Math 55b: Honors Real and Complex Analysis

Homework Assignment #7 (20 March 2017):
Univariate integral calculus

Bitte ve[r]giß alles, was Du auf der Schule gelernt hast;
denn Du hast is nicht gelernt. *Emil Landau*[1]

Admittedly that's a bit extreme, but it is true that for many of you Stieltjes integrals, especially of vector-valued functions, are a new path in the familiar territory of integration, and might require a different kind of thinking. This problem set consists of only eight problems including four from Rudin, but most are geared towards developing such "new kinds of thinking", and a few are somewhat open-ended to suggest further directions in analysis that we won't pursue in Math 55.

The first few problems are from Rudin pages 138–139:[2]

1. [Rudin #3] Define functions $\beta_j : \mathbf{R} \to \mathbf{R}$ $(i = 1, 2, 3)$ as follows: for each j, set $\beta_j(x) = 0$ for $x < 0$ and $\beta_j(x) = 1$ for $x > 0$; but $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = 1/2$. Let $f : [-1, 1] \to \mathbf{R}$ be any bounded function.
   a) Prove that $f \in \mathscr{R}(\beta_1)$ $\underline{\text{iff}}$ $f(0) = \lim_{x\to 0+} f(x)$, and then $\int_{-1}^{1} f\, d\beta_1 = f(0)$;
   b) State and prove a similar result for $\mathscr{R}(\beta_2)$;
   c) Prove that $f \in \mathscr{R}(\beta_3)$ $\underline{\text{iff}}$ f is continuous at 0, in which case $\int_{-1}^{1} f\, d\beta_j = f(0)$ for each $j = 1, 2, 3$.

2. [Rudin #8; "integral test" for convergence of a positive series $\sum_{n>n_0} f(n)$]
   Let $\alpha : [a, \infty) \to \mathbf{R}$ be any increasing function. Suppose $f : [a, \infty)$ is in $\mathscr{R}(\alpha)$ on $[a, b]$ for each $b > a$. The "improper Riemann-Stieltjes integral" $\int_a^\infty f(x)\, d\alpha(x)$ is then defined as $\lim_{b\to\infty} \int_a^b f(x)\, d\alpha(x)$ if the limit exists [and is finite]. In that case we say the integral *converges*; we say it *converges absolutely* if $\int_a^\infty |f(x)|\, d\alpha(x)$ also converges. Naturally the "improper Riemann integral" is the special case of this where $\alpha(x) = x$ for all x. [Likewise for $\int_{-\infty}^{a}$; and $\int_{-\infty}^{\infty} f\, d\alpha$ converges to $\int_{-\infty}^{0} f\, d\alpha + \int_0^\infty f\, d\alpha$ if both integrals converge.]
   Suppose further that $f(x) \geq 0$ and f is monotone decreasing on $x \geq 1$. Prove that $\int_1^\infty f(x)\, dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

3. [Integration by parts for improper integrals] Show that in some cases integration by parts can be applied to the "improper" integrals defined in the previous problem; that is, state appropriate hypotheses, formulate a theorem, and prove it. Your hypotheses should be applicable in the following special case: the improper integrals $\int_0^\infty \cos(x)\, dx/(x+1)$ and $\int_0^\infty \sin(x)\, dx/(x+1)^2$ converge and are equal. Show that one of these two integrals (which one?) conerges absolutely, but the other does not.

[1] Quote taken from Chapter 10 of M. Artin's *Algebra*. It roughly translates as "Please forget all that you have learned in school, for you haven't [really] learned it." Don't complain about the German transcription, which is presumably of some local dialect — even I recognize that this isn't the the German we *auf der Schule lernen*.

[2] For the first of these, cf. also Rudin #1: Suppose $\alpha : [a, b] \to \mathbf{R}$ is increasing, and continuous at x_0. Define $f : [a, b] \to \mathbf{R}$ by $f(x) = 0$ if $x \neq x_0$ and $f(x_0) = 1$. Then $f \in \mathscr{R}(\alpha)$ [i.e. f is integrable with respect to α], and $\int_a^b f(x)\, d\alpha = 0$.

