Math 55b: Honors Real and Complex Analysis

Homework Assignment #13 (22 April 2011):
Complex Analysis II: residues and contour integration

Who knows Thirteen? —*Echad Mi Yodea*[1]

Since the last day before Reading Period is Wednesday, and the take-home final will happen soon afterwards, this final problem set is shorter than usual and is be due *Wednesday, April 27*, at the beginning of class.

We start with a few examples of using contour integration to evaluate definite integrals:

1. [From our graduate Qualifying Exam, Spring 1997] For $b > 0$, compute $\int_0^\infty \log x \, dx/(x^2+b^2)$.

2. Prove using contour integration that $\int_{-\infty}^{\infty} \exp(-(x-w)^2)\, dx$ is independent of w even when w is allowed to be a complex number. Hence it is equal to $\sqrt{\pi}$, since we've seen (in several ways) that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. Derive from this Rudin's formula for $\int_0^\infty e^{-x^2} \cos cx \, dx$ (Example 9.43 on pages 237–238).

3. Recall that for $z \in \mathbf{C}$ the *hyperbolic cosine* $\cosh z$ is defined as $\cos(iz) = (e^z + e^{-z})/2$. [2] Prove that
$$\int_0^\infty \frac{\cos(mx)}{\cosh(\pi x)}\, e^{-mx^2}\, dx = \frac{1}{2}\, e^{-m/4}$$
for all $m > 0$. [From the Fall 1998 Qualifying Exam.] Can you evaluate any other such integrals this way (other than those obtained trivially from this formula by linear change of variable etc.)?

More about residues, contour integrals, and the complex Gamma function:

4. Determine for any entire function f the residue at $z = 0$ of the differential $f(\cot(z))\, dz$. In particular, what is the residue at the origin of $\sin(\cot(z))\, dz$?

Note that in general $f(\cot(z))$ has an essential singularity at $z = 0$. As far as I know, the other odd-order coefficients of the Laurent expansion of $\sin(\cot(z))$ about $z = 0$ are not known in closed form.

5. Let f be a nonconstant analytic function on a neighborhood of z_0, and let n be the multiplicity of the zero at z_0 of the function $f(z) - f(z_0)$. We have seen that, for any a sufficiently close to $f(z_0)$, the equation $f(z) = a$ has n solutions (counted with multiplicity) near z_0, call them $z_1, \ldots, z_n$ in some order. Prove that the coefficients of the polynomial $\prod_{j=1}^n (X - z_j)$ are analytic functions of a in that neighborhood of z_0. [These coefficients are (up to sign) the elementary symmetric functions in the z_j. For $n = 1$ the claim is equivalent to the existence of an analytic inverse function. To prove it in general, show that $\sum_{j=1}^n z_j^k$ is an analytic function of a for each $k = 1, 2, 3, \ldots$.]

6. Prove that for $c, x > 0$ the integral $\int_{-\infty}^{\infty} \Gamma(c+iy)\, x^{-iy}\, dy$ converges to $2\pi x^c e^{-x}$. [While we will not prove Stirling's approximation for the complex Gamma function, here it will be enough to use judiciously the elementary inequality $|\Gamma(z)| \le |\Gamma(\mathrm{Re}(z))|$ for $\mathrm{Re}(z) > 0$.]

[1] Hebrew for "Who Knows One?"; one of the traditional (post-)Seder songs, which (unlike the similarly-structured "Partridge in a Pear Tree", but like this semester's problem sets) goes to 13.

[2] The abbreviation "cosh" is from the Latin *cosinus hyperbolicus*, but is still often pronounced to rhyme with "mosh" or even expanded to "coshine"…