinant. Since χ is real, the representation M is orthogonal and gives a homomorphism

$$\varphi : D \longrightarrow SO(M).$$

We now prove a uniqueness result for L/k:

Lemma 6.3 *In a given algebraic closure $\bar{\mathbf{Q}}_2$, the field L is the unique Galois extension of $k = \mathbf{Q}_2$ with $\mathrm{Gal}(L/k) \simeq D$, inertia subgroup $D_0 \simeq BA$, wild inertia subgroup B and Swan conductor $b(M) = 1$.*

Proof: Any abelian extension L'/E which is Galois over k with $\mathrm{Gal}(L/E) \simeq \mathfrak{F}^+$ arises from a surjective mapping

$$\vartheta : E^\times \longrightarrow \mathfrak{F}^+$$

which is equivariant for the action of $\mathrm{Gal}(E/k) \simeq \Gamma_{\mathsf{d}}$. We show that ϑ is unique, as follows. Let $s \in \mathrm{Gal}(E/K)$ be a generator corresponding to $\zeta \in A$. For any $y \in E^\times$, we must have

$$\vartheta(s(y)) = \zeta \cdot \vartheta(y) \in \mathfrak{F}^+.$$

It follows that ϑ vanishes on $K^\times$. In particular ϑ vanishes on $\mathfrak{F}^\times$ and we also have

$$m\vartheta(\pi) = \vartheta(\pi^m) = \vartheta(2) = 0,$$

so $\vartheta(\pi) = 0$. But

$$E^\times = \langle \pi \rangle \times \mathfrak{F}^\times \times (1 + \pi A_E),$$

so ϑ is determined by its values on $1 + \pi A_E$. To have $b(M) = 1$ we must have $\vartheta(1 + \pi^2 A_E) = 0$. Hence we have two isomorphisms

$$(1 + \pi A_E)/(1 + \pi^2 A_E) \xrightarrow{\sim} \mathfrak{F}^+,$$

namely ϑ and the canonical map $1 + \pi y \mapsto y \mod \pi$. It follows that there is $\beta \in \mathfrak{F}^\times$ such that

$$\vartheta(1 + \pi y) = \beta y \mod \pi.$$

Since $2 \in k$, there exists a Frobenius element $F \in \mathrm{Gal}(E/k)$ fixing π, and we have

$$\beta \cdot F(y) \equiv \vartheta(1 + \pi F(y)) = F(\vartheta(1 + \pi y)) = F(\beta \cdot y) = F(\beta) \cdot F(y),$$

so $\beta \in \mathbf{F}_2$, which means $\beta = 1$. Hence ϑ is uniquely determined, as is the extension L/k. ■

Remark: The subfield L_0 of L fixed by AC has degree q over k, and has the form $L_0 = k[x]/(f(x))$ where $f(x)$ is an Eisenstein polynomial whose roots are permuted simply-transitively by B and generate L. When $m = q - 1$, David Roberts has pointed out that

$$f(x) = x^q + 2x + 2$$

(see [28] for the cases $m = 3, 5, 7$). For $q \geq 8$, the discriminant $\mathfrak{d}_0 = \mathfrak{d}_{L_0/k}$ of the ring of integers in L_0 has 2-adic valuation $\mathrm{val}_k(\mathfrak{d}_0)$ equal to the Artin conductor $a(V)$ of the permutation representation $V = \mathrm{Ind}_{AC}^{D}\,\mathbf{C}$ (cf. (2)). Since

$$V|_{BA} = \mathrm{Ind}_A^{BA}\,\mathbf{C}, \qquad V|_B = \mathrm{Ind}_1^B\,\mathbf{C},$$

we have $\dim V^{D_0} = \dim V^{D_1} = 1$ and

$$\mathrm{val}_k(\mathfrak{d}_0) = q - 1 + \frac{q-1}{m}.$$

Let G be a split group over k of one of the following types:

$$Sp_{2n}, \qquad SO_{2n+2}, \qquad PGL_{2n+1}, \qquad G_2, \qquad E_8.$$

We construct simple wild parameters for the above groups G, using our representation M, the natural inclusions of L-groups

$$i : SO(M) \hookrightarrow SO(M \oplus \mathbf{C}), \qquad j : SO(M) \hookrightarrow SL(M), \qquad g : G_2(\mathbf{C}) \hookrightarrow SO_7(\mathbf{C}),$$

and the Dempwolff subgroup in E_8 [59].

Proposition 6.4 *Let $\Gamma = \mathrm{Gal}(L/k) \simeq D$ be the Galois group of the unique extension L of $k = \mathbf{Q}_2$ constructed above, for $m = 2n+1$. Then*

1. *The map $\varphi : \Gamma \to SO(M)$ is a simple wild parameter for $G = Sp_{2n}$.*
2. *The composition*
$$\Gamma \xrightarrow{\varphi} SO(M) \overset{i}{\hookrightarrow} SO(M \oplus \mathbf{C})$$
is a simple wild parameter for $G = SO_{2n+2}$.
3. *The composition*
$$\Gamma \xrightarrow{\varphi} SO(M) \overset{j}{\hookrightarrow} SL(M)$$
is a simple wild parameter for $G = PGL_{2n+1}$.
4. *For $m = 7$, the map $\varphi : \Gamma \to SO(M) = SO_7(\mathbf{C})$ factors through an embedding $G_2(\mathbf{C}) \hookrightarrow SO_7(\mathbf{C})$ and is a simple wild parameter for $G = G_2$.*
5. *For $m = 31$, the group Γ embeds in $E_8(\mathbf{C})$ and gives is a simple wild parameter for $G = E_8$.*

In each case above, the upper breaks of Γ are at the points 0 and $1/m = 1/(2n+1)$. We note that this is only equal to $1/h$ when $G = PGL_{2n+1}$.

Proof: As representations of D, the Lie algebras $\hat{\mathfrak{g}}$ are given in the first three cases by

$$\mathfrak{so}(M) = \Lambda^2 M, \qquad \mathfrak{so}(M \oplus \mathbf{C}) = M \oplus \Lambda^2 M, \qquad \mathfrak{sl}(M) = \Lambda^2 M \oplus (S^2 M)_0,$$

where $\Lambda^2 M$ is the exterior square of M and $(S^2 M)_0 \simeq (\mathrm{Sym}^2 M)/\mathbf{C}$ is the trace-zero subspace of the symmetric square of M. Since M is irreducible and orthogonal for $D_0 = BA$, the spaces $\Lambda^2 M$ and $(S^2 M)_0$ have no nonzero invariants under D_0, so $\hat{\mathfrak{g}}^{D_0} = 0$ in all cases. Since M is multiplicity-free on B, we have $(\Lambda^2 M)^{D_1} = 0$, so

$$b(\Lambda^2 M) = \frac{m(m-1)/2}{m} = n = \mathrm{rank}(\mathfrak{so}(M)).$$

Since $b(M) = 1$ we have

$$b(M \oplus \Lambda^2 M) = n + 1 = \mathrm{rank}(\mathfrak{so}(M \oplus \mathbf{C}))$$

with an identical calculation for M' which is isomorphic to M on $D_0 = BA$. Since all characters of B have order two, we have

$$b((S^2 M)_0) = \frac{m(m+1)/2 - 1 - (m-1)}{m} = n$$

so

$$b(\Lambda^2 M \oplus (S^2 M)_0) = 2n = \mathrm{rank}(\mathfrak{sl}(M)),$$

proving the result in the first three cases.

In $\hat{G} = G_2(\mathbf{C})$ we have a subgroup $B \simeq \mathbf{F}_8^+$ generated by the 2-torsion subgroup $\hat{T}[2]$, along with an involution in $\hat{N}$ acting by inversion on $\hat{T}$. Each nontrivial character of η of B appears in $\hat{\mathfrak{g}}|_B$ and each eigenspace $\hat{\mathfrak{g}}_\eta$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$, so that the decomposition

$$\hat{\mathfrak{g}}|_B = \bigoplus_{1 \neq \eta \in \hat{B}} \hat{\mathfrak{g}}_\eta$$

expresses $\hat{\mathfrak{g}}$ as a direct sum of seven Cartan subalgebras.

The centralizer of any nontrivial element of B is isomorphic to $SO_4(\mathbf{C})$. Hence B may be viewed as the group of diagonal ± 1 matrices in any of these copies of $SO_4(\mathbf{C})$, in which each $SO_4(\mathbf{Z})$ (dot product form) normalizes B and is transitive on the noncentral elements in B. Since $SO_4(\mathbf{Z})$ is a nonsplit extension of S_4, it follows that the normalizer $\hat{N}(B)$ of B in $G_2(\mathbf{C})$ is transitive on $\mathbf{F}_8 - \{0\}$ and is a nonsplit extension $B \cdot SL(B) = 2^3 \cdot SL_3(2)$. This extension splits over the subgroup of $SL_3(2)$ of order $7 \cdot 3$ normalizing an anisotropic torus in $SL_3(2)$. Hence for $m = 7$ with $f = 3$, the group $D = BAC$ of structure $2^3 \cdot 7 \cdot 3$ is a subgroup of $\hat{G}$. Since $\hat{\mathfrak{g}}^B = 0$ and $[BA : B] = 7$, we have

$$b(\hat{\mathfrak{g}}) = \frac{14}{7} = 2.$$

It follows that the quotient $\Gamma \to \mathrm{Gal}(L/\mathbf{Q}_2) = D \hookrightarrow G_2(\mathbf{C})$ is a simple wild parameter for G_2. Composing with the map $G_2(\mathbf{C}) \to SO_7(\mathbf{C})$, we get a seven dimensional orthogonal representation of D which has $\det = 1$ and is nontrivial on B, hence is equivalent to M.

In $\hat{G} = E_8(\mathbf{C})$ we have a subgroup $B \simeq \mathbf{F}_{32}^{+}$ generated by a maximal isotropic subspace $U \subset \hat{T}[2]$, along with an involution in $\hat{N}$ acting by inversion on $\hat{T}$. Each nontrivial character of η of B appears in $\hat{\mathfrak{g}}|_B$ and each eigenspace $\hat{\mathfrak{g}}_\eta$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$, so that

$$\hat{\mathfrak{g}}|_B = \bigoplus_{1 \neq \eta \in \hat{B}} \hat{\mathfrak{g}}_\eta$$

expresses $\hat{\mathfrak{g}}$ as a direct sum of 31 Cartan subalgebras.

The Dempwolff group $\mathcal{D}$ is a nonsplit extension $\mathcal{D} = 2^5 \cdot SL_5(2)$ which embeds in the normalizer in $\hat{G}$ of B. The extension splits over the subgroup of $SL_5(2)$ of order $31 \cdot 5$ normalizing an anisotropic torus in $SL_5(2)$. Hence, for $m = 31$ with $f = 5$, the group $D = BAC$ of structure $2^5 \cdot 31 \cdot 5$ is a subgroup of $\mathcal{D}$. Since $\hat{\mathfrak{g}}^B = 0$ and $[BA : B] = 31$, we have

$$b(\hat{\mathfrak{g}}) = \frac{248}{31} = 8,$$

so the composition $\Gamma \to \mathrm{Gal}(L/k) = D \hookrightarrow E_8(\mathbf{C})$ is a simple wild parameter for E_8, as claimed. ■

Remark 1: For $\hat{G} = E_8$ (resp. G_2) one can check that the restriction of $\hat{\mathfrak{g}}$ to D is given by

$$\hat{\mathfrak{g}}|_D = 2(V_1 + V_2 + V_3 + V_4) \qquad \text{(resp.} \qquad \hat{\mathfrak{g}}|_D = V_1 + V_2),$$

where $V_i = \mathrm{Ind}_{BC}^{D} \chi_i$ are the irreducible representations of dimension 31 (resp. 7) other than M.

