Math 55b: Honors Real and Complex Analysis

Homework Assignments #5 and #6 (27 February 2017):
Topology coda; calculus prelude

> Q: What did the mathematician say as ϵ approached zero?
> A: "There goes the neighborhood."
> —*Hoary math joke*

The first part of this problem set comprises several standard problems on and around completeness and compactness.

1. Let d_1, d_2, d_3 be the following three metrics on $\mathbf{R}$:
   d_1 is the standard metric $d_1(x, y) = |x - y|$;
   d_2 is the discrete metric; and
   d_3 is the metric $d_3(x, y) = |x^3 - y^3|$.
   The identity function $x \mapsto x$ on $\mathbf{R}$ then gives rise to six functions $(\mathbf{R}, d_i) \to (\mathbf{R}, d_j)$ with $i \neq j$.
   i) Which of these six functions
   $$(\mathbf{R}, d_1) \rightleftarrows (\mathbf{R}, d_2), \quad (\mathbf{R}, d_2) \rightleftarrows (\mathbf{R}, d_3), \quad (\mathbf{R}, d_3) \rightleftarrows (\mathbf{R}, d_1)$$
   are continuous?
   ii) Of those, which are uniformly continuous?
   iii) For which $i, j \in \{1, 2, 3\}$ does there exist a subset $S \subseteq \mathbf{R}$ that becomes compact in $(\mathbf{R}, d_i)$ but not in $(\mathbf{R}, d_j)$? Give and justify an example of such an S.

2. Prove or disprove: a metric space is separable if and only if every open cover has a countable subcover. [As usual finite sets are regarded as countable.]

3. Prove or disprove: if X, Y, Z are metric spaces and $f : X \to Y$ and $g : Y \to Z$ are any uniformly continuous functions then the composite function $g \circ f : X \to Z$ is uniformly continuous.

4. Let X, Y metric spaces, and X^*, Y^* their completions. Prove that any uniformly continuous $f : X \to Y$ extends[1] uniquely to a continuous function $f^* : X^* \to Y^*$, and that f^* is still uniformly continuous. Show that if f is continuous, but not uniformly so, then there might not be a continuous f^* that extends f.

Problem 3 is the main thing to check if we want a "category of metric spaces with uniformly continuous functions"; if there is such a category then Problem 4 is the key step in constructing a "completion functor" to its subcategory of complete metric spaces.

5. Let Y be a metric space, X an arbitrary set, and $\{f_n\}$ a sequence of functions from X to Y. We saw that if the f_n are bounded then f_n approaches a function $f : X \to Y$ in the $\mathcal{B}(X, Y)$ metric if and only if $f_n \to f$ uniformly. What should it mean for a sequence $\{f_n\}$ to be "uniformly Cauchy"? Prove that if Y is complete and X is a topological space then a uniformly Cauchy sequence of continuous functions from X to Y converges uniformly to a continuous function.

[1] A function f^* on a set S^* "extends" a function f on a subset $S \subseteq S^*$ if $f^*(s) = f(s)$ for all $s \in S$. We also say (as in the footnoted word of this problem) that f "extends to f^*".

6. In the previous problem set we defined a metric

$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx$$

on the space $\mathcal{C}([0,1], \mathbf{C})$.

i) Show that $\mathcal{C}([0,1], \mathbf{C})$ is <u>not</u> complete under this metric.

ii) Fix $x \in [0,1]$. Is the map $f \mapsto f(x)$ from $\mathcal{C}([0,1], \mathbf{C})$ to $\mathbf{C}$ continuous with respect to the d_1 metric?

iii) Now fix a continuous function $m : [0,1] \to \mathbf{C}$, and define a map $I_m : \mathcal{C}([0,1], \mathbf{C}) \to \mathbf{C}$ by

$$I_m(f) := \int_0^1 f(x)\, m(x)\, dx.$$

Prove that this map is uniformly continuous.

[By problem 4, the map I_m extends to a uniformly continuous map on the completion $L_1([0,1])$ of $\mathcal{C}([0,1], \mathbf{C})$ under the d_1 metric. This map is also linear: it satisfies the identity $I_m(af + bg) = aI_m(f) + bI_m(g)$ for all $f, g \in L_1([0,1])$ and $a, b \in \mathbf{C}$. Are there any linear maps from $L_1([0,1])$ to $\mathbf{C}$ not of that form?]

7. Let X be a metric space, and f a function from X to itself such that

$$d(x,y) > d(f(x), f(y))$$

for all $x, y \in X$ such that $x \neq y$. [NB this is weaker than the notion of a "contraction map", which is a function $f : X \to X$ such that there exists $\theta < 1$ with $d(f(x), f(y)) \leq \theta \cdot d(x,y)$ for all $x, y \in X$.] Let $g : X \to \mathbf{R}$ be the real-valued function on X defined by

$$g(x) := d(x, f(x)).$$

i) Prove that f is uniformly continuous, and has at most one fixed point (that is, there is at most one $x_0 \in X$ such that $x_0 = f(x_0)$).

ii) Prove that g is continuous.

iii) Conclude that if X is compact then f has a fixed point. Must this still be true if X is complete but not necessarily compact?

Problems 1–7 are due Monday, March 6, at the beginning of class.

