Math 55b: Honors Real and Complex Analysis

Homework Assignment #7 (4 March 2011):
Univariate integral calculus

> Bitte ve[r]giß alles, was Du auf der Schule gelernt hast;
> denn Du hast is nicht gelernt. *Emil Landau*[1]

Admittedly that's a bit extreme, but it is true that for many of you Stieltjes integrals, especially of vector-valued functions, are a new path in the familiar territory of integration, and might require a different kind of thinking. This problem set consists of only eight problems including four from Rudin, but most are geared towards developing such "new kinds of thinking", and a few are somewhat open-ended to suggest further directions in analysis that we won't pursue in Math 55.

1.–4. Solve Problems 3, 4, 8, 9 on pages 138–9. [For #3 it may help to look at #1 as well, though #1 is not required. Problems #8 and #7 are similarly related; naturally in #9 you need only deal with the "improper integrals" of #8, not #7. For #4, your intuition might protest that since the vast majority of real numbers are irrational the integral "ought" to be zero; the Lebesgue integral does this, but its construction is nontrivial even for functions of one variable, and we won't cover it in Math 55. To find out more about Lebesgue integration see the final chapter of Rudin.]

5. [Bernoulli polynomials][2] Prove that for each positive integer m there exists a polynomial B_m such that $\sum_{i=1}^n i^{m-1} = B_m(n)$ for all positive integers n. [Hint: What must the polynomial $B_m(x) - B_m(x-1)$ be? The map taking any polynomial $P(x)$ to the polynomial $Q(x) := P(x) - P(x-1)$ is linear.] Determine the leading coefficient of B_m, and deduce the value of $\int_0^b x^{m-1}\,dx$ for any $b > 0$ (and thus also of $\int_a^b x^{m-1}\,dx$) without using the Fundamental Theorem of Calculus. Beyond the leading term, what further patterns can you detect in the coefficients of B_m? Can you prove any of these patterns? (You may need to go at least to $m = 6$ or $m = 7$ to see what's going on; a computer algebra system could help to handle the linear algebra manipulations.)

6. [Fermat] Prove *using the Riemann-sum definition of the integral* that $\int_a^b x^{r-1}\,dx = (b^r - a^r)/r$ for every nonzero rational number r and all real a, b such that $0 < a < b$. [Note: since Fermat predated Newton, the solution cannot use the Fundamental Theorem of Calculus. Besides the special case that r is a positive integer, addressed in the previous problem, you may also find a solution for

[1] Quote taken from Chapter 10 of M. Artin's *Algebra*. It roughly translates as "Please forget all that you have learned in school, for you haven't [really] learned it." Don't complain about the German transcription, which is presumably of some local dialect — even I recognize that this isn't the the German we *auf der Schule lernen*.

[2] Not that it matters for our purposes, but the "Bernoulli polynomials" usually seen in the literature differ from our B_m by an additive constant.

the special case that $1/(r-1)$ is a positive integer — but this will not directly lead you to a solution of the general case.]

7. In the `vint` handout on integration of vector-valued functions you might have expected a theorem to the effect that such a function is integrable (as defined there) with integral I if and only if for each $\epsilon > 0$ there exists a partition P all of whose Riemann sums differ from I by vectors of norm at most ϵ. Certainly the existence of such P is a consequence of integrability, but in fact the converse implication does not hold! Prove this by finding a normed vector space V and a function $f : [0,1] \to V$ such that $\Delta(P) = 1$ for any partition P (and thus $f \notin \mathscr{R}$), but nevertheless for each ϵ there exist partitions P such that every Riemann sum $R(P, \vec{t})$ for $\int_0^1 f(x)\,dx$ has norm at most ϵ. [Hints: f cannot be continuous or even nearly so, because then our vector version of Thm. 6.8 would yield integrability; in fact the function I have in mind is discontinuous everywhere. Moreover, V cannot be finite dimensional. Thus the example is rather pathological — but it is also simple enough that it can be described and proved in a short paragraph.]

8. Fix a bounded function f on an interval $[a, b]$. Let

$$M_f := \{\alpha - \beta : f \in \mathscr{R}(\alpha) \cap \mathscr{R}(\beta)\}.$$

Show that M_f is a vector space, and that

$$I_f : \alpha - \beta \mapsto \int_a^b f\,d\alpha - \int_a^b f\,d\beta$$

yields a well-defined linear map on M_f. Naturally we write this map as $I_f(\mu) = \int_a^b f\,d\mu$. Try to define M_f intrinsically, i.e., in such a way that one can recognize functions $\mu \in M_f$ directly and define $I_f(\mu)$ without finding α, β. (It is possible to do this in a way that generalizes smoothly to vector-valued f or μ, and makes the integration-by-parts formula

$$f(b)g(b) - f(a)g(a) = \int_a^b f\,dg + \int_a^b g\,df$$

work even when say f is vector valued. In that case we must of course start from the definition of $\int_a^b f\,d\alpha$ in the `vint` handout. You might also get a sense of what to do from Rudin's discussion of "rectifiable curves" at the end of Chapter 6.)

This problem set due Friday, 11 March, at the beginning of class. You may postpone at most two problems until 10AM on March 21, which is the first day of class after Spring Break; but you'd probably rather not use this extension unless you really must...