Remark 2: All of the simple wild parameters constructed in this paper have abelian wild inertia groups D_1, but there are simple wild parameters for which this is not the case. The simplest example is for G of type G_2 and $k = \mathbf{Q}_3$, where D_1 is the Heisenberg group 3^{1+2} of order 27 and $D = 3^{1+2} \cdot 8 \cdot 2$. The upper breaks of D are at the points $0, 1/8, 1/6$.

7 Euler-Poincaré measure and Formal Degrees

We say μ is an *invariant measure* on $G(k)$ if μ is a real multiple of a positive Haar measure on $G(k)$. The set of invariant measures forms a one-dimensional real vector space.

An irreducible admissible representation (π, V) of $G(k)$ belongs to the *discrete series* if π has a matrix coefficient which is square-integrable with respect to some (equivalently any) nonzero invariant measure μ on $G(k)$. When this holds, there is a real number $\deg_\mu(\pi)$ such that

$$\deg_\mu(\pi) \cdot \int_{G(k)} |\langle \pi(g)v, \tilde{v}\rangle|^2 \, \mu = |\langle v, \tilde{v}\rangle|^2, \tag{52}$$

for every $v \in V$ and $\tilde{v} \in \tilde{V}$, the contragredient representation space, where $\langle \cdot, \cdot \rangle$ is the natural pairing between V and $\tilde{V}$. According to [24], this is an unpublished result of Mackey; for a proof see [32].

The real number $\deg_\mu(\pi)$ is called *formal degree* of π. It depends on the choice of invariant measure μ. If we replace μ by a scalar multiple $c \cdot \mu$, the formal degree changes by:

$$\deg_{c\cdot\mu}(\pi) = c^{-1} \cdot \deg_\mu(\pi).$$

In other words, the discrete series representation π determines a positive Haar measure $\deg_\mu(\pi) \cdot \mu$ on $G(k)$ which is independent of μ.

If $\pi = \operatorname{ind}_J^{G(k)} V$ is compactly induced from a finite dimensional representation V of a compact open subgroup J of $G(k)$, then one can use (52) to show that

$$\deg_\mu(\pi) \cdot \int_J \mu = \dim V. \tag{53}$$

The Steinberg representation St_G of $G(k)$ is a discrete series representation which is canonical in the sense of being uniformly constructed for all groups. The ratio

$$\frac{\deg_\mu(\pi)}{\deg_\mu(St_G)} \tag{54}$$

is positive and independent of the choice of invariant measure μ, and may be taken as a normalized formal degree, equal to 1 when $\pi = St_G$. In fact, there is a natural choice of invariant measure on $G(k)$ which achieves this normalized formal degree, up to sign. This is the Euler-Poincaré measure μ_G, which Serre introduced in [46] to study the cohomology of discrete subgroups of $G(k)$. In this section we review this measure and use it to reformulate the formal degree conjecture in [26], in a manner consistent with (54).

7.1 Invariant measures on $G(k)$.

Serre proved the existence of an invariant measure μ_G on $G(k)$ with the property that for any discrete, co-compact torsion-free subgroup $\Gamma \subset G(k)$, the Euler-Poincaré characteristic of $H^\bullet(\Gamma, \mathbf{Q})$ is given by

$$\chi(H^\bullet(\Gamma, \mathbf{Q})) = \int_{\Gamma\backslash G(k)} \mu_G.$$

The measure μ_G, called *Euler-Poincaré measure*, is non-zero on $G(k)$ precisely when the connected center of G is anisotropic over k (which we have assumed). In this case, $(-1)^{r(G)}\mu_G$ is a positive Haar measure on $G(k)$, where $r(G)$ is the k-rank of G.

The Euler -Poincaré measure μ_G is the unique invariant measure on $G(k)$ for which

$$\deg_{\mu_G}(St_G) = (-1)^{r(G)}$$

(see [3] and [46]). Hence, for any discrete series representation π of $G(k)$, the canonical degree

$$\frac{\deg_\mu(\pi)}{\deg_\mu(St)} = (-1)^{r(G)} \deg_{\mu_G}(\pi)$$

is indeed achieved by μ_G, up to sign.

The volume of any open compact subgroup $J \subset G(k)$ with respect to μ_G is a rational number. Indeed, it suffices to check this for one open compact subgroup, since any two are commensurable. Let $I \subset G(k)$ be an Iwahori subgroup, which fixes (pointwise) a facet of maximal dimension in the building of $G(k)$. The group I is a semidirect product of its normal pro-p-Sylow subgroup I_+ and a finite subgroup $\mathsf{T}(q)$ of order prime to p:

$$I = I_+ \rtimes \mathsf{T}(q).$$

The group $\mathsf{T}(q)$ is the group of $\mathfrak{f}$-rational points in an algebraic torus T over $\mathfrak{f}$, where $\dim \mathsf{T} = \dim E^{\varphi_0(\mathcal{I})} = \ell_0$. Using [15, 4.11] we find that

$$e(G)\int_I \mu_G = \frac{L(\check{M}(1))}{L(M)\cdot|^L Z|}\cdot|\mathsf{T}(q)|\cdot q^{-m},$$

where $e(G) = (-1)^{r(G)-r(G_0)}$ is the Kottwitz sign of G and $m = \sum d\cdot\dim(E_d^{\varphi_0(\mathcal{I})})$. An equivalent formulation, using the L-function of the principal parameter φ_0, is

$$\int_I \mu_G = \frac{L(\varphi_0, \hat{\mathfrak{g}}, 1)}{L(\varphi_0, \hat{\mathfrak{g}}, 0)\cdot|^L Z|}\cdot|\mathsf{T}(q)|\cdot q^{-\ell_0}. \tag{55}$$

The right hand sides of these formulas are clearly rational numbers.

To reformulate the conjecture of Hiraga-Ichino-Ikeda [26] on formal degrees, we need to compare μ_G with a positive Haar measure ν_G defined in [18] as follows. Let G' be the split form of G over k, let $\phi : G \to G'$ be an isomorphism over $\bar{k}$, and let ω' be a differential form of top degree on the Chevalley model of G' over A, which has good reduction mod P. Let $\phi^*(\omega')$ be the pull-back of ω' to G. Then

$$\nu_G = |\phi^*(\omega')|.$$

Another measure

$$|\omega_G| := q^{a(M)/2}\nu_G$$

was used in [15], which gave the functional equation

$$L(\check{M}(1))\cdot|\omega_G| = e(G)\cdot|^L Z|\cdot L(M)\cdot\mu_G. \tag{56}$$

Since $|\varepsilon(M)| = q^{a(M)/2}$, we obtain the following relation between μ_G and ν_G:

$$(-1)^{r(G)}\cdot|^L Z|\cdot\mu_G = |\gamma(M)|\cdot\nu_G = |\gamma(\varphi_0)|\cdot\nu_G. \tag{57}$$

7.2 The Local Langlands correspondence

The conjectured degree formula in [26] depends on the conjectural local Langlands correspondence, of which there is more than one version. For example, the correspondence discussed in [10] and [21] involves $H^1(k, G_0)$, which parametrizes the pure inner forms of the quasi-split group G_0. Here however, we want to include all inner forms in our conjecture, so we use another version, influenced by Arthur [2] and involving $H^1(k, G_{ad})$, where $G_{ad} = G_0/Z$ is the adjoint group of G, which parametrizes the inner forms of G_0.

The dual group of G_{ad} is the simply-connected cover $\hat{G}_{sc}$ of the derived subgroup of $\hat{G}$. The pinned action of $\mathrm{Gal}(k_0/k)$ on $\hat{G}$ lifts to $\hat{G}_{sc}$, and ${}^L G_{ad} = \mathrm{Gal}(k_0/k) \ltimes \hat{G}_{sc}$ is the L-group of G_{ad}. Let ${}^L\mathcal{Z}$ be the center of ${}^L G_{ad}$. Explicitly, ${}^L\mathcal{Z}$ is the subgroup of $\mathrm{Gal}(k_0/k)$-invariants in the center $\hat{Z}_{sc}$ of $\hat{G}_{sc}$. We may identify [35]

$$H^1(k, G_{ad}) \simeq \mathrm{Hom}({}^L\mathcal{Z}, \mathbf{C}^\times). \tag{58}$$

Thus, the inner forms of G_0 are parametrized by irreducible characters of ${}^L\mathcal{Z}$, in such a way that G_0 itself corresponds to the trivial character of ${}^L\mathcal{Z}$. We fix a character ζ_G of $\hat{Z}_{sc}$ whose restriction to ${}^L\mathcal{Z}$ corresponds to the inner form G of G_0. If $G = G_0$ we take ζ_G to be the trivial character.

Recall our standing assumption that the maximal torus in the center of G is anisotropic over k, which is equivalent to assuming that the center ${}^L Z$ of ${}^L G$ is finite.

Given a discrete parameter $\varphi : \mathcal{W} \times SL_2 \to \hat{G}$, the group $A_\varphi/{}^L Z$ is a finite subgroup of the adjoint group $(\hat{G})_{ad} = \hat{G}/Z(\hat{G})$. Let $\mathcal{A}_\varphi$ be the full preimage of $A_\varphi/{}^L Z$ in $\hat{G}_{sc}$ under the adjoint isogeny $\hat{G}_{sc} \to (\hat{G})_{ad}$. Thus, $\mathcal{A}_\varphi$ is a central extension

$$1 \longrightarrow \hat{Z}_{sc} \longrightarrow \mathcal{A}_\varphi \longrightarrow A_\varphi/{}^L Z \longrightarrow 1.$$

We let $\mathrm{Irr}(\mathcal{A}_\varphi, \zeta_G)$ be the set of irreducible characters of $\mathcal{A}_\varphi$ whose restriction to $\hat{Z}_{sc}$ is a multiple of ζ_G. For the principal parameter φ_0, we have $\mathcal{A}_{\varphi_0} = \hat{Z}_{sc}$ and $\mathrm{Irr}(\mathcal{A}_{\varphi_0}, \zeta_G) = \{\zeta_G\}$.

Let $\Pi^2(G/k)$ be the set of equivalence classes of irreducible discrete series representations of $G(k)$ and let $\mathcal{L}(G/k)$ be the set of $\hat{G}$-conjugacy classes of pairs (φ, ρ), where $\varphi : \mathcal{W} \times SL_2 \to {}^L G$ is a discrete parameter and $\rho \in \mathrm{Irr}(\mathcal{A}_\varphi, \zeta_G)$.

Conjecture 7.1 *Let G be a connected reductive group over k and assume the maximal torus in the center of G is anisotropic. There is a bijection*

$$\Pi^2(G/k) \xrightarrow{\sim} \mathcal{L}(G/k),$$

which we denote by $\pi \mapsto (\varphi_\pi, \rho_\pi)$, with the following properties.

