Math 55b: Honors Real and Complex Analysis

Homework Assignment #8 (21 March 2011):
More univariate calculus

> Fejér discovered his theorem[1] at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.[2]

More about power series:

1. For p-adic numbers a_n $(n = 1, 2, 3, \ldots)$, prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ in $\mathbf{Q}_p$. For which $x \in \mathbf{Q}_p$ does the exponential series $E(x) = \sum_{n=0}^{\infty} x^n/n!$ converge? Which $a \in \mathbf{Q}_p$ can be written as $E(x)$ for some $x \in \mathbf{Q}_p$ such that the sum for $E(x)$ converges?

Some integration techniques. First we show how to integrate an arbitrary rational function:

2. [Partial fractions[3]] Let k be an algebraically closed field. Let $K = k(x)$, the field of rational functions in one variable x with coefficients in k. Show that the following elements of K constitute a basis for K as a vector space over k: x^n for $n = 0, 1, 2, 3, \ldots$, and $1/(x - x_0)^n$ for $x_0 \in k$ and $n = 1, 2, 3, \ldots$. (Linear independence is easy. To prove that the span is all of K, consider for any polynomial $Q \in k[x]$ the subspace $V_Q := \{P/Q : P \in k[x], \deg(P) < \deg(Q)\}$ of K, and compare its dimension with the number of basis vectors in V_Q.)

3. Prove that the integral of any $f \in \mathbf{R}(x)$ is a rational function plus a linear combination of functions of the form $\log|x - x_0|$, $\log((x - x_0)^2 + c)$, and $\tan^{-1}(ax + b)$ $(x_0, a, b, c \in \mathbf{R}$, $c > 0)$.

Next we derive some classical product formulas and integrals. Be careful about justifying all steps!

4. Prove that $\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$ for all $n \geq 2$. Deduce that

$$\int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{2}{3}\frac{4}{5}\frac{6}{7} \cdots \frac{n-1}{n}, & \text{if } n \text{ is odd;} \\ \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6} \cdots \frac{n-1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

5. It follows that

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5} \cdots \frac{2m}{2m-1}\frac{2m}{2m+1} \cdot \frac{\int_0^{\pi/2} \cos^{2m} x\, dx}{\int_0^{\pi/2} \cos^{2m+1} x\, dx}\,.$$

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[1] On Fourier series; see Rudin, pages 199–200.

[2] Körner, *Fourier Analysis*, p.294 (conclusion of Chapter 59: "Weierstrass's proof of Weierstrass's theorem").

[3] The decomposition of any $f \in K$ as a linear combination of the basis elements described in this problem is called the "partial fraction decomposition" of f.

Show that

$$1 < \frac{\int_0^{\pi/2} \cos^{2m} x\,dx}{\int_0^{\pi/2} \cos^{2m+1} x\,dx} < \frac{\int_0^{\pi/2} \cos^{2m-1} x\,dx}{\int_0^{\pi/2} \cos^{2m+1} x\,dx} = 1 + \frac{1}{2m}\,,$$

and therefore

$$\frac{\pi}{2} = \lim_{m\to\infty} \left(\frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5} \cdots \frac{2m}{2m-1}\,\frac{2m}{2m+1} \right).$$

[This is usually written as the "infinite product"

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots,$$

attributed to Wallis.]

6. Use the formulas of the previous problem to prove that

$$\lim_{n\to\infty} \int_0^{\sqrt{n}\pi/2} \cos^n \frac{x}{\sqrt{n}}\,dx = \sqrt{\pi/2}.$$

Now show that $\lim_{n\to\infty} \cos^n(x/\sqrt{n}) = \exp(-x^2/2)$ for any $x \geq 0$, and use this to prove that[4]

$$\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}.$$

7. Define $I_n(\lambda)$ for $0 < \lambda < 1$ by

$$I_n(\lambda) = \int_0^{\pi/2} \cos^n x \cos(\lambda x)\,dx \qquad (n = 0, 1, 2, \ldots).$$

Integrate by parts twice to prove that $(n^2 - \lambda^2) I_n(\lambda) = (n^2 - n) I_{n-2}(\lambda)$ for $n \geq 2$. Then evaluate $I_0(\lambda)$ and $I_1(\lambda)$ to obtain a formula for $I_n(\lambda)$ for all n. Deduce a product formula for $\tan(\pi\lambda/2)$, and verify that Wallis' product can be recovered from your formula by taking the limit as $\lambda \to 0$. Can you obtain any further formulas by investigating the behavior of $I_n(\lambda)$ as $n \to \infty$?

This problem set due Friday, 25 March, at the beginning of class. You may, however, postpone any one of these problems until the due date of the next problem set.

[4] As noted in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral $\int \exp(cx^2)\,dx$ cannot be simplified. We shall give another proof of this result when we come to the change of variable formula for multiple integrals.