4. [Bernoulli polynomials][3] Prove that for each positive integer m there exists a polynomial B_m such that $\sum_{i=1}^{n} i^{m-1} = B_m(n)$ for all positive integers n. [Hint: What must the polynomial $B_m(x) - B_m(x-1)$ be? The map taking any polynomial $P(x)$ to the polynomial $Q(x) := P(x) - P(x-1)$ is linear.] Determine the leading coefficient of B_m, and deduce the value of $\int_0^b x^{m-1}\,dx$ for any $b > 0$ (and thus also of $\int_a^b x^{m-1}\,dx$) without using the Fundamental Theorem of Calculus. Beyond the leading term, what further patterns can you detect in the coefficients of B_m? Can you prove any of these patterns? (You may need to go at least to $m = 6$ or $m = 7$ to see what's going on; a computer algebra system could help to handle the linear algebra manipulations.)

5. [Fermat] Prove *using the Riemann-sum definition of the integral* that $\int_a^b x^{r-1}\,dx = (b^r - a^r)/r$ for every nonzero rational number r and all real a, b such that $0 < a < b$. [Note: since Fermat predated Newton, the solution cannot use the Fundamental Theorem of Calculus. Besides the special case that r is a positive integer, addressed in the previous problem, you might also find a solution for the special case that $1/(r-1)$ is a positive integer — but this will not directly lead you to a solution of the general case.]

6. In the `vint` handout on integration of vector-valued functions, you might have expected a theorem to the effect that such a function is integrable (as defined there) with integral I if and only if for each $\epsilon > 0$ there exists a partition P all of whose Riemann sums differ from I by vectors of norm at most ϵ. Certainly the existence of such P is a consequence of integrability, but in fact the converse implication does not hold! Prove this by finding a normed vector space V and a function $f : [0,1] \to V$ such that $\Delta(P) = 1$ for any partition P (and thus $f \notin \mathscr{R}$), but nevertheless for each ϵ there exist partitions P such that every Riemann sum $R(P, \vec{t})$ for $\int_0^1 f(x)\,dx$ has norm at most ϵ. [Hints: f cannot be continuous or even nearly (e.g. piecewise) continuous, because then our vector version of Thm. 6.8 would yield integrability; in fact the function I have in mind is discontinuous everywhere. Moreover, V cannot be finite dimensional. Thus the example is rather pathological — but it is also simple enough that it can be described and proved in a short paragraph.]

Finally, (indefinite) integration of arbitrary rational functions:[4]

7. [Partial fractions] Let k be an algebraically closed field. Let $K = k(x)$, the field of rational functions in one variable x with coefficients in k. Show that the following elements of K constitute a basis for K as a vector space over k: x^n for $n = 0, 1, 2, 3, \ldots$, and $1/(x - x_0)^n$ for $x_0 \in k$ and $n = 1, 2, 3, \ldots$. (Linear independence is easy. To prove that the span is all of K, consider for any polynomial $Q \in k[x]$ the subspace $V_Q := \{P/Q : P \in k[x],\ \deg(P) < \deg(Q)\}$ of K, and compare its dimension with the number of basis vectors in V_Q.)

8. Prove that the integral of any $f \in \mathbf{R}(x)$ is a rational function plus a linear combination of functions of the form $\log|x - x_0|$, $\log((x-x_0)^2 + c)$, and $\tan^{-1}(ax + b)$ $(x_0, a, b, c \in \mathbf{R},\ c > 0)$.

This problem set due Monday, 27 March, at the beginning of class.

[3] Not that it matters for our purposes, but the "Bernoulli polynomials" usually seen in the literature differ from our B_m by an additive constant.

[4] Along the way we again encounter a natural example of a vector space with an uncountable algebraic basis (assuming k is uncountable, e.g. $k = \mathbf{C}$).