1. *If π_0 is the Steinberg representation of $G(k)$ then φ_{π_0} is the principal parameter φ_0 and $\rho_{\pi_0} = \zeta_G$.*
2. *If $\pi, \pi' \in \Pi^2(G/k)$ have $\varphi_\pi = \varphi_{\pi'}$ then π and π' have the same central character.*

3. *If ρ_π is trivial then π is generic.*
4. *For a given discrete parameter φ, the following are equivalent:*
   (i) All $\pi \in \Pi^2(G/k)$ with $\varphi_\pi = \varphi$ are supercuspidal.
   (ii) If $\varphi_\pi = \varphi$ and $\rho_\pi = 1$ then π is supercuspidal.
   (iii) $\varphi(SL_2) = 1$.
5. *The formal degree of $\pi \in \Pi^2(G/k)$, with respect to the Euler-Poincaré measure μ_G, is given by*
$$(-1)^{r(G)} \cdot \deg_{\mu_G}(\pi) = \pm \frac{\dim(\rho_\pi)}{|A_{\varphi_\pi}/{}^L Z|} \cdot \frac{\gamma(\varphi_\pi)}{\gamma(\varphi_0)}, \tag{59}$$
where the sign is that of $\omega(\varphi_\pi)/\omega(\varphi_0) = \pm 1$.

In part 2, note that we may identify the centers of the inner forms of G, so it makes sense to compare the central character on these various groups. We will specify this central character in the next section. In part 3, note that if $\rho_\pi = 1$ then $\zeta_G = 1$ so π is a representation of $G_0(k)$, which has generic representations since G_0 is quasi-split.

Formula (59) is a reformulation of the formal degree conjecture in [26]. To see this, we note that Hiraga et al. use the invariant measure ν_G to calculate formal degrees, and they conjecture that
$$\deg_{\nu_G}(\pi) = \frac{\dim \rho_\pi}{|A_{\varphi_\pi}|} \cdot |\gamma(\varphi_\pi)|. \tag{60}$$
Comparing conjecture (60) with the relation (57) between μ_G and ν_G, we obtain the formula
$$(-1)^{r(G)} \deg_{\mu_G}(\pi) = \frac{\dim \rho}{|A_{\varphi_\pi}/{}^L Z|} \cdot \frac{|\gamma(\varphi_\pi)|}{|\gamma(\varphi_0)|}. \tag{61}$$
We have seen that $\gamma(\varphi_\pi)/\gamma(\varphi_0)$ is a rational number. From equations (36) and (37) it follows that the sign of $\gamma(\varphi_\pi)/\gamma(\varphi_0)$ is that of $\omega(\varphi_\pi)/\omega(\varphi_0)$. Hence we may remove the absolute value signs in formula (61) to obtain the formula (59) in conjecture 7.1. In chapter 8, we will make a conjecture relating the sign $\pm 1 = \omega(\varphi_\pi)/\omega(\varphi_0)$ in (59) to the value of the central character of π on a certain element in $Z(k)$.

Remark: The bijection conjectured to exist in 7.1 has been established for tori [38], for the groups $GL_n(k)$ [23], [25] and thence for $SL_n(k)$ [27]. In the first two cases, the groups $\mathcal{A}_\varphi$ are trivial. In the general case, one expects to have a natural parametrization $\varphi \mapsto \Pi_\varphi$ for the L-packets
$$\Pi_\varphi(G/k) := \{\pi \in \Pi^2(G/k) : \ \varphi_\pi = \varphi\}$$
but the labeling of the individual representations in each L-packet by the irreducible representations $\rho \in \mathrm{Irr}(\mathcal{A}_\varphi, \zeta_G)$ is perhaps not canonical, and may depend on auxiliary choices.

We give a simple example for the group $G = SL_2$ to illustrate this ambiguity. Assume that k has odd residual characteristic. We then have a surjective homomorphism

$$\mathcal{W} \to \mathcal{W}^{ab} \to k^\times \to k^\times/k^{\times 2}$$

onto a finite group of type $(2,2)$ which embeds (uniquely up to conjugacy) as a finite subgroup of the complex Lie group ${}^L G = \hat{G} = SO_3(\mathbf{C})$, given by rotations by 180 degrees around three orthogonal axes. Hence there is a unique Langlands parameter

$$\varphi : \mathcal{W} \times SL_2 \to {}^L G,$$

trivial on SL_2, whose image is a finite group of type $(2,2)$. The subgroup $\mathcal{A}_\varphi < SL_2(\mathbf{C})$ is quaternion of order eight; it has four irreducible representations ρ of dimension one and trivial on $\hat{Z}_{sc} = \{\pm I\}$, and one irreducible representation of dimension two which is non-trivial on $\hat{Z}_{sc}$.

The L-packet $\Pi_\varphi(G/k)$ consists of the constituents of the restriction of a single irreducible representation of $GL_2(k)$. These constituents are given as follows. Let K_0, K_1 be hyperspecial maximal compact subgroups of $G(k) = SL_2(k)$ which fix adjacent vertices in the building (which is a homogeneous tree). Both K_i surject onto $SL_2(\mathbf{F}_q)$ via reduction modulo ϖ. Let κ_0 and κ_1 be the two irreducible cuspidal representations of $SL_2(\mathbf{F}_q)$ of dimension $(q-1)/2$. For $i, j \in \{0, 1\}$, let π_{ij} be the compactly-induced representation

$$\pi_{ij} = \mathrm{ind}_{K_i}^{G(k)} \kappa_j.$$

The L-packet of φ for $G/k = SL_2/k$ is

$$\Pi_\varphi(G/k) = \{\pi_{ij} : \ i, j \in \{0, 1\} \}.$$

The four representations π_{ij} have $\deg_{\mu_G}(\pi_{ij}) = -1/2$ and all have central character taking $-I$ to the scalar $(-1)^{(q+1)/2}$. Each π_{ij} is generic for one of the four orbits of generic characters of the unipotent radical of a Borel subgroup of $SL_2(k)$, but we see no natural way of deciding which π_{ij} is to have $\rho_{\pi_{ij}} = 1$, without making additional choices of a hyperspecial vertex or a generic character.

This ambiguity does not arise for the non-split inner form G'/k, where $G'(k)$ is the compact group $SL_1(D)$ of norm-one elements of a quaternion division algebra D over k. Here $G'(k)$ has a unique character π of order two which factors through the natural map

$$SL_1(D) \to U_1(\mathbf{F}_q),$$

where $U_1(\mathbf{F}_q)$ is the kernel of the norm mapping $\mathbf{F}_{q^2}^\times \to \mathbf{F}_q^\times$. We have $\deg_{\mu_G}(\pi) = 1$, in accordance with the formal degree conjecture (59), where ρ_π is the two-dimensional representation of $\mathcal{A}_\varphi$.

7.3 The Swan inequality revisited

The formal degree conjecture leads to our conjectured inequality for the Swan conductor in the following way. Suppose

$$\pi = \operatorname{ind}_K^{G(k)} V$$

is a supercuspidal representation compactly-induced from a finite dimensional representation V of a parahoric subgroup K of $G(k)$. Then

$$\dim V = \deg_{\mu_G}(\pi) \cdot \operatorname{vol}_{\mu_G}(K).$$

The term $\operatorname{vol}_{\mu_G}(K)$ belongs to the set $\mathbf{Q}_{p'}$ of rational numbers whose numerator and denominator are prime to p. If the degree conjecture holds, then by our calculation (41) of the leading term of $\Gamma_\varphi(x)$, the formal degree $\deg_{\mu_G}(\pi)$ is a rational number in $\mathbf{Q}_{p'}$ times

$$r_\varphi \cdot q^\nu,$$

where r_φ is a rational number depending on φ and ν is the order of Γ_φ at $x = 0$. There is a finite set S of primes p with the property that every r_φ is a product of powers of primes from S. Hence if p is sufficiently large, we have

$$\dim V = R_\varphi \cdot q^\nu,$$

where $R_\varphi \in \mathbf{Q}_{p'}$. Since $\dim V$ is an integer, it follows that $\nu \geq 0$, as conjectured in 5.2, and indicates that φ has minimal Swan conductor exactly when $\dim V$ is prime to p.

8 Central characters and root numbers

In view of part 2 of Conjecture 7.1, it is natural to try to predict the central character of the discrete series representations π of $G(k)$. Recall that G is quasi-split over k and split over k_0. We continue to assume that the maximal torus in the center Z of G is anisotropic over k. The aim of this section is to construct a character ω_φ of $Z(k)$ from a discrete parameter $\varphi : \mathcal{W} \times SL_2 \to {}^LG$. This was done by Langlands for real groups in [37]. In [4] Borel outlined a non-Archimedean version of Langlands' method, but Borel's account omits an essential point, namely the vanishing of the Schur multiplier of $\operatorname{Gal}(\bar{k}/k)$, due to Tate [47, Thm. 4].

Our approach to ω_φ relies instead on Tate local duality, and is more convenient for our later study of of root numbers. At first sight our approach may appear to be different from that of Langlands, so we will also complete the non-Archimedean version of Langlands' construction and we will show that the two constructions give the same character ω_φ.

8.1 Tate Duality

Recall that Φ is the set of roots of T in G and $\mathbb{Z}\Phi$ is the sublattice of $X^*(T)$ generated by Φ. The character group $A = X^*(Z) = X^*(T)/\mathbf{Z}\Phi$ is a finitely generated abelian group with an action of $\mathrm{Gal}(k_0/k)$ and we may identify

$$Z = \mathrm{Hom}(A, \mathbf{G}_m).$$

Since $Z(k)$ is compact, the natural pairing $A \times Z \to \mathbf{G}_m$ gives, by Tate local duality [50, II.5.8], a pairing

$$H^2(k, A) \times H^0(k, Z) \xrightarrow{\langle\ ,\ \rangle} H^2(k, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}$$

which identifies

$$H^2(k, A) = \mathrm{Hom}(Z(k), \mathbf{Q}/\mathbf{Z}).$$

Via the exponential map $x \mapsto \exp(2\pi i x)$ (which depends on a choice of $i = \sqrt{-1}$) we thus identify $H^2(k, A)$ with the group of characters $\omega : Z(k) \to \mathbf{C}^\times$ having finite order.

8.2 An approach via the fundamental group

Let $\pi_1(\hat{G})$ denote the fundamental group of the Lie group $\hat{G}$, the dual group of G, based at the identity of $\hat{G}$. We first show how to identify the group $A = X^*(Z)$ with the fundamental group $\pi_1(\hat{G})$. Let $p : \hat{G}_{sc} \to \hat{G}_{der}$ be the simply-connected covering of the derived group of $\hat{G}$, and let $\hat{T}_{sc}$ be the maximal torus of $\hat{G}_{sc}$ mapping onto $\hat{T} \cap \hat{G}_{der}$. The Lie algebra $\mathfrak{t}$ of $\hat{T}$ has a canonical decomposition

$$\hat{\mathfrak{t}} = \hat{\mathfrak{t}}_{sc} \oplus \hat{\mathfrak{z}},$$

where $\hat{\mathfrak{t}}_{sc}$ and $\hat{\mathfrak{z}}$ are the Lie algebras of $\hat{T}_{sc}$ and the center of $\hat{G}$, respectively. Let

$$\exp : \hat{\mathfrak{t}} \longrightarrow \hat{T}$$

be the exponential map, defined by

$$\lambda(\exp(x)) = e^{2\pi i \langle \lambda, x\rangle} \qquad \forall \lambda \in X^*(\hat{T}),$$

where we make the same choice of $i = \sqrt{-1}$ as before, to embed $\mathbf{Q}/\mathbf{Z}$ into $\mathbf{C}^\times$.

