Math 55b: Honors Real and Complex Analysis

Every differentiable function of a complex variable is analytic (outline)

Let R be a rectangle with sides parallel to the real and imaginary axes, i.e. of the form $R = \{x + iy : a \le x \le b,\ c \le y \le d\}$ for some a, b, c, d with $a < b$ and $c < d$. For a continuous function $f : R \to \mathbf{C}$, we define $\oint_{\partial R} f(z)\,dz$ by

$$\oint_{\partial R} f(z)\,dz = \int_a^b f(x+ic)\,dx + i\int_c^d f(b+iy)\,dy - \int_a^b f(x+id)\,dx - i\int_c^d f(a+iy)\,dy.$$

(Each of the four terms, one for each side γ of R, is what we'll call the "contour integral" $\int_\gamma f(z)\,dz$ with γ oriented in the counterclockwise direction around the boundary ∂R of R; the combination $\oint$ is the contour integral around the boundary ∂R of R.)

1. Suppose F is a complex-valued function on a neighborhood[1] of R that is differentiable *as a function of a complex variable*, with derivative F'. Then $\oint_{\partial R} F'(z)\,dz = 0$. [The terms corresponding to the integrals around the four sides are $F(b+ic) - F(a+ic)$, $F(b+id) - F(b-ic)$, $F(a+id) - F(b+id)$, and $F(a+ic) - F(a+id)$, and these sum to zero. NB the examples of $f(x) = \mathrm{Re}(z)$ and $f(x) = \mathrm{Im}(z)$ show that $\oint_{\partial R} f(z)\,dz$ isn't always zero; for those choices, $\oint_{\partial R} f(z)\,dz$ is a nonzero multiple of the area $(b-a)(d-c)$ of R.]

2. Divide R into rectangles R_1, R_2 by choosing some $x_1 \in (a, b)$ and setting

$$R_1 = \{x + iy : a \le x \le x_1,\ c \le y \le d\}, \quad R_2 = \{x + iy : x_1 \le x \le b,\ c \le y \le d\}.$$

Then $\oint_{\partial R} f(z)\,dz = \oint_{\partial R_1} f(z)\,dz + \oint_{\partial R_2} f(z)\,dz$. Likewise for $y_1 \in (c, d)$. It follows by induction that for any partitions $a = x_0 < x_1 < \ldots < x_M = b$ and $c = y_0 < y_1 < \ldots < y_N = d$ we can write $\oint_{\partial R} f(z)\,dz = \sum_{j=1}^M \sum_{k=1}^N \oint_{\partial R_{jk}} f(z)\,dz$ where R is the rectangle of $x + iy$ with $x \in [x_{j-1}, x_j]$ and $y \in [y_{j-1}, y_j]$. (Note that this is consistent with the formulas from the previous problem: certainly $0 = \sum_{j=1}^M \sum_{k=1}^N 0$, and also the area of R equals the sum of the areas of the R_{jk}.

3. [Goursat] Suppose f is a complex-valued function on a neighborhood of R that is differentiable *as a function of a complex variable*. Then we prove that $\oint_R f(z)\,dz = 0$ as follows.[2] Assume the integral is nonzero, and let C be its absolute value, with $C > 0$. Repeatedly applying the result of the previous problem, we obtain a sequence of rectangles R_n (with $R_0 = R$), with each R_n ($n > 0$) being one quarter of R_{n-1} and satisfying $\left|\oint_{\partial R_n} f(z)\,dz\right| \ge C/4^n$. Then there exists some $z^* \in R$ contained in each R_n. Since f is differentiable at z^* we have $f(z) = f(z^*) + f'(z^*)(z^* - z) + o(|z^* - z|)$ as $z^* \to z$. But the contour integral over ∂R of the error $o(|z^* - z|)$ is $o(1/4^n)$, because $|z^* - z| = O(1/2^n)$ uniformly on R and each of the edges of R has length $O(1/2^n)$. Moreover, $\oint_{R_n} f(z^*)\,dz = \oint_{R_n} f(z^*)f'(z^*)(z^* - z)\,dz = 0$, because each of $f(z^*)$ (as a constant function of z) and $f'(z^*)(z^* - z)$ is F' for some $F : \mathbf{C} \to \mathbf{C}$ with a complex derivative, namely $F(z) = f(z^*)z$ and $F(z) = f'(z^*)(z^* - z)^2/2$ respectively. Thus $\oint_{\partial R^n} f(z)\,dz = o(1/4^n)$, contradiction.

