Math 55b: Honors Real and Complex Analysis

Homework Assignment #6 (25 February 2011):
Univariate differential calculus

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out. — E. Artin, *Geometric Algebra.*

Most of this problem set concerns "differential algebra"; that is, familiar algebraic axioms of a (usually commutative) ring or field, extended by a map $D : f \mapsto f'$ satisfying the axioms $(f+g)' = f' + g'$ and $(fg)' = fg' + f'g$. Such a map is called a *derivation* of the ring or field. Note that in the case of a field, the formula $(f/g)' = (f'g - fg')/g^2$ holds automatically because the argument we gave in class starting from $f = g(f/g)$ uses only the field and derivation axioms. The topological considerations that arise in the definition of the derivative enter into some of the following problems but are not the main point except for the final problem.

1. i) If $f, g, h : [a, b] \to \mathbf{R}$ are differentiable at $x \in [a, b]$, prove that so is their product fgh, and find $(fgh)'(x)$.
   ii) If $f, g : [a, b] \to \mathbf{R}$ are thrice[1] differentiable at $x \in [a, b]$, prove that so is their product fg, and find $(fg)'''(x)$.
   iii) Generalize.

2. i) Let U, V, W be finite-dimensional vector spaces over $\mathbf{R}$ or $\mathbf{C}$, and f, g any functions from an interval $[a, b]$ to the finite-dimensional vector spaces $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(U, V)$ respectively. If for some $x \in [a, b]$ both f and g are differentiable at x, prove that so is $f \circ g$, and $(f \circ g)'(x) = (f'(x) \circ g(x)) + (f(x) \circ g'(x))$.
   ii) Now let V be a finite-dimensional real or complex vector space and assume that the function $f : [a, b] \to \mathrm{End}(V)$ is differentiable at some $x \in [a, b]$. Prove that if $f(x)$ is invertible then the function $[a, b] \to \mathrm{End}(V)$, $t \to (f(t))^{-1}$ is differentiable at x, and determine its derivative. [Hint: remember Artin's quote above.]

3. i) Prove that if K is a field equipped with a derivation D then $k := \ker D$ is a subfield of K. This is called the "constant subfield" of K. Show that $D : K \to K$ is k-linear.
   ii) Now suppose $K = F(X)$, the field of rational functions in one variable over some field F. Define $D : K \to K$ by the usual formula: if $P = \sum_n a_n X^n$ then $D(P) = \sum_n n a_n X^{n-1}$; and any $f \in K$ is the quotient P/Q of two polynomials, so we may write $D(P/Q) = (P'Q - PQ')/Q^2$. Show that this is well-defined (i.e. if $f = P_1/Q_1 = P_2/Q_2$ then the two definitions of $D(f)$ agree), and yields a derivation of K. What is the constant subfield?

4. [Wronskians[2]] The numerator $f'g - fg'$ of the formula for f/g is the case $n = 2$ of a *Wronskian.* In general, if $f_1, \ldots, f_n$ are scalar-valued functions on $[a, b]$, each of which is differentiable $n-1$ times at some $x \in [a, b]$, then their "Wronskian" at x is the determinant of the $n \times n$ matrix, call it $M_W(f_1, \ldots, f_n)$, whose (i, j) entry is the $(j-1)$-st derivative of f_i. In the context of an arbitrary differential field, we likewise let $M_W(f_1, \ldots, f_n)$ be the matrix whose (i, j) entry is $D^{j-1}(f_i)$.
   i) Suppose each f_i is differentiable $n-1$ times on all of $[a, b]$. Prove that if the f_i are

[1] once : twice : thrice :: 1 : 2 : 3. Look it up if you don't believe me. As far as I know the sequence "once, twice, thrice" has no fourth term in English (though it does have a zeroth term of sorts in "never").

[2] I believe that this is pronounced as if it were "Vronskians", but I could be vrong.

linearly dependent over the scalar field then their Wronskian vanishes. Likewise show in the algebraic setting of a differential field K that if the f_i are linearly dependent over the constant field k then their Wronskian vanishes.

ii) In the algebraic setting, construct a homomorphism $w : K^* \to \mathrm{GL}_n(K)$ such that

$$M_W(cf_1, cf_2, \ldots, cf_n) = M_W(f_1, f_2, \ldots, f_n)\, w(c)$$

for all $f_1, \ldots, f_n \in K$ and $c \in K^*$. Use this homomorphism to show the converse of part (i): if $\det M_W(f_1, \ldots, f_n) = 0$ then the f_i are linearly dependent over k.

iii) Construct differentiable real-valued functions f, g on some interval I such that f, g are linearly independent but their Wronskian $f'g - fg'$ vanishes on all of I.

One of the few cases where the theory developed so far lets us prove that a Taylor expansion of some function f actually represents f:

5. Assume[3] that the standard formula $d(x^r)/dx = rx^{r-1}$ for the derivative of a power holds for all $r \in \mathbf{R}$ as long as $x > 0$. Prove that the binomial expansion

$$(1+x)^r = 1 + rx + r(r-1)\frac{x^2}{2!} + r(r-1)(r-2)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n-1} (r-j) \cdot \frac{x^n}{n!} \right)$$

holds for all $r \in \mathbf{R}$ and $x \in (-1, 1)$, and that the convergence is uniform in compact subsets of $(-1, 1)$.

The remaining two problems explicitly concern the topological underpinnings of differentiation:

6. [Rudin, problem 8 on page 114–115]

i) Suppose $f : [a, b] \to \mathbf{R}$ has a continuous derivative f'. Prove that f is "uniformly differentiable on $[a, b]$": for all $\epsilon > 0$ there exists $\delta > 0$ such that $\left|(f(t) - f(x))/(t - x) \; - \; f'(x)\right| < \epsilon$ for all distinct $x, t \in [a, b]$ such that $|t - x| < \delta$.

ii) Does this remain true if f takes values in $\mathbf{R}^n$, or in an arbitrary inner-product space V?

7. [a preview of part of multivariate differential calculus] Recall that if f is a function on some subset E of $\mathbf{R}^2$, and (x_0, y_0) is an interior point of U, then the *partial derivative* $\partial f/\partial x$ of f at (x_0, y_0) is the value at $x = x_0$ of the derivative of the function $f(x, y_0)$ of x, if that derivative exists; the partial derivative $\partial f/\partial y$ is defined similarly.

Let E be the closed unit circle $x^2 + y^2 \le 1$, and assume $f : E \to \mathbf{R}$ is continuous, and that both $\partial f/\partial x$ and $\partial f/\partial y$ exist at all interior points of E. Show that if $f(x, y) = x$ for all x, y such that $x^2 + y^2 = 1$ then there exists some interior point $(x_0, y_0) \in E$ at which $\partial f/\partial x = 1$ and $\partial f/\partial y = 0$.

This problem set due Friday, 4 March, at the beginning of class, together with any problems you postponed from Problem Set 5.

[3] This result is true, but we cannot readily prove it yet. It's not too hard when $r \in \mathbf{Q}$, but the standard trick of writing an arbitrary real number as the limit of a sequence of rational numbers is not enough: we could use it to define x^r for any $r \in \mathbf{R}$ but not (without further work) to differentiate it, because in general differentiation does not commute with pointwise or even uniform limits.