Let

$$\widetilde{G} = \hat{G}_{sc} \times \hat{\mathfrak{z}}$$

and extend p to the analytic surjective mapping

$$\tilde{p} : \widetilde{G} \longrightarrow \hat{G} \tag{62}$$

given by $\tilde{p}(t, z) = p(t) \cdot \exp(z)$. Since $\widetilde{G}$ is simply-connected of the same dimension as $\hat{G}$, it follows that the mapping (62) is the universal covering of $\hat{G}$, and that

$$\ker \tilde{p} = \pi_1(\hat{G}).$$

It is clear that $\ker\tilde{p}$ is contained in $\hat{T}_{sc}\times\hat{\mathfrak{z}}$. Since $\mathbf{Z}\Phi = X_*(\hat{T}_{sc})$, the restriction of $\tilde{p}$ to $\hat{T}_{sc}\times\hat{\mathfrak{z}}$ fits into a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & X_*(\hat{T})/\mathbf{Z}\Phi \arrow[r] \arrow[d, "\exp", "\simeq"'] & (\hat{\mathfrak{t}}_{sc}/\mathbf{Z}\Phi)\oplus\hat{\mathfrak{z}} \arrow[r] \arrow[d, "\exp\times 1", "\simeq"'] & \hat{\mathfrak{t}}/X_*(\hat{T}) \arrow[r] \arrow[d, "\exp", "\simeq"'] & 1 \\
1 \arrow[r] & \ker\tilde{p} \arrow[r] & \hat{T}_{sc}\times\hat{\mathfrak{z}} \arrow[r, "\tilde{p}"] & \hat{T} \arrow[r] & 1
\end{tikzcd}$$

where the rows are exact and the vertical maps are isomorphisms. Since $X_*(\hat{T}) = X^*(T)$, we have $A = X_*(\hat{T})/\mathbf{Z}\Phi$. It follows that the exponential map gives an isomorphism

$$A \xrightarrow{\exp} \ker\tilde{p} = \pi_1(\hat{G}).$$

Since the isomorphism is canonical, it respects the action of $\Gamma = \mathrm{Gal}(k_0/k)$ on $\hat{G}$ and $\tilde{G}$ by pinned automorphisms. Thus we have a Γ-equivariant exact sequence

$$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow \hat{G} \longrightarrow 1. \tag{63}$$

Now, given a parameter $\varphi : \mathrm{Gal}(\bar{k}/k) \to {}^LG$, any set-theoretic lifting

$$\tilde{\varphi} : \mathrm{Gal}(\bar{k}/k) \longrightarrow \Gamma \ltimes \widetilde{G}$$

gives a well-defined cohomology class $c_\varphi \in H^2(k, A)$ which measures the failure of $\tilde{\varphi}$ to be a homomorphism. In other words, $c_\varphi = \delta(\varphi)$, where

$$\delta : H^1(k, \hat{G}) \to H^2(k, A)$$

is the coboundary map of the exact sequence (63). By Tate duality as in section 8.1, we get a character

$$\omega_\varphi : Z(k) \to \mathbf{C}^\times$$

of finite order whose inverse is given by

$$\omega_\varphi(z)^{-1} = \langle c_\varphi, z\rangle, \quad \forall z \in Z(k).$$

As a special case, suppose Z is connected. Then

$$\hat{Z} = \hat{G}_{ab}$$

is the quotient of $\hat{G}$ by its derived group. Let $\zeta : \hat{G} \longrightarrow \hat{Z}$ be the natural projection. We have a smaller exact sequence

$$1 \longrightarrow A \longrightarrow \hat{\mathfrak{z}} \longrightarrow \hat{Z} \longrightarrow 1$$

with coboundary $\delta_Z : H^1(k, \hat{Z}) \to H^2(k, A)$. The composition

$$H^1(k, Z) \xrightarrow{\delta_Z} H^2(k, A) \longrightarrow \mathrm{Hom}(Z(k), \mathbf{C}^\times)$$

is the Langlands correspondence for the torus Z. From the commutative diagram

$$\begin{tikzcd} H^1(k, \hat{G}) \arrow[r, "\delta"] \arrow[d, "\zeta"'] & H^2(k, A) \arrow[d, equal] \\ H^1(k, \hat{Z}) \arrow[r, "\delta_Z"] & H^2(k, A), \end{tikzcd}$$

it follows that ω_φ is the inverse of the character of $Z(k)$ corresponding to the parameter $\zeta \circ \varphi \in H^1(k, \hat{Z})$.

8.3 Langlands' approach

For $k = \mathbf{R}$, Langlands had a different way to construct central characters If G has connected center, the example at the end of the previous section is identical to Langlands' construction, now in the p-adic case. We consider the opposite case, where G is semisimple, which we now assume. We adapt Langlands' method to the p-adic case, and compare with our previous construction.

We start by embedding G in a group G_1 with connected center such that the quotient G_1/G is an induced torus. The eventual central character is independent of G_1. Our aim is to relate this construction to the previous one. For concreteness, we take the group

$$G_1 := G \times_Z T,$$

with center

$$Z \times_Z T \simeq T.$$

The quotient $G_1/G = T/Z$ is a maximal torus in the adjoint group G_{ad}, hence is induced, since Γ permutes a basis of $X^*(T/Z)$ consisting of simple roots of T/Z in G_{ad}. The dual group of $\hat{G}_1$ is

$$\hat{G}_1 \simeq \widetilde{G} \times_A \widetilde{T},$$

where

$$\widetilde{G} = \hat{G}_{sc} \times \hat{\mathfrak{z}}, \quad \text{and} \quad \widetilde{T} = \hat{T}_{sc} \times \hat{\mathfrak{z}}$$

and

$$A = \ker[\hat{G}_{sc} \to \hat{G}] = \pi_1(\hat{G})$$

are as in the previous section. The natural maps

$$\tau : \hat{G}_1 \to \widetilde{T}/A = \hat{T} \quad \text{and} \quad \hat{\iota} : \hat{G}_1 \to \widetilde{G}/A = \hat{G}$$

are the abelianization quotient and the map dual to the embedding $\iota : G \hookrightarrow G_1$, respectively. From Shapiro's lemma and Tate's result $H^2(k, \mathbf{C}^\times) = 0$, it follows that

$$H^2(k, \widetilde{T}) = H^2(k, \hat{T}_{sc}) = 0,$$

so that any parameter $\varphi \in H^1(k, \hat{G})$ may be lifted to $\varphi_1 \in H^1(k, \hat{G}_1)$ (cf. [37, Lemma 2.10] for $k = \mathbf{R}$). As in the last paragraph of the previous section, the image of φ_1 in $H^1(k, \hat{Z}_1) = H^1(k, \hat{T})$ gives a character ω_{φ_1} of $T(k)$, which we then restrict to get a character $\omega'_\varphi : Z(k) \to \mathbf{C}^\times$. Two lifts φ_1, φ'_1 are related by:

$$\varphi'_1 = \eta \cdot \varphi_1,$$

where η is a parameter for T/Z. Applying functorality to the map $T \to T/Z$, we see that ω_{φ_1} agrees with $\omega_{\varphi'_1}$ on $Z(k)$, so the character ω'_φ is well-defined.

We now show that ω'_φ is is equal to the character ω_φ constructed in the previous section. Consider the diagram

$$\begin{tikzcd}
 & & & \Gamma \arrow[d, "\tau\circ\varphi_1"] & \\
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \widetilde{T} \arrow[r] \arrow[d] & \hat{T} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & A \arrow[r] & \widetilde{G} \arrow[r] & \hat{G} \arrow[r] & 1 \\
 & & & \Gamma \arrow[u, "\varphi"'] &
\end{tikzcd}$$

For each $\gamma \in \Gamma$, we choose $\tilde{\varphi}(\gamma) \in \widetilde{G}$, $\tilde{\tau}(\gamma) \in \tilde{T}$ such that $\varphi_1(\gamma)$ is the class of $(\tilde{\varphi}(\gamma), \tilde{\tau}(\gamma))$ in $\hat{G}_1 = \widetilde{G} \times_A \widetilde{T}$. Then

$$\tilde{\varphi} : \Gamma \to \widetilde{G} \quad \text{and} \quad \tilde{\tau} : \Gamma \to \widetilde{T}$$

are set-theoretic liftings of φ and $\tau \circ \varphi_1$ which give rise to the 2-cocycles in A whose classes in $H^2(k, A)$ give ω_φ^{-1} and ω'_φ, respectively. Since φ_1 does not fail to be a homomorphism, it follows that $\omega_\varphi = \omega'_\varphi = 1$, as claimed.

8.4 Deligne's formula for orthogonal root numbers

Recall that k has characteristic zero, with Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let

$$\varrho : \Gamma \longrightarrow O(V)$$

be an orthogonal complex representation of Γ. The root number $w(\varrho)$ satisfies

$$w(\varrho)^2 = \det \varrho(-1) = \pm 1.$$

Since $\det\varrho$ is also an orthogonal representation of Γ with the same determinant as ϱ, the quotient

$$c(\varrho) = \frac{w(\varrho)}{w(\det\varrho)} = \pm 1.$$

Let $w_2(\varrho)$ be the second Stiefel-Whitney class of ϱ in $H^2(k, \mathbf{Z}/2)$. If $\det\varrho = 1$, then $w_2(\varrho) = \delta\varrho$, where

$$\delta : \mathrm{Hom}(\Gamma, SO(V)) = H^1(k, SO(V)) \to H^2(k, \mathbf{Z}/2)$$

is the coboundary map of the central extension

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow \mathrm{Spin}(V) \longrightarrow SO(V) \longrightarrow 1$$

with trivial Γ-action. We map $\mathbf{Z}/2 \to \mu_2$, taking $a \mapsto (-1)^a$ and let $c_2(\varrho)$ be the image of $w_2(\varrho) \in H^2(k, \mu_2) = \{\pm 1\}$. Deligne's formula for orthogonal root numbers [12] is as follows.

Proposition 8.1 *We have*

$$c(\varrho) = c_2(\varrho) \in H^2(k, \mu_2) = \{\pm 1\}.$$

Corollary 8.2 *Assume that* $\det\varrho = 1$*, so that* $c(\varrho) = w(\varrho) = \pm 1$*. Then* $w(\varrho) = +1$ *if and only if the homomorphism* $\varrho : \Gamma \to SO(V)$ *lifts to a homomorphism* $\tilde{\varrho} : \Gamma \to \mathrm{Spin}(V)$.

