Math 55b: Honors Real and Complex Analysis

Homework Assignments #3 and #4 (4 February 2011):
Metrics, sequences, compactness, and a bit of general topology

> **Sketch of a proof** *n.* I couldn't verify all the details, so I'll break it down into the parts I couldn't prove.[1]

Please avoid merely "sketching" (as defined in the above quote) a proof. In all problem sets, you may use the result in one problem (or problem part) to solve another, even if you have not proved the first one, unless this becomes circular [EXCEPTION: when problem B is clearly a generalization of A, don't use B to solve A unless you've solved B!]. As in Math 55a, the problems are generally not in order of difficulty.

More about the topology of $\mathbf{R}$, and a connection with continuity:

1. i) Prove that the only subsets of $\mathbf{R}$ that are simultaneously open and closed are $\emptyset$ and $\mathbf{R}$.
   ii) Suppose X, Y are metric spaces, and that X has the discrete metric. Find all continuous maps from X to Y. Find all continuous maps from $\mathbf{R}$ to X.

Another characterization of convergence:

2. Fix a sequence $\{r_n\}$ of positive real numbers such that $r_1 > r_2 > r_3 > \cdots$ and $r_n \to 0$. Let $\widetilde{\mathbf{N}}$ be the metric space consisting of $1, 2, 3, \ldots$ together with a symbol ∞, with the distance function defined by

$$d(m,n) = |r_m - r_n|, \quad d(n,\infty) = d(\infty,n) = r_n, \quad d(\infty,\infty) = 0.$$

   [In other words, d is defined so that the map $\rho : \widetilde{\mathbf{N}} \to \mathbf{R}$ given by $n \mapsto r_n$, $\infty \mapsto 0$ is an isometry to $\rho(\widetilde{\mathbf{N}})$.] Let $\mathbf{N}$ be the subspace $\{1, 2, 3, \ldots\}$ of $\widetilde{\mathbf{N}}$, so $\widetilde{\mathbf{N}}$ is the disjoint union of $\mathbf{N}$ with $\{\infty\}$.
   i) Which subsets of $\mathbf{N}$ are open?
   ii) Which subsets of $\widetilde{\mathbf{N}}$ are open?
   iii) Let $\{s_n\}$ be a sequence in an arbitrary metric space X. Let $\sigma : \mathbf{N} \to X$ be the map that takes n to s_n. Show that $\{s_n\}$ converges if and only if σ extends to a continuous function $\tilde{\sigma} : \widetilde{\mathbf{N}} \to X$ (that is, if and only if there exists a continuous $\tilde{\sigma} : \widetilde{\mathbf{N}} \to X$ such that $\tilde{\sigma}(n) = \sigma(n)$ for all $n \in \mathbf{N}$), in which case $\tilde{\sigma}(\infty) = \lim_{n\to\infty} s_n$.

More about sequences and $\mathcal{C}(X, Y)$:

3. Prove that[2]

$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx$$

   is a metric on the space $\mathcal{C}([0,1], \mathbf{C})$ of (bounded) continuous functions $f : [0,1] \to \mathbf{C}$ on the closed unit interval $[0,1]$. [That is, the vector space $\mathcal{C}([0,1], \mathbf{C})$ has a norm $\|\cdot\|_1$ defined by $\|f\|_1 = \int_0^1 |f(x)|\, dx$.]

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

[2] Yes, I know: we have yet to officially define $\int_0^1$. For this problem, though, only the most basic facts are needed, such as the existence of $\int_0^1 F(x)\, dx$ for any continuous function $F : [0,1] \to \mathbf{R}$, and the fact that if $F(x) \le G(x)$ for all $x \in [0,1]$ then $\int_0^1 F(x)\, dx \le \int_0^1 G(x)\, dx$. Only ϵ more is needed for the next problem.

4. Let X be our metric space $\mathcal{C}([0,1],\mathbf{R})$ of continuous functions on $[0,1]$ with $d_X(f,g) = \max_{0\le x\le 1}|f(x)-g(x)|$.
   i) Find an infinite set $S \subset X$ such that the restriction of d_X to S is the discrete metric. Could you do the same for the metric d_1 of the previous problem?
   ii) Let $\mathbf{0} \in X$ be the zero function. Is the closed unit ball $\overline{B}_1(\mathbf{0})$ in X compact? Why?

5. Define sequences $\{f_n\}$, $\{g_n\}$ $(n = 1,2,3,\ldots)$ of functions from $\mathbf{R}$ to $\mathbf{R}$ by

$$f_n(x) = \frac{n}{x^2+n^2}, \qquad g_n(x) = \frac{n^2}{x^2+n^2}.$$

   i) Do these sequences of functions f_n and g_n converge pointwise?
   ii) Do they converge uniformly on $\mathbf{R}$? Explain.