[1] For any subset S of a metric space (or even a general topological space) X, a "neighborhood of S" is an open set N of X such that $N \supseteq S$. See the footnote to problem 4 below.

[2] Note that this proof manages to avoid any continuity hypothesis on f', so you cannot obtain the same result by appealing to Green's theorem even if you already know that approach.

Remarks: i) An important use of contour integration is the evaluation of definite integrals over real intervals. We can already give an example: having shown $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, we can compute $\int_{-\infty}^{\infty} e^{-x^2+2icx}\,dx$ for any $c > 0$ by applying $\oint_{R_n} f(z)\,dz = 0$ to $f(z) = \exp(-z^2)$ and $R = \{x + iy : -M \le x \le M,\ 0 \le y \le c$, and letting $M \to \infty$. The vertical contributions approach zero, and we're left with

$$\int_{-\infty}^{\infty} e^{-(x+ic)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

whence

$$\int_{-\infty}^{\infty} e^{-x^2+2icx}\,dx = \sqrt{\pi}\,e^{-c^2}.$$

Averaging x and $-x$ yields the equivalent form $\int_{-\infty}^{\infty} e^{-x^2}\cos cx\,dx = \sqrt{\pi}\,e^{-c^2}$, and thus $\int_0^{\infty} e^{-x^2}\cos cx\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-c^2}$ because the integrand is even.

ii) In general, if γ is a line segment from z_1 to z_2 then $\int_\gamma f(z)\,dz$ can be defined as the definite integral $\int_0^1 f(z_1 + (z_2 - z_1)t)\,dt$. This lets us define contour integrals over arbitrary polygonal paths, as the sum of the integrals over their component line segments. If F is a function from a neighborhood of γ to $\mathbf{C}$ with a complex derivative F', then the formula $\int_\gamma F'(z)\,dz = F(z_2) - F(z_1)$ still holds (again by the Fundamental Theorem of Calculus). Thus the integral of $F'(z)\,dz$ over a closed polygonal contour vanishes. If Δ is a triangle and f is a function from a neighborhood of Δ to $\mathbf{C}$, then we can tile Δ by four half-size copies $\Delta_1, \ldots, \Delta_4$ of Δ (one in the opposite orientation), and check that $\oint_{\partial\Delta} f(z)\,dz = \sum_{j=1}^4 \oint_{\partial\Delta_j} f(z)\,dz$ If f is differerentiable as a function of a complex variable, then the Goursat trick applies and we deduce that $\oint_\Delta f(z)\,dz = 0$. We can then do the same with Δ replaced by any simple polygon, by tiling the polygon with finitely many triangles.

4. With f as in the previous problem, we can now construct an antiderivative $F : R \to \mathbf{C}$, defined by

$$F(u+iv) = \int_a^u f(x+ic)\,dx + i\int_c^v f(u+iy)\,dy = \int_a^u f(x+iv)\,dx + i\int_c^v f(a+iy)\,dy.$$

(these expressions are equal because they differ by $\oint_{R_1} f(z)\,dz$ for some rectangle $R_1 \subseteq R$, and we already know that such an integral is zero). Better yet, since f is defined and complex-differentiable on some neighborhood N of R, we can find[3] a rectangle $R' \subset N$ whose interior contains R, and make the same definition on R', so that it makes sense to differentiate F also on the boundary of R. Now if $a_1 + ib, a_2 + ib \in R'$ with $a_1 < a_2$ then $F(a_2 + ib) - F(a_1 + ib) = \int_{a_1}^{a_2} f(x + ib)\,dx$, and likewise if $a + ib_1, a + ib_2 \in R'$ with $b_1 < b_2$ then $F(a + ib_2) - F(a + ib_1) = i\int_{b_1}^{b_2} f(a + iy)\,dy$. Since f is continuous, we can find for all $z \in R$ and $\epsilon > 0$ some $\delta > 0$ such that $N_\delta(z) \subset R'$ and $|f(w) - f(z)| < \epsilon$ for all $w \in N_\delta(z)$. It soon follows that $|F(w) - F(z) - (w - z)f'(z)| < 2\epsilon|w - z|$ for all $w \in N_\delta(z)$ (note that all the