8.5 Lifting Ad to Spin

The adjoint representation $\mathrm{Ad} : \hat{G} \longrightarrow SO(\hat{\mathfrak{g}})$ lifts uniquely to a homomorphism $\widetilde{\mathrm{Ad}} : \widetilde{G} \to \mathrm{Spin}(\hat{\mathfrak{g}})$, where $\widetilde{G}$ is the simply-connected cover of $\hat{G}$, as in section 8.2. Let $A = X_*(\hat{T})/\mathbb{Z}\hat{\Phi}$ be the fundamental group of $\hat{G}$; we obtain a homomorphism

$$e : A \longrightarrow \mathbf{Z}/2$$

making the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\exp} & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & {\scriptstyle e}\big\downarrow & & {\scriptstyle \widetilde{\mathrm{Ad}}}\big\downarrow & & {\scriptstyle \mathrm{Ad}}\big\downarrow & & \\
1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathrm{Spin}(\hat{\mathfrak{g}}) & \longrightarrow & SO(\hat{\mathfrak{g}}) & \longrightarrow & 1.
\end{array}$$

The value $\epsilon(a) = (-1)^{e(a)}$ is the scalar by which $a \in A$ acts on the space V of spinors of $\hat{\mathfrak{g}}$, via the homomorphism $\widetilde{\mathrm{Ad}}$. This scalar can be calculated as follows. Choose maximal torus and Borel subgroup $\hat{T} \subset \hat{B}$ in $\hat{G}$ and let $2\check{\rho}$ be the sum of the roots of $\hat{T}$ in $\hat{B}$. Let $\hat{T}_{sc}$ be the preimage $\hat{T}$

in $\widetilde{G}$. Then $X^*(\hat{T}) \subseteq X^*(\hat{T}_{sc})$ and $\check{\rho} \in X^*(\hat{T}_{sc})$ is a weight of $\hat{T}_{sc}$ in V. Given $\lambda \in X_*(\hat{T})$ with coset $a_\lambda \in A$, we have

$$\epsilon(a_\lambda) = \check{\rho}(\exp(a_\lambda)) = e^{2\pi i \langle \lambda, \check{\rho} \rangle} = e^{\pi i \langle \lambda, 2\check{\rho} \rangle} = (-1)^{\langle \lambda, 2\check{\rho} \rangle}. \tag{64}$$

This proves, in particular, the well-known fact that $\mathrm{Ad} : G \to SO(\hat{\mathfrak{g}})$ lifts to a homomorphism $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\hat{\mathfrak{g}})$ if and only if $\check{\rho} \in X^*(\hat{T})$.

Recall that $A = \mathrm{Hom}(Z, \mathbf{G}_m)$, where Z is the center of our p-adic group G whose dual group is $\hat{G}$, so that the character ϵ may be viewed as an element of Z. The torus $\hat{T}$ is also dual to our maximal torus T in G, so $2\check{\rho} \in X_*(T)$. Equation (64) shows that

$$\epsilon = 2\check{\rho}(-1) \in Z. \tag{65}$$

This involution in Z does not depend on any choices of maximal tori or Borel subgroups. Since the Γ action on Z is independent of inner twisting, and Γ preserves a pinning on the quasi-split form, we see that ϵ is k-rational. Thus, we have a *canonical involution* $\epsilon = 2\check{\rho}(-1) \in Z(k)$, of order one or two.

8.6 On the root number of the adjoint representation

In this section we conjecture a relation between root numbers and central characters. To motivate this conjecture, let us first suppose that G is an inner form of a split group over k, so that ${}^LG = \hat{G}$. Let $\varphi : \Gamma \to \hat{G}$ be a discrete parameter and let $w(\varphi) = w(\hat{\mathfrak{g}})$ be the root number of the orthogonal Galois representation

$$\mathrm{Ad}\,\varphi : \Gamma \xrightarrow{\varphi} \hat{G} \xrightarrow{\mathrm{Ad}} SO(\hat{\mathfrak{g}}).$$

Then $w(\varphi) = \pm 1$ and $w(\varphi) = +1$ if and only if $\mathrm{Ad}\,\varphi$ lifts to $\mathrm{Spin}(\hat{\mathfrak{g}})$, by Deligne's formula 8.1. On the other hand, from the diagram

$$\begin{tikzcd}
 & & & \Gamma \arrow[d, "\varphi"'] & \\
1 \arrow[r] & A \arrow[r, "\exp"] \arrow[d, "e"'] & \widetilde{G} \arrow[r] \arrow[d, "\widetilde{\mathrm{Ad}}"'] & G \arrow[r] \arrow[d, "\mathrm{Ad}"'] & 1 \\
1 \arrow[r] & \mathbf{Z}/2 \arrow[r] & \mathrm{Spin}(\hat{\mathfrak{g}}) \arrow[r] & SO(\hat{\mathfrak{g}}) \arrow[r] & 1,
\end{tikzcd}$$

we see that $\mathrm{Ad}\,\varphi$ lifts to $\mathrm{Spin}(\hat{\mathfrak{g}})$ if and only if the class $c_\varphi \in H^2(k, A)$ from section 8.2 belongs to the kernel of the map

$$\epsilon_* : H^2(k, A) \to H^2(k, \mu_2) = \{\pm 1\}$$

induced by ϵ. Viewing $\epsilon = 2\check{\rho}(-1)$ as the canonical involution in Z, this means that $\mathrm{Ad}\,\varphi$ lifts to $\mathrm{Spin}(\hat{\mathfrak{g}})$ if and only if $\omega_\varphi(\epsilon) = +1$, where ω_φ is the character of $Z(k)$ corresponding to c_φ under Tate duality. Combining these two criteria for $\mathrm{Ad}\,\varphi$ lifting to $\mathrm{Spin}(\hat{\mathfrak{g}})$, we find that

$$w(\varphi) = \omega_\varphi(\epsilon) = \pm 1$$

when G is an inner form of a split group.

This suggests the following more general conjecture.

Conjecture 8.3 *Let G be a connected reductive group over k and assume the maximal torus in the center of G is anisotropic. Let $\pi \in \Pi^2(G/k)$ be a discrete series representation corresponding to the discrete parameter $\varphi_\pi : \mathcal{W} \times SL_2 \to {}^LG$ as in conjecture 7.1 and let φ_0 be the principal parameter for G. Then the ratio*

$$\frac{\omega(\varphi_\pi)}{\omega(\varphi_0)} = \pm 1$$

of orthogonal root numbers is equal to the scalar by which the canonical involution $\epsilon \in Z(k)$ acts on the representation π.

We note that D. Prasad [40] has shown that the action of ϵ on a self-dual irreducible generic representation often determines the sign of the invariant bilinear form on the space of the representation.

9 Minimal Swan conductors and simple supercuspidal representations

By Prop. 4.1, the formal degree conjecture in part 4 of Conjecture 7.1 can be expressed as

$$(-1)^{r(G)} \cdot \deg_{\mu_G}(\pi) = \pm \frac{\dim(\rho_\pi)}{|A_{\varphi_\pi}/{}^LZ|} \cdot \Gamma_{\varphi_\pi}(q), \tag{66}$$

where $\Gamma_{\varphi_\pi}(x) \in \mathbf{Q}(x)$ is a certain rational function whose leading term was studied in section 5.1. In section 5.3, this analysis led to the notion of discrete parameters φ with "minimal Swan conductor"; recall that these are the parameters φ for which

$$\dim \hat{\mathfrak{g}}_N^{\varphi(\mathcal{I})} + b(\varphi) = \ell_0 + b(\varphi_0). \tag{67}$$

If G is split and $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$, then $N = 0$ and the condition (67) simplifies to

$$b(\varphi) = \ell, \tag{68}$$

where ℓ is the rank of G.

The tension between the two conditions (66) and (68) is strong enough to force π to be a particularly simple kind of supercuspidal representation. We give here the construction of these simple supercuspidal representations, which is based on a curious fact about the geometry of affine Coxeter groups.

9.1 Affine Coxeter groups

As a general reference for this section we recommend [7]. Let Ψ be an affine root system on a Euclidean affine space $\mathcal{A}$. Each root $\psi \in \Psi$ is a non-zero affine function on $\mathcal{A}$, denoted by $x \mapsto \langle \psi, x \rangle$. Let $H_\psi := \{x \in \mathcal{A} : \langle \psi, x \rangle = 0\}$ denote the corresponding root hyperplane, with positive side $H_\psi^+ := \{x \in \mathcal{A} : \langle \psi, x \rangle > 0\}$. An *alcove* is a connected component of the set $\{x \in \mathcal{A} : \langle \psi, x \rangle \neq 0 \quad \forall \psi \in \Psi\}$ of points in $\mathcal{A}$ not lying on any root hyperplane. A *wall* of an alcove is a root hyperplane H_ψ whose intersection with the closure of the alcove contains an open subset of H_ψ. Every alcove is the intersection of the positive sides of its walls.

Fix an alcove C and let

$$\Psi^+ := \{\psi \in \Psi : \langle \psi, x \rangle > 0 \quad \forall x \in C\}$$

be the corresponding *positive* affine roots. There is a unique finite subset $\Pi \subset \Psi^+$, the *simple* affine roots, such that Ψ^+ consists of those affine roots of the form

$$\sum_{\psi \in \Pi} n_\psi \psi,$$

with n_ψ integers ≥ 0. There are unique positive integers a_ψ such that the sum

$$\sum_{\psi \in \Pi} a_\psi \psi = 1$$

is the constant function $\equiv 1$ on $\mathcal{A}$. The walls of C are the hyperplanes H_ψ for $\psi \in \Pi$. The aim of this section is to prove: *Each alcove C' lies on the positive side of some wall of C that is not a wall of C'.*

This is reformulated and proved in Lemma 9.1 below, after some preparation.

The closure of C is partitioned into a disjoint union

$$\bar{C} = \bigcup_{J \subsetneq \Pi} C_J,$$

of non-empty facets C_J, indexed by the subsets $J \subset \Pi$ with $J \neq \Pi$, and C_J is the set of points $x \in \bar{C}$ where $\langle \psi, x \rangle = 0$ for $\psi \in J$ and $\langle \psi, x \rangle > 0$ for $\psi \in \Pi - J$ (see [7, V.1.6]). For $\psi \in \Psi$, let r_ψ denote the orthogonal reflection in $\mathcal{A}$ about the hyperplane H_ψ. These reflections generate a group

$$W(\Psi) := \langle r_\psi : \psi \in \Psi \rangle$$

of Euclidean affine transformations of $\mathcal{A}$. In fact, $W(\Psi)$ is an affine Coxeter group generated by $\{r_\psi : \psi \in \Pi\}$, and $W(\Psi)$ acts simply-transitively on the set of alcoves [7, V.3.2]. The corresponding action of $W(\Psi)$ on Ψ is given by:

$$\langle w\psi, x \rangle = \langle \psi, w^{-1}x \rangle, \qquad \forall \psi \in \Psi,\ x \in \mathcal{A}.$$

We reformulate the above geometric statement as follows.