More about compact metric spaces:

6. Show directly that a sequentially compact subset of a metric space is closed and totally bounded.

7. Say that a subset E of a metric space X is "totally bounded relative to X" if, for each $\epsilon > 0$, there is a finite cover of E by ϵ-neighborhoods in X. Prove that E is totally bounded relative to X if and only if E is totally bounded. [That is, allowing centers of the ϵ-neighborhoods to be in a larger ambient metric space does not change the notion of total boundedness.]

8. Let $\{U_\alpha\}$ be an open cover of the compact metric space X. Show that there exists $r > 0$ such that, for every $x \in X$, the r-ball $B_r(x)$ is contained in some U_α. [Proceed by contradiction, assuming no such r exists. Construct a sequence $\{x_n\}$ in X such that $B_{1/n}(x_n)$ is contained in no U_α. Let x be the limit of a convergent subsequence. Show that $B_\rho(x) \subseteq U_\alpha$ for some $\rho > 0$ and α. Obtain the contradiction by showing that $B_\rho(x)$ contains some $B_{1/n}(x_n)$.]

Finally, a bit more about topological spaces that need not be metric. Recall that a *topological space* is an ordered pair $(X,\mathcal{T})$ where $\mathcal{T}$ is a *topology* on X, that is, a family of subsets of X (the "open sets" of the topology) that contains $\emptyset$, X, and the finite intersection and arbitrary union of any sets in $\mathcal{T}$. We noted that the open sets in a metric space constitute a topology, but not all topologies arise in this way; for instance, for any set X with more than 1 element, $\{\emptyset, X\}$ is a non-metric topology, because in a metric topology all one-point sets are closed.

9. A topological space $(X,\mathcal{T})$ is said to be *Hausdorff* if, for any two distinct $p,q \in X$, there are disjoint $U,V \in \mathcal{T}$ with $U \ni p$ and $V \ni q$. For instance, a metric space is automatically Hausdorff, since we may take U and V to be the open balls of radius $\frac{1}{2}d(p,q)$ about p and q.
   i) Prove that in a Hausdorff space every single-point set is closed.
   ii) Now let X,Y be topological spaces with Y Hausdorff, and let f,g be any continuous functions from X to Y. If $S \subset X$ is a dense subset such that $f(s) = g(s)$ for all $s \in S$, prove that $f = g$, i.e., that $f(x) = g(x)$ for all $x \in X$. [Naturally you must use the topological definition of denseness: "S is dense in X" means that the only open set in X disjoint from S is $\emptyset$.]

NB one useful strategy for obtaining results about general topological spaces is to prove them first in the special and more intuitive case of metric spaces, and then try to adapt the proof to an arbitrary topological space. Even if this doesn't work directly, it might give you a better sense of just where in the argument a topological space that need not be metric requires a new approach.

Problems 1–9 are due at 10AM on Friday, February 11. You may, however, postpone any one or two of these problems until the due date of problems 10–13.

Applications of Heine-Borel:

10. i) Prove that (as noted but not proved in Math 55a) all norms on a finite-dimensional vector space are equivalent (see ⟨http://www.math.harvard.edu/∼elkies/M55a.10/norm.html⟩ if you need a reminder of what this means).
    ii) Use (i) to solve the following Putnam problem: Prove that for each positive integer n there exists a constant C_n such that, if $P(x)$ is a polynomial of degree at most n, then $|P(0)| \leq C_n \int_{-1}^{1} |P(x)|\, dx$.

11. Prove that the closed unit ball in an inner-product space V over either $\mathbf{R}$ or $\mathbf{C}$ is compact (in the metric topology associated to the usual norm defined by $\|x\| = (x,x)^{1/2}$) if and only if V is finite dimensional.

More about separation properties in metric spaces and general topological spaces:

12. Let X be a metric space and A, B disjoint closed subsets. Prove that there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$.

13. A topological space X is said to be *normal* if it is Hausdorff and has the property that for every disjoint closed subsets A, B there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$. Show that if A, B are disjoint closed subsets of a normal space X then there exists a continuous function $f : X \to \mathbf{R}$ such that $f(X) \subseteq [0,1]$ and $f(x) = 0$ if $x \in A$ while $f(x) = 1$ if $x \in B$.
    [Hint: If $S \subset [0,1]$ is a dense subset then a function $f : X \to [0,1]$ is specified completely by the sets $f^{-1}([s,1])$ for $s \in S$.]

To put #13 in context: the distance function on a metric space gives us a ready source of continuous functions from the space to $\mathbf{R}$; in particular, enough such functions to "separate points": if $x_0 \neq x_1$ then there's a continuous function taking x_0 to 0 and x_1 to 1. [For instance, the function $x \mapsto d(x,x_0)/d(x_1,x_0)$ does the trick.] According to Problem 13, the space $\mathcal{C}(X,\mathbf{R})$ separates points also under the hypothesis of normality, which is weaker than (well, at least as weak as) metrizability by #12.

Problems 10–13, and any problems you postponed from 1–9, are due Friday, February 18, at the beginning of class.