[3] For each z on the boundary of R, find an open square centered on z and contained in N. These form an open cover of the compact set R, so there is a finite subcover. Add to this subcover the squares centered at the corners of R, if they are not in it already. Choose $\epsilon > 0$ such that each of these squares has side $> 2\epsilon$. Then R' can be $\{x + iy : a - \epsilon \le x \le b + \epsilon,\ c - \epsilon \le y \le d + \epsilon\}$.

More generally: let S be any compact subset of an open set N in any metric space X. Then there exists $\epsilon > 0$ such that N contains the "ϵ-neighborhood" $N_\epsilon(S) := \cup_{x\in S} B_\epsilon(x)$ of S. Proof: if not, find $x_n \in S$ and $y_n \notin N$ such that $d(x_n, y_n) \to 0$; extract a convergent subsequence $\{x_{n_i}\} \to x \in S \subseteq N$, and then $y_{n_i} \to x$, contradiction. (For our application $S = R$ and it is convenient to use the sup metric on $\mathbf{C} \cong \mathbf{R}^2$.)

horizontal and vertical contours we need are contained in $N_\delta(z)$). Because ϵ was arbitrary we conclude that $F'(z) = f(z)$ as claimed.

(Once we define $\int_\gamma f(z)\,dz$ for general contours γ we can use this result to show that such an integral vanishes over any closed contour contained in R.)

5. We next use the complex exponential function to deduce results on integrals over circular contours from our results on contour integrals over rectangular contours.

i) Suppose $0 < r_0 < r$, and let f be a complex-valued function on a neighborhood N of the annulus $\{z \in \mathbf{C} : r_0 \le |z| \le r\}$. Assume again that f is differentiable on N as a function of a complex variable. Then

$$\int_0^{2\pi} f(r_0 e^{i\theta})\,d\theta = \int_0^{2\pi} f(re^{i\theta})\,d\theta.$$

Indeed let R be the rectangle with $[a, b] = [\log r_0, \log r]$ and $[c, d] = [0, 2\pi]$, and consider the function $g(z) = f(e^z)$ on a neighborhood of R whose image under $z \mapsto e^z$ is contained in N. This function is differentiable as a function of a complex variable, because it is the product of e^{-z} and $f(e^z)$, each of which is differentiable by the complex chain rule. Hence $\oint_{\partial R} g(z)\,dz = 0$. But the horizontal sides of R both map to the interval $[r_0, r] \in \mathbf{R}$, and their contributions $\pm \int_{r_0}^r f(x)\,dx/x$ to $\oint_{\partial R} g(z)\,dz$ cancel out. The vertical sides contribute $i\big(\int_0^{2\pi} f(re^{i\theta})\,d\theta - \int_0^{2\pi} f(r_0)e^{i\theta}\,d\theta\big)$. The claimed equality follows.

ii) If furthermore N contains the full disc $\{z \in \mathbf{C} : |z| \le r\}$, then we can let $r_0 \to 0$ in $\int_0^{2\pi} f(r_0 e^{i\theta})\,d\theta = \int_0^{2\pi} f(re^{i\theta})\,d\theta$ and deduce that

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta.$$

(Note that we never use the existence of $f'(z)$ at $z = 0$ itself, only the continuity of f at $z = 0$. We shall soon see that even if f is merely bounded on $0 < |z| < r$ then we can define $f(0)$ by setting $f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta$ to obtain a function that's not just continuous but even complex-analytic (that is, equal to a convergent power series) at $z = 0$.

