**Math 55b: Honors Real and Complex Analysis**

Homework Assignment #5 (18 February 2011):
Uniform continuity, completeness, and compactness: the grand finale

Q: What did the mathematician say as $\epsilon$ approached zero?
A: "There goes the neighborhood."
—*Hoary math joke*

This final topology homework contains several standard problems on completeness and compactness, and several which show these concepts in action in various contexts.

1. Let $d_1, d_2, d_3$ be the following three metrics on $\mathbf{R}$:
   $d_1$ is the standard metric $d_1(x,y) = |x-y|$;
   $d_2$ is the discrete metric; and
   $d_3$ is the metric $d_3(x,y) = |x^3 - y^3|$.
   The identity function $x \mapsto x$ on $\mathbf{R}$ then gives rise to six functions $(\mathbf{R}, d_i) \rightarrow (\mathbf{R}, d_j)$ with $i \neq j$.
   i) Which of these six functions
   $$(\mathbf{R}, d_1) \rightleftarrows (\mathbf{R}, d_2), \quad (\mathbf{R}, d_2) \rightleftarrows (\mathbf{R}, d_3), \quad (\mathbf{R}, d_3) \rightleftarrows (\mathbf{R}, d_1)$$
   are continuous?
   ii) Of those, which are uniformly continuous?
   iii) For which $i, j \in \{1, 2, 3\}$ does there exist a subset $S \subseteq \mathbf{R}$ that becomes compact in $(\mathbf{R}, d_i)$ but not in $(\mathbf{R}, d_j)$? Give and justify an example of such an $S$.

2. Prove or disprove: a metric space is separable if and only if every open cover has a countable subcover. [As usual finite sets are regarded as countable.]

3. For a topological space $X$, a function $f : X \rightarrow \mathbf{R}$ is said to have a "local maximum" at $x \in X$ if there is some neighborhood $N$ of $x$ such that $f(x) = \max_{y \in N} f(y)$. A "local minimum" is defined similarly.
   Now let $f \in \mathbf{C}[z]$ be a nonconstant polynomial. We shall show that $f(z) = 0$ for some $z \in \mathbf{C}$. As we already know, this yields the Fundamental Theorem of Algebra.
   i) Prove that the functions $\mathrm{Re}(f(\cdot))$, $\mathrm{Im}(f(\cdot))$, $|f(\cdot)|$ have no local maximum, and no local minimum except for the local minimum of $|f(\cdot)|$ at any $z \in \mathbf{C}$ such that $f(z) = 0$. (Hint: it may help to note that it is enough to check for a local maximum or minimum at $z = 0$.)
   ii) Show that for all $M > 0$ there exists $R > 0$ such that $|f(z)| > M$ for all $z \in \mathbf{C}$ such that $|z| > R$. (Intuitively, "$|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$".) Combine this with the previous part and Heine-Borel to complete the proof that $f(z) = 0$ has a solution $z \in \mathbf{C}$.

4. Prove or disprove: if $X, Y, Z$ are metric spaces and $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are any uniformly continuous functions then the composite function $g \circ f : X \rightarrow Z$ is uniformly continuous.

5. Let $X, Y$ metric spaces, and $X^*, Y^*$ their completions. Prove that any uniformly continuous $f : X \rightarrow Y$ extends[1] uniquely to a continuous function $f^* : X^* \rightarrow Y^*$, and that $f^*$

[1] A function $f^*$ on a set $S^*$ "extends" a function $f$ on a subset $S \subseteq S^*$ if $f^*(s) = f(s)$ for all $s \in S$. We also say (as in the footnoted word of this problem) that $f$ "extends to $f^*$".

is still uniformly continuous. Show that if $f$ is continuous, but not uniformly so, then there might not be a continuous $f^*$ that extends $f$.

Problem 4 is the main thing to check if we want a "category of metric spaces with uniformly continuous functions"; if there is such a category then Problem 5 is the key step in constructing a "completion functor" to its subcategory of complete metric spaces.

