Math 55b: Honors Real and Complex Analysis

Homework Assignment #9 (5 April 2017):
Multivariate differentiation cont'd; integration in $\mathbf{R}^k$

Similarly, *adv.*: At least one line of the proof of this case is the same as before.[1]

Multivariate differentiation of functions coming from linear algebra:

1. Let E be the open subset of the $(n+1)$-dimensional real vector space $\mathcal{P}_n$ consisting of the polynomials of degree n, i.e. $E = \{\sum_{j=0}^{n} a_j T^j : a_n \neq 0\}$. Fix $P_0 \in E$ and a real root t_0 of P_0. Give necessary and sufficient conditions on P_0, t_0 for there to exist a $\mathscr{C}^1$ real-valued function t on a neighborhood of P_0 such that $t(P_0) = t_0$ and $t(P)$ is a root of P for each P in the neighborhood. What is the derivative $t'(P_0)$?

2. Let E be the set $\mathrm{GL}_n(\mathbf{R})$ of invertible matrices in the n^2-dimensional vector space $\mathcal{M}_n$ of $n \times n$ real matrices. Then E is open (why?). Let $f : E \to \mathcal{M}_n$ be the map taking any matrix $A \in E$ to A^{-1}. Equivalently, f takes A to the solution of $AX = I$. Use the Implicit Function Theorem to show that f is differentiable, and compute its derivative, i.e. give a formula for $f'(A)B$ for any $A \in E$ and $B \in \mathcal{M}_n$. Check that your formula is consistent with the identities $f(TA) = f(A)T^{-1}$, $f(AT) = T^{-1}f(A)$ for all $T \in \mathrm{GL}_n(\mathbf{R})$. [NB the maps $A \mapsto TA$ and $A \mapsto AT$ are linear.]

3. Recall that we proved the Implicit Function Theorem via the Inverse Function Theorem, and thus by constructing $f(A)$ for A near A_0 as the fixed point of some contraction mapping ϕ_A on a neighborhood of A_0. Having constructed $f(A) = A^{-1}$ this way as the solution of $AX = I$, take $A_0 = I$, determine ϕ_A, and check directly that its iterates converge to A^{-1} in some neighborhood of I.

4. Let M_0 be an $n \times n$ matrix, and $\lambda_0 \in \mathbf{R}$ an eigenvalue of M_0 that is a simple root of the characteristic polynomial of M_0. Let v_0 be a λ_0-eigenvector of M_0. Regard M_0 as an element of the n^2-dimensional vector space $\mathcal{M}_n$ of $n \times n$ matrices. Prove that there is a neighborhood U of M_0 in $\mathcal{M}_n$ and $\mathscr{C}^1$ functions $\lambda : U \to \mathbf{R}$ and $v : U \to \mathbf{R}^n$ such that $\lambda(M_0) = \lambda_0$, $v(M_0) = v_0$, and $\forall M \in U : \lambda(M)$ is an eigenvalue of M with eigenvector $v(M)$. What is the derivative $\lambda'(M_0)$? What can you say about $v'(M_0)$? (This is the basis for "perturbation methods" for approximating eigenvalues of operators near M_0, commonly used in quantum mechanics and other applications.)

Multivariate integration basics:

5. [Rudin p.288 #1] Let H be a compact convex set in $\mathbf{R}^k$ with nonempty interior, and $B = \prod_{i=1}^{k} [a_i, b_i]$ any box containing H. Suppose $f : H \to \mathbf{R}$ is continuous, and extend f to a

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. See Problem 6. Note that this does not define an equivalence relation.

(possibly discontinuous) function on B by setting $f(x) = 0$ if $x \notin H$. Define $\int_H f = \int_B f$, with the latter integral defined as we did for a continuous function on B. Show that the result is independent of the order of integration (and on the choice of B). [Hint: approximate f by a sequence of continuous functions supported on H, as we did for the integral over a parallelogram.]

6. [Rudin p.288 #2, extended] For $i = 1, 2, 3, \ldots$, let ϕ be a continuous function on $\mathbf{R}$ supported[2] on $(1/2^i, 1/2^{i-1})$, such that $\int_0^1 \phi_i(x)\,dx = 1$. For $(x, y) \in \mathbf{R}^2$ define

$$f(x,y) = \sum_{i=1}^{\infty} [\phi(x) - \phi_{i+1}(x)]\phi_i(y).$$

Show that $f : \mathbf{R}^2 \to \mathbf{R}$ has compact support and is continuous except at $(0, 0)$, and moreover that f must be unbounded in every neighborhood of $(0, 0)$; and that $\int_0^1 \left(\int_0^1 f(x,y)\,dx\right)\,dy = 0$ but $\int_0^1 \left(\int_0^1 f(x,y)\,dy\right)\,dx = 1$. Likewise, construct a *bounded* function $f : [0,1] \times [0,1] \to \mathbf{R}$ such that: for each x, the function $y \mapsto f(x,y)$ is Riemann integrable with $\int_0^1 f(x,y)\,dy = 0$ (from which it follows that $\int_0^1 [\int_0^1 f(x,y)\,dy]dx = 0$); but there exist y such that the function $y \mapsto f(x,y)$ is not Riemann integrable (whence $\int_0^1 [\int_0^1 f(x,y)\,dx]dy$ doesn't even make sense).

7. Let $Q : \mathbf{R}^n \to \mathbf{R}$ be a positive-definite form. Show that the integral of $e^{-Q(x)}$ over $x \in \mathbf{R}^n$ converges, and evaluate this integral. For $y \in \mathbf{R}^n$ determine the integral of $\exp(-Q(x) + i\langle x, y\rangle)$ over $x \in \mathbf{R}^n$, where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product.

8. Let n be a positive integer, V the vector space of symmetric $n \times n$ matrices, and $E \subset V$ the set of positive-definite matrices. Prove that the function $A \mapsto 1/\det(A)$ on E is logarithmically convex. [Use the first part of Problem 7. I learned this proof from Don Zagier; I do not know its original source.]

This problem set due Friday, April 14, at the beginning of class.

[2] Recall that "f is supported on S" means "$f(x) = 0$ if $x \notin S$."