**Math 55b: Honors Real and Complex Analysis**

Homework Assignment #10 (17 April 2017):
Integration in $\mathbf{R}^k$ and $\mathbf{C}$, etc.

Old MacDonald had a form: $e_i \wedge e_i = 0$.
*—Mike Stay (October 2009), in a* `mathoverflow` *thread "Do good math jokes exist?".*

More about Bohr-Mollerup and logarithmic convexity:

1. [Cf. PS8, problem 4] For real $\nu, \lambda$ with and $|\lambda| < 1$ and $\nu > |\lambda|$, prove that

$$\int_0^{\pi/2} \cos^\nu x \cos(\lambda x)\, dx = \frac{\pi}{2^{\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma\Big(1+\frac{\nu-\lambda}{2}\Big)\,\Gamma\Big(1+\frac{\nu+\lambda}{2}\Big)}.$$

[The hypotheses on $\nu$ and $\lambda$ can be relaxed further, as we'll see when we develop complex analysis.]

2. The logarithmic convexity of $\Gamma(x)$, or more generally of any function of the form $f(x) = \int (\alpha(t))^x \beta(t)\, dt$, can be interpreted as the nonnegativity of the determinant of a symmetric $2 \times 2$ matrix. Generalize this to larger determinants. For instance, prove that for any positive reals $a_1, \ldots, a_n$ the determinant of the $n \times n$ matrix with entries $\Gamma(a_i + a_j)$ is nonnegative, as is the determinant with entries $(a_i + a_j)^{-k}$ for any $k > 0$. [Hint for this last part: remember $\int_0^\infty t^{x-1} e^{-ct}\, dt$?]

Change of variable and related ideas:

3. [Calabi] Prove that

$$\int_0^1\!\!\int_0^1 \frac{dx\, dy}{1 - x^2 y^2} = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{n=1}^\infty \frac{1}{n^2}.$$

(Note that this is an improper integral (why?) so some care will be needed here.) Now let $\Delta$ be the triangle $\{(u,v) \in \mathbf{R}^2 | u, v > 0, u + v < \pi/2\}$. Prove that the map $T : \Delta \to \mathbf{R}^2$ defined by $T(u,v) = (\sin u / \cos v, \sin v / \cos u)$ is a $\mathscr{C}^1$ and $\mathscr{C}^1$-invertible map of $\Delta$ to the open unit square $0 < x, y < 1$, and compute its Jacobian determinant. Conclude that

$$\int_0^1\!\!\int_0^1 \frac{dx\, dy}{1 - x^2 y^2} = \iint_\Delta 1\, du\, dv = \frac{\pi^2}{8}$$

as desired.
[One way to prove $T(\Delta)$ covers the square is to construct for each $(u,v)$ a suitable contraction map on the square. Can you generalize this to evaluate $\sum_{n=1}^\infty 1/n^4$, or show more generally that $\sum_{n=1}^\infty 1/n^s$ is a rational multiple of $\pi^s$ for all positive even $s$? Can you prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - \cdots = \frac{\pi^3}{32}$$

using this method?]

4. [Newton] Let $V$ be the vector space $\mathbf{R}^3$ with the usual Euclidean norm, and let $\omega = dx_1 \wedge dx_2 \wedge dx_3$ be the standard volume form on $V$. Define a vector-valued function $g$ from $V - \{\mathbf{0}\}$ to $V$ by $g(v) = v/|v|^3$. Fix $r_1, r_2 \in \mathbf{R}$ with $0 < r_1 < r_2$, and let $E$ be the spherical shell $\{x \in V : r_1 < |x| < r_2\}$.
   i) Prove that if $v_0 \in V$ with $|v_0| < r_1$ then $\int_E g(x - v_0)\,\omega = 0$.
   ii) Prove that if $v_0 \in V$ with $|v_0| > r_2$ then $\int_E g(x - v_0)\,\omega = I g(-v_0)$, where $I = \int_E \omega = 4\pi(r_2^3 - r_1^3)/3$ is the volume of $E$.

Note: The manipulations required, while straightforward, may be somewhat lengthy. I don't assign many such problems, but this one has special significance, both theoretical and historical: Newton had observed that the force of gravity near the earth's surface and the acceleration of the moon in its orbit around the earth are consistent with a universal inverse-square law of gravitation, provided the gravitational force of a spherical body was equivalent to that of an equal point mass; but it was only some twenty years later that he succeeded in proving this result and thus clinching the inverse-square law. That's the significance of part (ii); part (i) also figures in the physics of electrostatic forces: a uniformly charged sphere exerts no force on its interior. (For the effect on the exterior of the sphere, see again part (ii).) It is now known that this can be proved in a more "conceptual" way, albeit at the cost of introducing more machinery (surface integrals, etc.), from the fact that the inverse-square force exerted by a point mass is the gradient of a potential function $G(x) = C/|x - x_0|$ satisfying the Laplace equation $\Delta G(x) = 0$ for all $x \neq x_0$.

Some more basic properties of analytic functions:

5. (Reflection principles)
   i) Let $E$ be an open set in $\mathbf{C}$, and $E' = \{z \in \mathbf{C} : \bar{z} \in E\}$ (I can't call this "$\overline{E}$" because that looks like topological closure). Prove that $f : E \to \mathbf{C}$ is differentiable if and only if the function $E' \to \mathbf{C}$ defined by $z \mapsto \overline{f(\bar{z})}$ is differentiable. Deduce that if $E$ is an open rectangle or circle symmetric about the real axis and $f(z) \in \mathbf{R}$ for all $z \in E \cap \mathbf{R}$ then $f(\bar{z}) = \overline{f(z)}$ for all $z \in E$.
   ii) Suppose now that $r > 1$ and let $E$ be the annulus $\{z \in \mathbf{C} : 1/r < |z| < r\}$. If $f : E \to \mathbf{C}$ is a differentiable function such that $|f(z)| = 1$ for all $z$ on the unit circle $|z| = 1$, what can you deduce about $f$?

6. Let $E$ be an open set in $\mathbf{C}$, fix some real number $B$, and let $\mathcal{F}$ be the family of analytic functions $f : E \to \mathbf{C}$ such that $|f(z)| \leq B$ for all $z \in E$. If $K$ is any compact subset of $E$, prove that $\{f'(z) : f \in \mathcal{F},\, z \in K\}$ is bounded. Deduce that "$\mathcal{F}|_K$ is equicontinuous": for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z')| < \epsilon$ for all $f \in \mathcal{F}$ and any $z, z' \in K$ such that $|z - z'| < \delta$.

This problem set due Monday, April 24, at the beginning of class (but you can probably do it by Friday, in which case I encourage you to do so).