6. Let $Y$ be a metric space, $X$ an arbitrary set, and $\{f_n\}$ a sequence of functions from $X$ to $Y$. We saw that if the $f_n$ are bounded then $f_n$ approaches a function $f : X \to Y$ in the $\mathcal{B}(X, Y)$ metric if and only if $f_n \to f$ uniformly. What should it mean for a sequence $\{f_n\}$ to be "uniformly Cauchy"? Prove that if $Y$ is complete and $X$ is a topological space then a uniformly Cauchy sequence of continuous functions from $X$ to $Y$ converges uniformly to a continuous function.

7. In the previous problem set we defined a metric

$$d_1(f, g) := \int_0^1 |f(x) - g(x)| \, dx$$

on the space $\mathcal{C}([0, 1], \mathbf{C})$.
i) Show that $\mathcal{C}([0, 1], \mathbf{C})$ is <u>not</u> complete under this metric.
ii) Fix $x \in [0, 1]$. Is the map $f \mapsto f(x)$ from $\mathcal{C}([0, 1], \mathbf{C})$ to $\mathbf{C}$ continuous with respect to the $d_1$ metric?
iii) Now fix a continuous function $m : [0, 1] \to \mathbf{C}$, and define a map $I_m : \mathcal{C}([0, 1], \mathbf{C}) \to \mathbf{C}$ by

$$I_m(f) := \int_0^1 f(x) \, m(x) \, dx.$$

Prove that this map is uniformly continuous.

[By problem 5, the map $I_m$ extends to a uniformly continuous map on the completion $L_1([0, 1])$ of $\mathcal{C}([0, 1], \mathbf{C})$ under the $d_1$ metric. This map is also linear: it satisfies the identity $I_m(af + bg) = aI_m(f) + bI_m(g)$ for all $f, g \in L_1([0, 1])$ and $a, b \in \mathbf{C}$. Are there any linear maps from $L_1([0, 1])$ to $\mathbf{C}$ not of that form?]

The next two problems describe, for a compact space $X$, certain compact spaces of functions on $X$ or subsets of $X$; these are very useful in rigorous treatments of the calculus of variations and isoperimetric problems, respectively. The last problem (similar to one used some years ago in the qualifying exam for graduate students here) is a version of the contraction mapping theorem; later in the course we'll prove and use the usual contraction mapping theorem to show existence and uniqueness of solutions of certain differential equations.

8. Let $X$ be a compact topological space, and $\mathcal{F} \subseteq \mathcal{C}(X, \mathbf{C})$ any family of continuous functions. We say $\mathcal{F}$ is *equicontinuous* if, for each $\epsilon > 0$ and any $x \in X$, there exists an open set $U \subseteq X$ containing $x$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $f \in \mathcal{F}$ and $y \in U$. Prove that $\mathcal{F}$ is bounded if and only if

$$\{f(x) : x \in X, f \in \mathcal{F}\}$$

is a bounded subset of $\mathbf{C}$. Prove that $\mathcal{F}$ is totally bounded if and only if it is bounded and equicontinuous. What happens if $\mathbf{C}$ is replaced by an arbitrary complete metric space?

9. Recall that for any metric space $X$ we gave the set $\mathcal{X}$ of nonempty, bounded, closed subsets of $X$ the structure of a metric space using the Minkowski distance.
   i) Prove that if $X$ is complete then so is $\mathcal{X}$.
   ii) Prove that if $X$ is totally bounded then so is $\mathcal{X}$.

10. Let $X$ be a metric space, and $f$ a function from $X$ to itself such that

$$d(x, y) > d(f(x), f(y))$$

for all $x, y \in X$ such that $x \neq y$. Let $g : X \to \mathbf{R}$ be the real-valued function on $X$ defined by

$$g(x) := d(x, f(x)).$$

   i) Prove that $f$ is continuous, and has at most one fixed point (that is, there is at most one $x_0 \in X$ such that $x_0 = f(x_0)$).
   ii) Prove that $g$ is continuous.
   iii) Conclude that if $X$ is compact then $f$ has a fixed point. Must this still be true if $X$ is complete but not necessarily compact?

Problem set is due Friday, February 25, at the beginning of class. You may, however, postpone at most three problems until March 4, which will be the due date of the next problem set.