Using $[c, d] = [0, \pi]$ instead of $[0, 2\pi]$, we find instead that

$$\int_{-r}^{-r_0} f(z)\,\frac{dz}{z} + \int_{r_0}^{r} f(z)\,\frac{dz}{z} = i\Big(\int_0^{\pi} f(r_0 e^{i\theta})\,d\theta - \int_0^{\pi} f(re^{i\theta})\,d\theta\Big).$$

Now let $f(z) = e^{icz}$ for some $c > 0$, and let $r_0 \to 0$ and $r \to \infty$. Then the left-hand side is $\int_{r_0}^r 2i \sin cz \frac{dz}{z} \to \int_0^\infty 2i \sin cz \frac{dz}{z}$. In the right-hand side, the first integral approaches π, and the second goes to zero (because e^{icz} is small except for θ near 0 or π). Therefore $\int_0^\infty 2i \sin cz \frac{dz}{z} = i\pi$, so we have obtained the famous integral formula

$$\int_0^\infty \frac{\sin cz}{z}\,dz = \frac{\pi}{2} \quad (c > 0).$$

(It is easy to see that the integral does not depend on the choice of c, but the fact that it equals π is far from obvious.)

6. A nice property of circular discs $D \subset \mathbf{C}$ is that they support non-obvious bijections w that are complex-differentiable on a neighborhood of D and *do not* preserve the center. We have just proved a formula that, for an arbitrary complex-differentiable function f, expresses its value at the center of D as the average of the boundary

values. We next construct w and use it to generalize our formula from the centeral value of f to its value at any interior point of D.

For simplicity we work with the unit disc (so take $r = 1$ in the previous problem), and choose real z_0 with $|z_0| < 1$. (Once we have the result for z_0 real, the general case will follow by applying the formula to the function $f(cz)$ for suitable $c \in \mathbf{C}^*$ with $|c| = 1$.) Then $|z| = 1$ implies $z\bar{z} = 1$, and thus also

$$|z + z_0| = |\bar{z} + z_0| = |z^{-1} + z_0| = |z|^{-1}|1 + z_0 z| = |1 + z_0 z|,$$

whence $\big|(z + z_0)/(1 + z_0 z)\big| = 1$. Define

$$w(z) = \frac{z + z_0}{1 + z_0 z}$$

for $z \in \mathbf{C}$ such that $z \neq -z_0^{-1}$. Then we have just shown $|z| = 1 \Longrightarrow |w(z)| = 1$. We claim that also $|z| < 1 \Longleftrightarrow |w(z)| < 1$. One way to see this is to compute

$$|1 + z_0 z|^2 = 1 + |z z_0|^2 + 2\,\mathrm{Re}(z z_0), \quad |z + z_0|^2 = |z|^2 + z_0^2 + 2\,\mathrm{Re}(z z_0),$$

and thus

$$|1 + z_0 z|^2 - |z + z_0|^2 = (1 + |z z_0|^2) - (|z|^2 + z_0^2) = (1 - z_0^2)(1 - |z|^2),$$

which has the same sign as $1 - |z|^2$; thus if $|z| < 1$ then $|z + z_0|^2 < |1 + z_0 z|^2$, so $|w(z)|^2 < 1$ and $|w(z)| < 1$, while if $|z| > 1$ then likewise $|w(z)|^2 > 1$.

It follows that w is a bijection on the unit circle $|z| = 1$, and a bijection on the closed unit disc $D = \{z \in \mathbf{C} : |z| \leq 1\}$ that sends 0 to z_0. Moreover, w is differentiable as a function of a complex variable; so if f is a complex-valued function with a complex derivative on a neighborhood of D, then the same is true of $f \circ w$. But $(f \circ w)(0) = f(w(z_0)) = f(z_0)$. Since $f \circ w$ has a complex derivative, we know that $(f \circ w)(0)$ is the average of the values of $(f \circ w)$ on $|z| = 1$. It follows that $f(z_0)$ can also be given by such a formula, though it will be a weighted average: $f(z_0) = (2\pi)^{-1} \int_0^{2\pi} c(\theta) f(e^{i\theta})\, d\theta$, for some function c depending on z_0. We next outline the computation of this function c.