Lemma 9.1 *Given $w \in W(\Psi)$ with $w \neq 1$, there exists $\psi \in \Pi$ such that $w\psi \in \Psi^+ - \Pi$.*

Proof: Since $w \neq 1$, the alcove $C_w := w^{-1}C$ is distinct from C. Each root $\psi \in \Psi$ is either always positive or always negative on C_w; ψ is positive on C_w if and only if $w\psi \in \Psi^+$. Write Π as a disjoint union $\Pi = \Pi_+ \sqcup \Pi_-$, where Π_+ and Π_- are the sets of roots in Π which are positive and negative, respectively, on C_w. Since $C_w \neq C$, the set Π_- is non-empty,

We must show that there exists $\psi \in \Pi_+$ for which $w\psi \notin \Pi$. Suppose on the contrary that $w\Pi_+ \subseteq \Pi$. Since Π_- is non-empty, we have $|\Pi_+| < |\Pi|$. Hence $w\Pi_+ \subset \Pi$ and $w\Pi_+ \neq \Pi$. Taking $J = w\Pi_+$, we have a (non-empty) facet $C_J \subset \bar{C}$. Let $x \in C_J$ and let $y = w^{-1}x \in \bar{C}_w$. For all $\psi \in \Pi_+$ we have $\langle \psi, y\rangle = \langle w\psi, x\rangle = 0$. But then

$$1 = \sum_{\psi \in \Pi} a_\psi \langle \psi, y\rangle = \sum_{\psi \in \Pi_-} a_\psi \langle \psi, y\rangle \leq 0,$$

a contradiction. Therefore $w\Pi_+$ is contained in Ψ^+ but is not contained in Π. ■

Remark: The lemma is false for finite root systems: If Δ is a base of a finite root system Φ, then there is a unique element $w_0 \in W(\Phi)$ such that $w_0\Delta = -\Delta$.

9.2 Affine generic characters

In this section we assume that G is simply-connected, almost simple, and split over k. Let $\ell = r(G)$ denote the rank of G. Let T be a maximal k-split torus in G, let T_0 be the maximal compact subgroup of $T(k)$ and let N be the normalizer of T in $G(k)$. Let Φ be the set of roots of T in G. Fix a Chevalley basis in the Lie algebra of G. This determines, for each root $\alpha \in \Phi$, an embedding

$$x_\alpha : k^+ \hookrightarrow G(k)$$

such that $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$, for all $t \in T$ and $c \in k$. Our choice of T determines an *apartment* $\mathcal{A}$, which is an affine Euclidean space under the vector space $\mathbf{R} \otimes X_*(T)$, together with a system Ψ of affine roots on $\mathcal{A}$. The choice of Chevalley basis determines a base-point $o \in \mathcal{A}$, which we use to identify

$$\mathcal{A} = \mathbf{R} \otimes X_*(T), \quad \text{and} \quad \Psi = \{\alpha + n :\ \alpha \in \Phi, n \in \mathbf{Z}\}.$$

Each affine root $\psi = \alpha + n$ indexes an *affine root group* $U_\psi = x_\alpha(P^n)$ which pointwise-fixes the positive side of the hyperplane H_ψ in $\mathcal{A}$ and is normalized by T_0. An element $t \in T_0$ acts on the quotient

$$U_\psi/U_{\psi+1} \simeq P^n/P^{n+1} \simeq \mathfrak{f}^+$$

as scalar multiplication by the image of $\alpha(t)$ in $\mathfrak{f}^\times$. The canonical action of N on $\mathcal{A}$ identifies the affine Weyl group N/T_0 with $W(\Psi)$; if $n \in N$ has image w in $W(\psi)$, we have

$$nU_\psi n^{-1} = U_{w\psi}.$$

Fix an alcove C in $\mathcal{A}$ with corresponding simple and positive affine roots $\Pi \subset \Psi^+$. The stabilizer of C in $G(k)$ is the Iwahori subgroup

$$I := \langle T_0,\ U_\psi :\ \psi \in \Psi^+ \rangle.$$

Let

$$T_1 := \langle t \in T_0 :\ \lambda(t) \in 1 + P \quad \forall \lambda \in X^*(T) \}.$$

The subgroup

$$I_+ := \langle T_1,\ U_\psi :\ \psi \in \Psi^+ \rangle \subseteq I$$

is the pro-p-Sylow subgroup of I and we have

$$I = T(q) \ltimes I_+, \qquad G(k) = I_+ N I_+,$$

where $T(q) = \{t \in T_0 :\ t^q = t\}$ projects isomorphically onto T_0/T_1. Finally, we consider the subgroup

$$I_{++} := \langle T_1,\ U_\psi :\ \psi \in \Psi^+ - \Pi \rangle.$$

Lemma 9.2 *The subgroup I_{++} is normal in I_+ with quotient*

$$I_+/I_{++} \simeq \bigoplus_{\psi \in \Pi} U_\psi / U_{\psi+1}$$

as T_0-modules.

Proof: Each affine root $\phi \in \Psi^+$ may be uniquely written as

$$\phi = \sum_{\psi \in \Pi} n_\psi \psi,$$

where the n_ψ are integers ≥ 0. We have $\phi \in \Psi^+ - \Pi$ if and only if $\sum n_\psi > 1$. The commutator $[U_\phi, U_\eta]$ is contained in the subgroup of I_+ generated by $U_{i\phi + j\eta}$ for i, j positive integers with $i\psi + j\eta \in \Psi$. Also $[T_1, U_\phi] \subset U_{\phi+1}$. It follows that I_{++} is normal in I_+ and contains the commutator subgroup of I_+, so that I_+/I_{++} is abelian.

For each $\psi \in \Pi$, the inclusion $U_\psi \hookrightarrow I_+$ factors through a map

$$\mathfrak{f} \simeq U_\psi / U_{\psi+1} \longrightarrow I_+/I_{++}.$$

Since I_+/I_{++} is abelian, we get a well-defined homomorphism

$$\bigoplus_{\psi \in \Pi} U_\psi / U_{\psi+1} \longrightarrow I_+/I_{++}, \tag{69}$$

which is surjective, from the definition of I_{++}. From uniqueness-of-expression [60, 3.1.1] the product $\prod_{\psi\in\Pi} U_\psi$ (taken in any order) intersects I_{++} trivially. It follows that the map in (69) is also injective.

■

Remark: For every point x in the Bruhat-Tits building of $G(k)$, Moy and Prasad have defined filtration subgroups $G(k)_{x,r+} \leq G(k)_{x,r}$ indexed by $r \in \mathbf{R}_{\geq 0}$, of the parahoric subgroup $G(k)_x$ at x. Let x_0 be the unique point in the alcove C on which all simple affine roots $\psi \in \Pi$ take the same value. This common value is $1/h$, where h is the Coxeter number of G, and we have

$$I_+ = G(k)_{x_0,0+} = G(k)_{x_0,1/h}, \qquad I_{++} = G(k)_{x_0,1/h+} = G(k)_{x_0,2/h}.$$

Recall that Z is the center of G. Let $Z(q) = Z \cap T(q)$. Note that $Z(k)I_+ = Z(q) \times I_+$. We say that a character $\chi : Z(k)I_+ \to \mathbf{C}^\times$ is *affine generic* if

(i) χ is trivial on I_{++} and

(ii) χ is nontrivial on U_ψ for every $\psi \in \Pi$.

There are $|Z(q)| \cdot (q-1)^{\ell+1}$ affine generic characters. They are permuted freely by the group $T(q)/Z(q)$ (see the proof of Prop. 9.3 below). Hence, there are $|Z(q)|^2 \cdot (q-1)$ orbits of $T(q)$ on the set of affine generic characters of $Z(k)I_+$.

9.3 Simple supercuspidal representations

For each affine generic character $\chi : Z(k)I_+ \to \mathbf{C}^\times$, let π_χ be the compactly-induced representation

$$\pi_\chi := \operatorname{ind}_{Z(k)I_+}^{G(k)} \chi \tag{70}$$

of $G(k)$ in the space of functions $f : G(k) \to \mathbf{C}$ having the properties:

- $f(hg) = \chi(h)f(g)$ for all $h \in Z(k)I_+,\ g \in G(k)$.
- The support of f consists of finitely many left cosets of $Z(k)I_+$ in $G(k)$.

Proposition 9.3 *Let χ and η be affine generic characters of $Z(k)I_+$. Then*

1. *the representation π_χ is irreducible (and supercuspidal) for $G(k)$;*
2. *the representations π_χ and π_η are equivalent as representations of $G(k)$ if and only if $\eta = \chi^t$ for some $t \in T(q)$.*

Proof: To ease the notation we write (in this proof only) $G = G(k),\ H = Z(k)I_+$, so that $G = HNH$, by the affine Bruhat decomposition [60, 3.3.1]. We first prove the second assertion. By Mackey's theorem [36], we have

$$\operatorname{Hom}_G(\pi_\eta, \pi_\chi) \simeq \bigoplus_{n \in H\backslash G/H} \operatorname{Hom}_{H\cap H^n}(\eta, \chi^n).$$

Suppose $n \in N$ and $\eta = \chi^n$ on $H \cap H^n$. Let w be the image of n in $W(\Psi)$. Assume first that $w \neq 1$. By Lemma 9.1 there exists $\psi \in \Pi$ such that $w\psi \in \Psi^+ - \Pi$. Then we have $U_\psi = U^n_{w\psi} \in H \cap H^n$. Since $\eta = \chi^n$ on $H \cap H^n$ we also have $\eta = \chi^n$ on U_ψ.

But $nU_\psi n^{-1} = U_{w\psi}$ is contained in I_{++}, since $w\psi \in \Psi^+ - \Pi$. And $I_{++} \subset \ker \chi$ by condition (i) above. Therefore χ is trivial on $nU_\psi n^{-1}$, so χ^n is trivial on U_ψ. But we have seen that $\eta = \chi^n$ on U_ψ. So η is trivial on U_ψ, contradicting condition (ii) above for η. Therefore $w = 1$, so n, which we now call t, lies in T_0. Since $T_0 = T(q)T_1$ and $T_1 \subset I_+$, we may take $t \in T(q)$. The second assertion is proved.

For irreducibility we take $\chi = \eta$. It suffices to show that the element t above lies in Z. [8, 3.11.4] Choose $\psi \in \Pi$ and let α be the gradient of ψ. On U_ψ, both χ and χ^t may be viewed as nontrivial characters of $U_\psi/U_{\psi+1} \simeq \mathfrak{f}$. If $t \in T(q)$ is such that $\chi = \chi^t$ on H then $\chi^{-1} \cdot \chi^t$ is the trivial character of $\mathfrak{f}$. For $x \in \mathfrak{f}$, we have $\chi^{-1} \cdot \chi^t(x) = \chi((\alpha(t) - 1)x)$. Since χ is nontrivial on $\mathfrak{f}$, we must have $\alpha(t) = 1$. Therefore t is in the kernel of every root of T in G, so $t \in Z$, as claimed. ∎

The irreducible representations π_χ constructed in Prop. 9.3 will be called *simple supercuspidal representations* of $G(k)$. By part 2 of Prop. 9.3 there are $|Z(q)|^2 \cdot (q-1)$ equivalence classes of simple supercuspidal representations.

Remark: The "spherical" analogue of π_χ, where I_+ is replaced by the unipotent radical of a Borel subgroup of $G(k)$, is highly reducible. Indeed, π_χ is then a Gelfand-Graev representation, whose constituents are the χ-generic representations of $G(k)$. See the remark after Lemma 9.1.