The remaining problems concern "differential algebra"; that is, familiar algebraic axioms of a (usually commutative) ring or field, extended by a map $D : f \mapsto f'$ satisfying the axioms $(f+g)' = f' + g'$ and $(fg)' = fg' + f'g$. Such a map is called a *derivation* of the ring or field. Note that in the case of a field, the formula $(f/g)' = (f'g - fg')/g^2$ holds automatically because the argument we gave in class starting from $f = g(f/g)$ uses only the field and derivation axioms. The topological considerations that arise in the definition of the derivative enter into some of the following problems but are not the main point except for the final problem.

8. i) If $f, g, h : [a, b] \to \mathbf{R}$ are differentiable at $x \in [a, b]$, prove that so is their product fgh, and find $(fgh)'(x)$.
   ii) If $f, g : [a, b] \to \mathbf{R}$ are thrice[2] differentiable at $x \in [a, b]$, prove that so is their product fg, and find $(fg)'''(x)$.
   iii) Generalize.

9. i) Let U, V, W be finite-dimensional vector spaces over $\mathbf{R}$ or $\mathbf{C}$, and f, g any functions from an interval $[a, b]$ to the finite-dimensional vector spaces $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(U, V)$ respectively. If for some $x \in [a, b]$ both f and g are differentiable at x, prove that so is $f \circ g$, and $(f \circ g)'(x) = (f'(x) \circ g(x)) + (f(x) \circ g'(x))$.
   ii) Now let V be a finite-dimensional real or complex vector space and assume that the function $f : [a, b] \to \mathrm{End}(V)$ is differentiable at some $x \in [a, b]$. Prove that if $f(x)$ is invertible then the function $[a, b] \to \mathrm{End}(V)$, $t \to (f(t))^{-1}$ is differentiable at x, and determine its derivative. [Hint: heed Artin's admonition (from *Geometric Algebra*), "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."]

10. i) Prove that if K is a field equipped with a derivation D then $k := \ker D$ is a subfield of K. This is called the "constant subfield" of K. Show that $D : K \to K$ is k-linear.
    ii) Now suppose $K = F(X)$, the field of rational functions in one variable over some field F. Define $D : K \to K$ by the usual formula: if $P = \sum_n a_n X^n$ then $D(P) = \sum_n n a_n X^{n-1}$; and any $f \in K$ is the quotient P/Q of two polynomials, so we may write $D(P/Q) = (P'Q - PQ')/Q^2$. Show that this is well-defined (i.e. if $f = P_1/Q_1 = P_2/Q_2$ then the two definitions of $D(f)$ agree), and yields a derivation of K. What is the constant subfield?

11. [Wronskians[3]] The numerator $f'g - fg'$ of the formula for f/g is the case $n = 2$ of a *Wronskian*. In general, if $f_1, \ldots, f_n$ are scalar-valued functions on $[a, b]$, each of which is differentiable $n - 1$ times at some $x \in [a, b]$, then their "Wronskian" at x is the determinant of the $n \times n$ matrix, call it $M_W(f_1, \ldots, f_n)$, whose (i, j) entry is the $(j - 1)$-st derivative of f_i. In the context of an arbitrary differential field, we likewise let $M_W(f_1, \ldots, f_n)$ be the matrix whose (i, j) entry is $D^{j-1}(f_i)$.
    i) Suppose each f_i is differentiable $n - 1$ times on all of $[a, b]$. Prove that if the f_i are linearly dependent over the scalar field then their Wronskian vanishes. Likewise show in the algebraic setting of a differential field K that if the f_i are linearly dependent over the constant field k then their Wronskian vanishes.
    ii) In the algebraic setting, construct a homomorphism $w : K^* \to \mathrm{GL}_n(K)$ such that

$$M_W(cf_1, cf_2, \ldots, cf_n) = M_W(f_1, f_2, \ldots, f_n)\, w(c)$$

    for all $f_1, \ldots, f_n \in K$ and $c \in K^*$. Use this homomorphism to show the converse of part (i): if $\det M_W(f_1, \ldots, f_n) = 0$ then the f_i are linearly dependent over k.
    iii) Construct differentiable real-valued functions f, g on some interval I such that f, g are linearly independent but their Wronskian $f'g - fg'$ vanishes on all of I.

[2] once : twice : thrice :: 1 : 2 : 3. Look it up if you don't believe me. As far as I know the sequence "once, twice, thrice" has no fourth term in English (though it does have a zeroth term of sorts in "never").

[3] I believe that this is pronounced as if it were "Vronskians", but I could be vrong.

We conclude with one of the few cases where the theory developed so far lets us prove that a Taylor expansion of some function f actually represents f:

12. Assume[4] that the standard formula $d(x^r)/dx = rx^{r-1}$ for the derivative of a power holds for all $r \in \mathbf{R}$ as long as $x > 0$. Prove that the binomial expansion

$$(1+x)^r = 1 + rx + r(r-1)\frac{x^2}{2!} + r(r-1)(r-2)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n-1} (r-j) \cdot \frac{x^n}{n!} \right)$$

holds for all $r \in \mathbf{R}$ and $x \in (-1, 1)$, and that the convergence is uniform in compact subsets of $(-1, 1)$.

Problems 8–12 are due Friday, March 10, at the beginning of class.

[4]This result is true, but we cannot readily prove it yet. It's not too hard when $r \in \mathbf{Q}$, but the standard trick of writing an arbitrary real number as the limit of a sequence of rational numbers is not enough: we could use it to define x^r for any $r \in \mathbf{R}$ but not (without further work) to differentiate it, because in general differentiation does not commute with pointwise or even uniform limits.