7. Before setting out on the calculation, note that we have a very strong hint and sanity check on the result: it must work for power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that converge on $|z| < R$ for some $R > 1$. Indeed we know that

$$(*) \qquad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\, d\theta$$

for such f, and that the integral vanishes for $n < 0$. (Note that the case $n = 0$ recovers the formula for $f(0)$ that we've already proved.) Thus

$$(**) \qquad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} e^{-in\theta} z_0^n \right) f(e^{i\theta})\, d\theta$$

(the exchange of sum and integral is easily justified here), and the sum over n is a geometric series, equal to $1/(1 - z_0 e^{-i\theta})$. This can't quite be right, not even for $z_0 = 0$, because it's not real-valued; but we can fix it by adding $\sum_{n=1}^{\infty} e^{+in\theta} z_0^n$ (which is the complex conjugate minus 1), because $f(z) \sum_{n=1}^{\infty} z^n z_0^n$ is a differentiable function on $|z| < R$ that vanishes at $z = 0$, so its average over $|z| = 1$ vanishes. This gives

$$\frac{1}{1 - z_0 e^{-i\theta}} + \frac{1}{1 - z_0 e^{i\theta}} - 1 = \frac{1 - z_0^2}{1 + z_0^2 - 2z_0 \cos\theta},$$

which is indeed the formula we shall obtain for the multiplier of $(2\pi)^{-1} f(e^{i\theta})\, d\theta$.

Now we have

$$f(z_0) = f(w(0)) = (f \circ w)(0) = \frac{1}{2\pi} \int_0^{2\pi} (f \circ w)(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(w(e^{i\theta}))\, d\theta,$$

and we saw that we can write $w(e^{i\theta}) = e^{i\psi}$ for some real ψ, so

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\psi}) \frac{d\psi}{\psi'(\theta)}$$

if we regard ψ as a function of θ. By the chain rule (which we know applies also to complex analytic functions),

$$ie^{i\psi(\theta)}\psi'(\theta) = ie^{i\theta} w'(e^{i\theta}),$$

and $w'(z) = (1 - z_0^2)/(1 + z_0 z)^2$ (in general the derivative of $(az + b)/(cz + d)$ is $(ad - bc)/(cz + d)^2$). We write everything in terms of $w = e^{i\psi}$, including $z = e^{i\theta} = (w - z_0)/(1 - z_0 w)$. This gives

$$\psi'(\theta) = \frac{w/z}{w'(z)} = \frac{w(1 + z_0 z)^2}{(1 - z_0^2)z},$$

which can be written as $(1 - z_0^2)/\big((1 - wz_0)(1 - w^{-1}z_0)\big)$. The denominator expands to $1 + z_0^2 - (w + w^{-1}z_0) = 1 + z_0^2 - 2z_0 \cos\psi$, which matches our prediction once we change the variable name from ψ to θ:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - z_0^2}{1 + z_0^2 - 2z_0 \cos\theta}\, d\theta.$$

To get from this "Poisson integral formula" to (**), we subtract

$$0 = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \sum_{n=1}^{\infty} e^{in\theta} z_0^n\, d\theta$$

and finally obtain the power series $f(z_0) = \sum_{n=0}^{\infty} a_n z_0^n$ with a_n given by the integral (*).

8. Some additional properties of analytic functions soon follow. For example, an analytic function f on some disc $B_r(z_0)$ cannot have a sequence of zeros z_k (solutions of $f(z) = 0$) such that $z_k \to z_0$, unless f is the zero function. Indeed if f is not the zero function then it has a power series $\sum_{n=n_0}^{\infty} c_n(z - z_0)^n = (z - z_0)^{n_0} f_1(z)$ with $c_{n_0} \neq 0$ and $f_1(z) = \sum_{n=0}^{\infty} c_{n_0+n}(z - z_0)^n$; since f_1 is itself an analytic function on $B_r(z_0)$, it is *a fortiori* continuous, and then $f_1(z_0) = c_{n_0} \neq 0$ implies that f_1 is nonzero in some neighborhood of z_0, whence $f(z) \neq 0$ for all $z \neq z_0$ in that neighborhood. This means that if f, g are analytic functions on $B_r(z_0)$, and $\{z_k\}$ is a sequence in $B_r(z_0)$ such that $z_k \to z_0$, then $f(z_k) = g(z_k)$ for all k (or even for infinitely many k) implies $f(z) = g(z)$ for all $z \in B_r(z_0)$. (Proof: consider the analytic function $f - g$.) This soon gives the "reflection principle(s)" for analytic functions and the process of "analytic continuation".