9.4 Formal degrees of simple supercuspidal representations

Recall that G is split and simply-connected and that Z is the center of G. Since G is simply-connected, the Iwahori subgroup I is its own normalizer. Formula (55) gives the volume of I with respect to Euler-Poincaré measure as

$$(-1)^{r(G)} \int_I \mu_G = \frac{L(\varphi_0, \hat{\mathfrak{g}}, 1)}{L(\varphi_0, \hat{\mathfrak{g}}, 0)} \cdot |T(q)| \cdot q^{-\ell}, \tag{71}$$

where φ_0 is the principal parameter. Since G is split, the Artin conductor $\alpha(\varphi_0) = 2N$ is the number of roots of T in G, while the root number $\omega(\varphi_0) = 1$. Hence the gamma factor of φ_0 is

$$\gamma(\varphi_0) = q^N \cdot \frac{L(\varphi_0, \hat{\mathfrak{g}}, 1)}{L(\varphi_0, \hat{\mathfrak{g}}, 0)}$$

and we may write

$$(-1)^{r(G)} \int_I \mu_G = q^{-(N+\ell)} \cdot \gamma(\varphi_0) \cdot |T(q)|.$$

Let $\chi : Z(k)I_+ \to \mathbf{C}^\times$ be an affine generic character. From Prop. 9.3 we have a simple supercuspidal representation π_χ compactly induced from χ on $Z(k)I_+$. Let us view π_χ as induced from I:

$$\pi_\chi = \mathrm{ind}_I^{G(k)} \kappa, \quad \text{where} \quad \kappa = \mathrm{ind}_{Z(k)I_+}^I \chi.$$

We write

$$Z(k) = Z_+ \times Z(q),$$

where $Z_+ = Z(k) \cap I_+$ is a p-group and $Z(q) = Z(k) \cap T(q)$ has order prime to p. We have

$$Z(k)I_+ = Z(q) \times I_+ \quad \text{and} \quad [I : Z(k)I_+] = [T(q) : Z(q)].$$

It follows that κ has dimension

$$\dim \kappa = [I : ZI_+] = [T(q) : Z(q)],$$

so the formal degree of π_χ is given by

$$(-1)^\ell \deg_{\mu_G}(\pi_\chi) = \frac{\dim \kappa}{\mathrm{vol}_{\mu_G}(I)} = \frac{q^{N+\ell}}{|Z(q)| \cdot \gamma(\varphi_0)}. \tag{72}$$

9.5 Predictions for the Langlands parameter

While guided by conjecture 7.1, our construction of simple supercuspidal representations did not depend on any conjectures. In this section we again use the degree conjecture 7.1 to analyze the putative Langlands parameter φ_π for a simple supercuspidal representation $\pi = \pi_\chi$. Since G is split and simply-connected we have

$${}^LG = \hat{G} = \mathrm{Aut}(\hat{\mathfrak{g}})^\circ$$

and ${}^LZ = 1$. As in Prop. 4.1, we view formal degrees as rational functions. This means we must consider simple supercuspidal representations of the groups $G(k_m)$ where k_m/k is the unramified extension of degree $m \geq 1$. Let K be the maximal unramified extension of k (in a given algebraic closure of k) and let F be the endomorphism of $G(K)$ given by Frob, so that $G(k_m) = G(K)^{\mathsf{F}^m}$. Then F preserves $T(K)$ and each affine root group $U_\psi(K)$, for $\psi \in \Psi$. For each integer $m \geq 1$, let $I^{(m)}$, $I_+^{(m)}$, and $I_{++}^{(m)}$ be the subgroups of $G(k_m)$, defined as in section 9.2, with k replaced by k_m. Then

$$I_+^{(m)}/I_{++}^{(m)} \simeq \bigoplus_{\psi \in \Pi} \mathfrak{f}_m,$$

where $\mathfrak{f}_m$ is the residue field of k_m. The mapping $g \mapsto g\mathsf{F}(g) \cdots \mathsf{F}^{m-1}(g)$ on $I^{(m)}$ gives a group homomorphism

$$\tau_m : Z(k_m)I_+^{(m)}/I_{++}^{(m)} \longrightarrow Z(k)I_+/I_{++}$$

whose image contains I_+/I_{++}. Given an affine generic character χ of $Z(k)I_+$, we get an affine generic character

$$\chi_m := \chi \circ \tau_m : Z(k_m)I_+^{(m)} \longrightarrow \mathbf{C}^\times$$

and a simple supercuspidal representation $\pi_m := \pi_{\chi_m}$ of $G(k_m)$ whose formal degree is given by

$$(-1)^{\ell} \deg_{\mu_G^m}(\pi_m) = \frac{q^{m(N+\ell)}}{|Z(q^m)| \cdot \gamma(\varphi_0^m)}, \tag{73}$$

where μ_G^m is Euler-Poincaré measure on $G(k_m)$. The right side of equation (73) determines a rational function $\Delta(x) \in \mathbf{Q}(x)$ such that

$$(-1)^{\ell} \deg_{\mu_G^m}(\pi_m) = \Delta(q^m), \tag{74}$$

for all $m \equiv 1 \mod e$, where F^e is trivial on $Z(K)$.

For each such m, conjecture 7.1 gives a discrete parameter $\varphi^m = \varphi_{\pi_m}$ We also assume that these parameters are compatible, via the base-change formula

$$\varphi_{\pi_m} = \varphi_\pi|_{\mathcal{W}(k_m)}. \tag{75}$$

Since $F = \varphi(\mathrm{Frob})$ has finite order, there is a positive integer f such that $\mathcal{A}_{\varphi^m} = \mathcal{A}_\varphi$ for all $m \equiv 1 \mod f$. Then it makes sense to require that

$$\rho_{\pi_m} = \rho_\pi. \tag{76}$$

Then conjecture 7.1 predicts the formula

$$(-1)^{\ell} \deg_{\mu_G^m}(\pi_m) = \omega(\varphi_\pi) \frac{\dim(\rho_\pi)}{|A_{\varphi_\pi}|} \cdot \frac{\gamma(\varphi^m)}{\gamma(\varphi_0^m)}, \tag{77}$$

for all $m \equiv 1 \mod 4def$, [1] In terms of rational functions, this means that

$$\Delta(x) = \omega(\varphi_\pi) \frac{\dim(\rho_\pi)}{|A_{\varphi_\pi}|} \cdot \Gamma_{\varphi_m}(x).$$

Comparing equations (73) and (77) we find that we must have

$$x^{\alpha(\varphi_\pi)/2} \cdot \frac{L_{\varphi_\pi}(x,1)}{L_{\varphi_\pi}(x,0)} = \omega(\varphi_\pi) \cdot \gamma(\varphi_\pi) = x^{N+\ell} \frac{|A_{\varphi_\pi}|}{|Z(q)| \cdot \dim \rho}, \tag{78}$$

where $L_{\varphi_\pi}(x,s)$ is the rational L-factor defined in (33). Since the right side of (78) has no poles, it follows from (33) that φ_π is trivial on SL_2 and that $L_{\varphi_\pi}(x,s) \equiv 1$, forcing

$$\hat{\mathfrak{g}}^{D_0} = 0. \tag{79}$$

The derived subalgebra of $\hat{\mathfrak{g}}^{D_1}$ has zero invariants under the cyclic group D_0/D_1, hence this derived subalgebra is zero. It follows that $\hat{\mathfrak{g}}^{D_1}$ is the Lie algebra of a torus. We must also have

$$a(\varphi_\pi) = 2(N+\ell) = \dim \hat{\mathfrak{g}} + \ell.$$

[1]Note that $\omega(\varphi_0) = 1$ and $\omega(\varphi_\pi) = \pm 1$ since G is split over k.

since $a(\varphi_\pi) = \dim \hat{\mathfrak{g}} + b(\varphi_\pi)$, this means that

$$b(\varphi_\pi) = \ell. \tag{80}$$

Thus, the degree conjecture 7.1 implies that the parameter of a simple supercuspidal representation is a simple wild parameter, as defined in section 6.

To say more about φ_π, we confine ourselves here to the simplest case, namely when p does not divide the order of the Weyl group W of G. In this case we have, by Prop. 5.6, that $D_2 = 1$, D_1 is contained in a maximal torus of $\hat{G}$ and the tame quotient D_0/D_1 is generated by a Coxeter element w in the Weyl group of $\hat{G}$. Since G is split and the exponent of $Z(K)$ divides the Coxeter number h of G, the congruence condition in (77) can be simplified to $m \equiv 1 \mod 2d$, where d is the order of q in $(\mathbf{Z}/h)^\times$.

To complete our analysis of the parameter of a simple supercuspidal representation, we now turn to the group A_{φ_π}, still under the assumption that $p \nmid |W|$. We have seen that D_1 is an elementary abelian p-group. By [57, Thm 2.28(c)], the centralizer $C_{\hat{G}}(D_1)$ is connected. Since $\hat{\mathfrak{g}}^{D_1} = \hat{\mathfrak{t}}$, we have

$$C_{\hat{G}}(D_1) = \hat{T} \quad \text{and} \quad C_{\hat{G}}(D_0) = \hat{T}^w.$$

The image $\varphi_\pi(F)$ of Frobenius normalizes D_1, so we have $\varphi_\pi(F) \in \hat{N}$, projecting to an element $u \in W$ such that $u^{-1}wu = w^q$. Recall that G is simply-connected and $\hat{G}$ is adjoint. Let $Y = X^*(T) = X_*(\hat{T})$, and identify $\hat{T} = \mathbf{C}^\times \otimes Y$. We have canonical isomorphisms

$$\hat{T}^w \xleftarrow{\exp} (1-w)^{-1}Y/Y \xrightarrow{1-w} Y/(1-w)Y = \mathrm{Hom}(T^w, \mathbf{G}_m).$$

Since w is a Coxeter element, $(1-w)Y$ is the root lattice of T and u acts trivially on $Y/(1-w)Y$. From the equation

$$u^{-1}(1-w) = (1-w)(1 + w + \cdots + w^{q-1})u^{-1},$$

it follows that qu^{-1} acts trivially on $(1-w)^{-1}Y/Y$, so that u acts as the q-power map on $\hat{T}^w$. Thus, we find that

$$A_{\varphi_\pi} = [\hat{T}^w]^{q=1}.$$

Since A_{φ_π} is abelian, we have $\dim\rho = 1$. Since $T^w = Z$ is the center of G, it also follows that

$$|A_{\varphi_\pi}| = |\operatorname{Irr}(Z(k))|.$$

The assumption that $p \nmid |W|$ implies that $p \nmid |Z|$, so $Z(q) = Z(k)$. Thus, in equation (78) we have

$$|A_{\varphi_\pi}| = |Z(q)| = |Z(k)| \qquad \text{and} \quad \dim(\rho_\pi) = 1.$$

We summarize what we have proved in this section:

Proposition 9.4 *Suppose that G is simply-connected, almost simple and is k-split of rank ℓ. Assume that the simple supercuspidal representation $\pi = \pi_\chi$ of $G(k)$ corresponds to the pair (φ_π, ρ_π) as in conjecture 7.1. We also assume that conjecture 7.1 and the base-change relations (75) and (76) hold for π_{χ_m} and φ^m when $m \equiv 1 \mod 2df$. Let L be the fixed-field of $\ker \varphi_\pi$ in $\bar{k}$. Then φ_π is a simple wild parameter:*

$$\hat{\mathfrak{g}}^{\varphi_\pi(\mathcal{I})} = 0 \quad \textit{and} \quad b(\varphi_\pi) = \ell.$$

If moreover p does not divide the order of the Weyl group of G, then we also have:

1. *The image $\varphi_\pi(\mathcal{W}) = \mathrm{Gal}(L/k)$ is contained in the normalizer $N(\hat{T})$ of a maximal torus $\hat{T}$ in $\hat{G}$ and has ramification filtration of the form*

$$\mathrm{Gal}(L/k) = D \geq D_0 > D_1, \qquad D_2 = 1.$$

2. *Let h be the Coxeter number of G and let d be the order of q modulo h. Then D/D_0 is cyclic of order dc where c divides the exponent of Z.*
3. *We have $D_0 = D_1 \rtimes \langle\sigma\rangle$, where $\sigma \in N(\hat{T})$ has order h and projects to a Coxeter element in W.*
4. *The wild inertia group D_1 has order p^a, where a is the order of p in $(\mathbf{Z}/h)^\times$, and D_1 is the unique simple $\mathbf{F}_p[\sigma]$-submodule of $\hat{T}[p]$ containing the σ-eigenvalue $\zeta = \sigma(\pi)/\pi$ where π is a uniformizing parameter of the tame extension $E = L^{D_1}$ of k such that $\pi^h \in E^{D_0}$. Moreover, we have*

$$\hat{\mathfrak{g}}^{D_1} = \hat{\mathfrak{t}} \quad \textit{and} \quad C_{\hat{G}}(D_1) = \hat{T}.$$

5. *The centralizer $A_{\varphi_\pi} = \{t \in \hat{T}^\sigma : t = t^q\}$ is abelian and has order equal to that of $Z(q)$.*

Remark: We have seen that $G(k)$ has $|Z(q)|^2 \cdot (q-1)$ simple supercuspidal representations. One can check that each orbit of simple supercuspidal representations under $G_{ad}(k)$, where G_{ad} is the adjoint group of G, has cardinality $|Z(q)|$. Prop. 9.4 suggests that for split simply-connected groups G the simple supercuspidal representations of $G(k)$ should be partitioned into $|Z(q)|\cdot(q-1)$ distinct L-packets, each of cardinality $|Z(q)|$ and consisting of a single $G_{ad}(k)$-orbit. On the arithmetic side, we believe that further analysis of our construction of the extension L/k and the embeddings of $\mathrm{Gal}(L/k)$ into $\hat{G}$ will show that there are exactly $|Z(q)| \cdot (q-1)$ equivalence classes of simple wild parameters.

References

[1] J. D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math., **185** (1998), pp. 1–32.

[2] J. Arthur, *A Note on L-packets*, Pure and App. Math Q., **2** (2006), pp. 199–217.

[3] A. Borel, *Admissible representations of a semisimple group over a local field with vectors fixed under an Iwahori subgroup*, Inventiones Math., **35** (1976), pp. 233-259.

[4] ______, *Automorphic L-functions*, Proc. Symp. Pure Math., vol. 33, part 2, Amer. Math. Soc., Providence, RI (1979), pp. 27–61.

[5] A. Borel, J.P. Serre, *Sur certains sous-groupes des groupes de Lie compacts*, Comment. Math. Helv., **27** (1953), pp. 128-139.

[6] N. Bourbaki, *Lie groups and Lie algebras*, Chap. 1-3, Springer-Verlag, Berlin, 2002.

[7] ______, *Lie groups and Lie algebras*, Chap. 4-6, Springer-Verlag, Berlin, 2002.

[8] C. Bushnell, G. Henniart *The Local Langlands conjecture for $GL(2)$*, Springer-Verlag, Berlin, 2006.

[9] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Wiley, 1985.

[10] S. DeBacker and M. Reeder, *Depth-Zero Supercuspidal L-packets and their Stability*, Annals of Math., **169**, No.3, (2009), pp. 795–901.

[11] P. Deligne, *Les constants des équations fonctionnelles des fonctions L*, Lecture Notes in Math., **349** (1973) pp. 501–597.

[12] P. Deligne, *Les constantes locales de l'équation fonctionnelle de la fonction L d'Artin d'une reprsentation orthogonale*, Invent. Math., **35** (1976) pp. 299–316.

[13] ______, *Application de la formule des traces aux sommes trigonométriques*, in Cohomologie Étale (SGA4 1/2), Lecture Notes in Math. **569**, Springer-Verlag, Berlin (1977) pp. 168–232.

[14] E. Frenkel and B. Gross *A rigid irregular connection on the projective line*, Annals of Math., **107** (2009), pp. 1469–1512.

[15] B. Gross, *On the motive of a reductive group*, Invent. Math., **130** (1997), pp. 287-313.

[16] ______, *On the motive of G and the principal homomorphism $SL_2 \to \hat{G}$*, J. Asian. Math. Soc., **351** (1997), pp. 208-213.

[17] ______, *Irreducible cuspidal representations with prescribed local behavior*, Amer. J. Math., to appear. Preprint available at http://www.math.harvard.edu/~gross/preprints.

[18] B. Gross and W.T. Gan, *Haar measure and the Artin conductor*, Trans. Amer. Math. Soc., **351** (1999), pp. 1691-1704.

[19] B. Gross and D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}*, Canad. J. Math., **44** (1992), pp. 974-1002.

[20] ———, *On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$, Canad. J. Math., **46** (1994), no. 5, pp. 930-950.

[21] B. Gross and M. Reeder, *From Laplace to Langlands via representations of orthogonal groups*, Bull. Amer. Math. Soc., **43** (2006), pp. 163-205.

[22] B. Gross and N. Wallach, *Restriction of small discrete series representations to symmetric subgroups*, Proc. Symp. Pure Math., **68** (2000), pp. 255–272.

[23] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties* , Annals of Mathematics Studies, **151**. Princeton, NJ, (2001).

[24] Harish-Chandra, *Harmonic Analysis on Reductive p-adic Groups*, Springer Lecture Notes, **162** (1970).

[25] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique.*, Invent. Math., **139** (2000), pp. 439–455.

[26] K. Hiraga, A. Ichino, T. Ikeda, *Formal degrees and adjoint gamma factors (and errata)*, J. Amer. Math. Soc., **21** (2008), no. 1, pp. 283–304.

[27] K. Hiraga, H. Saito, *On L-packets for inner forms of SL_n*, preprint, 2008.

[28] J.W. Jones, D.P. Roberts, *Database of Local Fields*, http://math.asu.edu/~jj/localfields.

[29] V. Kac *Infinite dimensional Lie algebras*, 3rd ed., Cambridge, 1995.

[30] N. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, Ann. of Math Studies, **116** Princeton Univ. Press, Princeton, N.J. 1988.

[31] J.-L. Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc., **20** (2007), no. 2, pp. 273–320.

[32] A. Knapp, *Representation theory of semisimple groups: An overview based on examples*, Princeton Math. Series **36** Princeton Univ. Press, Princeton, N.J. 2001.

[33] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math., **81** (1959), pp. 973–1032.

[34] B. Kostant, S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math., **93** (1971), pp. 753–809.

[35] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J., **51** (1984), pp. 611–650.

[36] P. Kutzko, *Mackey's Theorem for Nonunitary Representations*, Proc. Amer. Math. Soc., **64**, no. 1, (1977), pp. 173–175.

[37] R.P. Langlands, *On the classification of Irreducible Representations of Real Algebraic Groups*, Algebraic Groups, A.M.S. Math. Surveys and Monographs 31 (1988).

[38] R. Langlands, *Representations of abelian algebraic groups*, Pacific J. Math., (1997), pp. 231–250.

[39] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, Proc. Symp. Pure Math., **47**, Amer. Math. Soc., Providence, RI (1987), pp. 235–262.

[40] D. Prasad, *On the self-dual representations of a p-adic group*, Int. Math. Res. Not., (1999), pp.443-452.

[41] M. Reeder, *Formal degrees and L-packets of unipotent discrete series of exceptional p-adic groups*, J.Reine Angew. Math., **520** (2000), pp.37-93.

[42] ______, *Supercuspidal L-packets of positive depth and twisted Coxeter elements*, J.Reine Angew. Math., **620** (2008), pp.1-33.

[43] ______, *Torsion automorphisms of simple Lie algebras*, L'Enseignement Math., to appear.

[44] D. E. Rohrlich, *Elliptic Curves and the Weil-Deligne Group*, C.R.M. Proc. Lecture Notes, **4**, Amer. Math. Soc., Providence, RI, 1994. pp. 125–157.

[45] J.-P. Serre, *Conducteurs d'Artin des caracteres reels*, Invent. Math., **14** (1971), pp. 173–183.

[46] ______, *Cohomologie des groupes discrets*, Ann. of Math. Studies, **70** (1971), pp. 77-169.

[47] ______, *Modular forms of weight one and Galois representations*, Algebraic Number Fields, A. Fröhlich ed., Acad. Press (1977), pp. 193–268.

[48] ______, *Local fields*, Springer-Verlag, 1979.

[49] ______, *Exemples de plongements des groupes $\mathbf{PSL}_2(\mathbf{F}_p)$ dans des groupes de Lie simples*, Invent. Math., **124** (1996), pp. 525–562.

[50] ______, *Galois cohomology*, Springer-Verlag (2002).

[51] ______, *Complète Réductibilité*, Sem. Bourbaki, **932** (2004).

[52] N. Spaltenstein, *Classes unipotentes et sous groups de Borel*, Lecture Notes in Math., **946**, Springer Verlag (1982).

[53] T.A. Springer, *Regular elements in finite reflection groups*, Inv. Math., **25** (1974), pp. 159–198.

[54] ______, *Reductive groups*, Proc. Symp. Pure Math., **33**, Part 1, Amer. Math. Soc., Providence, RI (1979), pp. 3–27.

[55] ______, *A construction of representations of Weyl groups*, Invent. Math., **44** (1978), pp. 279–293.

[56] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., **80** (1968).

[57] ———, *Torsion in reductive groups*, Advances in Math., **15** (1975), pp. 63–92.

[58] J. Tate, *Number Theoretic Background*, Proc. Symp. Pure Math., **33**, Part 2, Amer. Math. Soc., Providence, RI (1979), pp. 2–26.

[59] J. G. Thompson, *A conjugacy theorem for E_8*, J. Algebra **15** (1976), no. 2, pp. 525–530.

[60] J. Tits, *Reductive groups over p-adic fields*, Proc. Symp. Pure Math., **33**, Part 1, Amer. Math. Soc., Providence, RI (1979), pp. 29–69.

[61] D. Vogan, *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., **145**, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.

[62] J.-K. Yu, *Construction of tame supercuspidal representations*, J. of the Amer. Math. Soc., **14** (2001), no. 3, pp. 579–622.

[63] ——— *On the motive and cohomology of a reductive group*, preprint, 2006

[64] A. Weil, *Exercices dyadiques*, Inventiones math., **27** (1974), pp. 1–22.